Pseudocompact C^*-Algebras

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August 4, 2017

Finite-Dimensional C*-algebras and Their Limits

- Finite-dimensional C*-algebras are just finite direct sums of matrix algebras.
- $\mathcal{K}(\mathcal{H})$ – the algebra of compact operators (norm-limits of finite-rank operators) on a Hilbert space $\mathcal{H}$.
- Uniformly hyperfinite or UHF algebras – inductive limits of matrix algebras with unital embeddings. Classified by their supernatural number. (Glimm)
- Approximately finite-dimensional or AF-algebras – inductive limits of finite-dimensional algebras. Classified by their augmented K_0 group. (Bratteli, Elliott)
- The pseudocompact algebras are *logical* limits of finite-dimensional C*-algebras.

Pseudofiniteness & Pseudocompactness

- A field K is *pseudofinite* if each classical first-order statement which is true in every finite field is also true in K. (Ax) There is also interest in pseudofinite groups.
- The analogous property to pseudofiniteness was given by Goldbring and Lopes: A C*-algebra $\mathfrak{A}$ is *pseudocompact* if whenever a continuous first-order property holds in every finite-dimensional C*-algebra then it holds in $\mathfrak{A}$.

Definition of Pseudocompact C*-algebras

- $\mathfrak{A}$ is a *pseudocompact* C*-algebras if it satisfies any of the following equivalent conditions:
  - If $\varphi^{\mathcal{F}} = 0$ for all finite-dimensional $\mathcal{F}$ then $\varphi^{\mathfrak{A}} = 0$.
  - If $\psi^{\mathfrak{A}} = 0$ then for all $\varepsilon > 0$ there is a finite-dimensional $\mathcal{F}$ so that $|\psi^{\mathcal{F}}| < \varepsilon$.
  - $\mathfrak{A}$ is elementarily equivalent to an ultraproduct of finite-dimensional C*-algebras.
- The pseudocompacts are the smallest axiomatizable class containing the finite-dimensional C*-algebras.
- Similarly we define *pseudomatrical* C*-algebras by replacing "finite-dimensional C*-algebra" with "matrix algebra".
- We are specifically interested in separable, infinite-dimensional pseudocompact C*-algebras.

(Bad) Examples of Pseudocompact C*-algebras

Let $\mathcal{U}$ be a free ultrafilter on the natural numbers.

- $\prod_{\mathcal{U}} M_n$ is a pseudomatricial C*-algebra. But this is non-separable. Use the Löwenheim-Skolem theorem to get a separable elementary subalgebra.
- $\prod_{\mathcal{U}}(M_2)^{\oplus n}$ is a pseudocompact C*-algebra. It is homogeneous of degree 2.

These are not concrete examples - they depend on the choice of the ultrafilter $\mathcal{U}$!

Commutative Pseudocompact C*-Algebras

- We know commutative, unital C*-algebras are of the form $\mathcal{C}(K)$ for compact Hausdorff K.
- If K_n are compact Hausdorff spaces, then $\prod_{\mathcal{U}} \mathcal{C}(K_n)$ is a commutative unital C*-algebra. Thus there is a compact Hausdorff space K so that

$$\prod_{\mathcal{U}} \mathcal{C}(K_n) \cong \mathcal{C}(K).$$

- The set-theoretic ultraproduct $\prod_{\mathcal{U}} K_n$ is canonically homeomorphic to a dense subset of K. (Henson)
- If $\mathcal{C}(K_n) \cong \mathbb{C}^{k_n}$ is finite-dimensional, then K_n is a finite discrete space.

- Theorem (Henson/Moore, Eagle/Vignati)

  $\mathcal{C}(K)$ is pseudocompact if and only if K is totally disconnected with a dense subset of isolated points.

Commutative Pseudocompact C*-Algebras

There is an explicit axiomatization of commutative pseudocompact C*-algebras:

- $\phi_c^{\mathfrak{A}} = \sup_{||x||,||y|| \leq 1} ||xy - yx|| = 0$.
  This guarantees that the algebra is commutative.
- $\phi_u^{\mathfrak{A}} = \inf_{||e|| \leq 1} \sup_{||x|| \leq 1} ||ex - x|| = 0$.
  This guarantees that the algebra is unital.
- $\phi_{rr0}^{\mathfrak{A}} = \sup_{x,y \text{ s.a.}} \inf_{p \text{ proj.}} \max\left(||px||, ||1-p||y|| \right)^2 \dot{-} ||xy|| = 0$.
  This guarantees that the algebra is real rank zero, so the underlying space is totally disconnected.
- $\sup_{||x|| \leq 1} \inf_{p \text{ proj}} \sup_{||y|| \leq 1} \inf_{|\lambda| \leq 1} ||pyp - \lambda p|| + \left| ||x|| - ||xp|| \right| = 0$.
  This says every element can be normed by minimal projections. This guarantees that the underlying space has dense isolated points.

Examples of Commutative Pseudocompact C*-Algebras

- $\mathcal{C}(\beta\mathbb{N}) \cong \ell^\infty(\mathbb{N})$ is pseudocompact.
- $\mathcal{C}(\mathbb{N} \cup \{\infty\}) \cong c$, the space of convergent sequences, is pseudocompact.
- $\mathcal{C}$(Cantor set) is AF but not pseudocompact.
- There is a totally disconnected compact Hausdorff space with dense isolated points which quotients onto the Cantor set.
- Subalgebras and quotients of pseudocompact C*-algebras need not be pseudocompact.

(Lack of) Examples

- Very little is known about pseudocompact Banach spaces, for instance it is not known if ℓ^p are pseudocompact or not.
- In the tracial von Neumann algebra setting, the hyperfinite II_1 factor is not pseudocompact since it has property Γ. (Fang/Hadwin and Farah/Hart/Sherman) We do not know concrete examples of pseudocompact II_1 factors.
- We do not know concrete examples of pseudomatricial algebras! However we can show that several natural candidates are not pseudomatrical.

Basic Properties

- Direct sums of pseudocompact C*-algebras are pseudocompact.
- Corners of pseudocompact C*-algebras are pseudocompact. That is, if $\mathfrak{A}$ is pseudocompact and $p \in \mathfrak{A}$ is a projection, then $p\mathfrak{A}p$ is pseudocompact.
- Matrix amplifications of pseudocompact C*-algebras are pseudocompact. That is, if $\mathfrak{A}$ is pseudocompact $M_n(\mathfrak{A}) \cong M_n \otimes \mathfrak{A}$ is pseudocompact.
- MF algebras are exactly those that admit norm microstates. (Brown/Ozawa) A separable C*-algebra is MF if and only if it is a (not necessarily unital) subalgebra of a pseudocompact C*-algebra. (Farah)

Properties of Pseudocompact C*-Algebras

Farah et al. showed the following properties are axiomatizable:

- Unital.
- Admitting a tracial state.
- Finite – left invertible elements are right invertible. Equivalently, isometries are unitaries. Thus pseudocompact algebras are stably finite.
- Stable rank one – the invertible elements are dense.
- Real rank zero – the self-adjoint elements with finite spectrum are dense in the self-adjoint elements of $\mathfrak{A}$. In particular, the span of the projections is dense.

Pseudomatricial C*-algebras are never nuclear!

Admitting a Tracial State is Axiomatizable

- Recall that we can show a property is axiomatizable if it is closed under $*$-isomorphisms, ultraproducts, and ultraroots, that is, if an ultrapower of $\mathfrak{A}$ has the property then $\mathfrak{A}$ has the property.
- Admitting a tracial state is clearly invariant under $*$-isomorphism.
- If τ_i is a tracial state on $\mathfrak{A}_i$, τ defined by $\tau(a_i)_{\mathcal{U}} = \lim_{\mathcal{U}} \tau_i(a_i)$ is a tracial state on $\prod_{\mathcal{U}} \mathfrak{A}_i$.
- If $\tau_{\mathcal{U}}$ is a tracial state on $\mathfrak{A}^{\mathcal{U}}$ we get a tracial state τ on $\mathfrak{A}$ defined by $\tau(a) = \tau_{\mathcal{U}}(a)_{\mathcal{U}}$.
- This does not give us an explicit set of conditions! But Farah et al. found an explicit set of conditions: for all n

$$\sup_{x_1,\ldots,x_n} \left(1 \dot{-} \| I - \sum_{i=1}^{n} [x_i, x_i^*] \| \right)$$

Finiteness is Axiomatizable

- Recall $\mathfrak{A}$ is finite if left-invertible elements are invertible.
- It is clear that finiteness is invariant under $*$-isomorphism.
- Proposition: $(a_i)_{\mathcal{U}} \in \prod_{\mathcal{U}} \mathfrak{A}_i$ is invertible if and only if there is an $S \in \mathcal{U}$ and an N so for all $i \in S$, a_i is invertible and $||a_i^{-1}||_{\mathcal{U}} < N$.
- Suppose for all i, $\mathfrak{A}_i$ is finite, and $(a_i)_{\mathcal{U}} \in \prod_{\mathcal{U}} \mathfrak{A}_i$ is left-invertible. Then there are $b_i \in \mathfrak{A}_i$ so that

$$(b_i a_i)_{\mathcal{U}} = (b_i)_{\mathcal{U}}(a_i)_{\mathcal{U}} = (I_i)_{\mathcal{U}}.$$

  There is a set $S \in \mathcal{U}$ so for all $i \in S$, $||b_i a_i - I_i||_{\mathcal{U}} < \frac{1}{2}$. This means that $b_i a_i$ is invertible (and the inverses have uniformly bounded norms!), so a_i is left-invertible, so a_i is invertible. Thus $(a_i)_{\mathcal{U}}$ is invertible.

Finiteness is Axiomatizable, continued

- Suppose $\mathfrak{A}^{\mathcal{U}}$ is finite and $a \in \mathfrak{A}$ is left-invertible. Then there is some $b \in \mathfrak{A}$ so $ba = I$, so $(a)_{\mathcal{U}} \in \mathfrak{A}^{\mathcal{U}}$ is left-invertible, thus invertible. So there are $b_i \in \mathfrak{A}$ so $(a)_{\mathcal{U}}(c_i)_{\mathcal{U}} = (ac_i)_{\mathcal{U}} = (I)_{\mathcal{U}}$. Proceed as above.
- This does not give us an explicit set of conditions! But Farah et al. found an explicit definable predicate:

$$\sup_{x \text{ isometry}} ||xx^* - I||$$

Properties of Pseudocompact C*-Algebras, continued

Another way to find properties of pseudocompact C*-algebras is to find properties of matrices that are independent of dimension:

- If A is a self-adjoint trace-zero matrix then there is a matrix B with $||B|| \leq \sqrt{2}||A||$ so $A = [B, B^*]$ (Thompson, Fong). Thus self-adjoint trace-zero elements in pseudomatricial C*-algebras are also self-commutators.
- Almost-normal elements in matrix algebras are close to normal elements (Lin, Friss/Rørdam). The same thing holds in pseudocompact C*-algebras.
- Matrix algebras have highly irreducible elements (von Neumann, Herrero/Szarek). That is, there is a $\varepsilon > 0$ so that

$$\inf_{||a|| \leq 1} \sup_{p \text{ non-trivial proj.}} ||ap - pa|| > \varepsilon$$

  in every matrix algebra and thus in every pseudocompact C*-algebra.

Properties of Pseudocompact C*-Algebras, continued

- Pseudocompact C*-algebras have the Dixmier property:

$$\forall a \in \mathfrak{A}, \qquad \overline{\mathrm{conv}(\mathcal{U}(a))}^{||\cdot||} \cap \mathcal{Z}(\mathfrak{A}) \neq \emptyset.$$

- If $\mathfrak{A}$ has the Dixmier property, $\mathrm{dist}(a, \mathcal{Z}(\mathfrak{A})) \leq \sup_{||x|| \leq 1} ||xa - ax||$ (Ringrose). For pseudocompact C*-algebras $\mathfrak{A}_n$, $\mathcal{Z}(\prod_{\mathcal{U}} \mathfrak{A}_n) = \prod_{\mathcal{U}} \mathcal{Z}(\mathfrak{A}_n)$. Not all AF algebras have this property!
- Centers of pseudocompact C*-algebras are pseudocompact.
- The pseudomatricial C*-algebras are the pseudocompact C*-algebras with trivial centers.

Unitaries

Theorem (Ge/Hadwin)

Let $\mathcal{U}$ be an ultrafilter on I, and for all $i \in I$ let $\mathfrak{A}_i$ be a non-trivial C^*-algebra. Consider the ultraproduct $\prod_{\mathcal{U}} \mathfrak{A}_i$. Then $(x_i)_{\mathcal{U}}$ is a unitary if and only if there is a representative sequence $(x_i)_{\mathcal{U}} = (u_i)_{\mathcal{U}}$ where the u_i are unitaries.

- Unitaries play nicely with continuous logic. That is, the unitaries form a definable set.
- In matrix algebras, unitaries are all of the form $\exp(ih)$ for self-adjoint h. In pseudocompact C*-algebras, unitaries are norm limits of unitaries of the form $\exp(ih)$ for self-adjoint h. Thus the connected component of the identity is the whole unitary group. This means the K_1 groups of pseudocompact C*-algebras are trivial.

Projections

Theorem (Ge/Hadwin)

Let $\mathcal{U}$ be an ultrafilter on I, and for all $i \in I$ let $\mathfrak{A}_i$ be a non-trivial C^*-algebra. Consider the ultraproduct $\prod_{\mathcal{U}} \mathfrak{A}_i$.

- *$(x_i)_{\mathcal{U}}$ is a projection if and only if there is a representative sequence $(x_i)_{\mathcal{U}} = (p_i)_{\mathcal{U}}$ where the p_i are projections. In fact, if p, and q are projections in $\prod_{\mathcal{U}} \mathfrak{A}_i$ with $q \leq p$, then for all i there are projections p_i, and $q_i \in \mathfrak{A}_i$ with $q_i \leq p_i$ so that $p = (p_i)_{\mathcal{U}}$ and $q = (q_i)_{\mathcal{U}}$.*
- *If $p = (p_i)_{\mathcal{U}}$ and $q = (q_i)_{\mathcal{U}}$ are Murray-von Neumann equivalent projections, then there are partial isometries v_i such that $v = (v_i)_{\mathcal{U}}$ and for $\mathcal{U}$-many i, $p_i = v_i^* v_i$ and $q_i = v_i v_i^*$.*

Projections

- Projections play nicely with continuous logic. That is, projections and partial isometries are definable sets.
- Finite-dimensional C*-algebras are determined by their matrix units.
- Projections are an important tool in understanding pseudocompact C*-algebras.

Projections in Pseudomatrical C*-Algebras

- Murray-von Neumann equivalence, unitary equivalence, and homotopy equivalence are all the same.
- Every non-zero projection dominates a minimal projection. UHF algebras are not pseudocompact.
- A non-zero projection p in a pseudomatricial C*-algebra $\mathfrak{A}$ is minimal if and only if $p\mathfrak{A}p = \mathbb{C}p$.
- All projections are comparable.
- All minimal projections are equivalent. Thus minimal projections in an infinite-dimensional pseudomatricial C*-algebra vanish under any tracial state. Infinite-dimensional pseudomatrical algebras are not simple.
- The trace ideal is maximal.

Projections in Pseudomatrical C*-Algebras, continued

- In a matrix algebra M_n, n is either even or odd.
- The identity in a pseudomatricial C*-algebra can be written as a sum of two orthogonal Murray-von Neumann equivalent projections, and maybe an orthogonal minimal projection. The unitization of the compacts $\mathcal{K}(\mathcal{H})^\sim$ is not pseudocompact.
- You can do this modulo any number!
- The tracial state is unique.
- There are uncountably many isomorphism classes of separable pseudomatricial C*-algebras.
- Conjecture: $\prod_{\mathcal{U}} M_{k_n} \equiv \prod_{\mathcal{V}} M_{j_m}$ if and only if for all d,

$$\lim_{\mathcal{U}} k_n \mod d = \lim_{\mathcal{V}} j_n \mod d$$

K_0 Groups of Pseudomatrical C*-Algebras

- Strict comparison of projections: if $\tau(q) < \tau(p)$ then $q \precsim p$.
- The K_0 group of a pseudomatricial C*-algebra is a totally-ordered abelian group with successors and predecessors. These are classified by Hahn's embedding theorem.
- The K_0 group of a pseudomatricial C*-algebra is of the form $G \oplus \ker(K_0(\tau))$ as ordered abelian groups, where G is a divisible subgroup of $\mathbb{R}$ and $\ker(K_0(\tau))$ is the subgroup generated by trace-zero projections.
- Let G be a countable divisible subgroup of $\mathbb{R}$ and S be a countable subset of $[0, 1]$. We can find a separable pseudomatricial C*-algebra $\mathfrak{A}$ so that $K_0(\mathfrak{A}) \supseteq G \oplus (\mathbb{Z}^S)$ as (lexicographically) ordered abelian groups.

K_0 Groups of Pseudomatrical C*-Algebras, Continued

(Proof sketch.)

Consider $\mathfrak{A} = \prod_{\mathcal{U}} M_n$ where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. For $s \in S$, let $p_n^{(s)}$ be a rank $\lfloor n^s \rfloor$ projection in M_n. Consider $P_s = (p_n^{(s)})_{\mathcal{U}}$, then $\{P_s\}_{s\in S}$ is a countable family of projections in $\mathfrak{A}$. Note that

$$\tau(P_s) = \lim_{\mathcal{U}} \tau_n(p_n^{(s)}) = \lim_{\mathcal{U}} \frac{\lfloor n^s \rfloor}{n} = 0.$$

If $s > r$, then for all $m \in \mathbb{N}$, eventually $x^s > mx^r$. P_s dominates m orthogonal copies of P_r. In $K_0(\mathfrak{A})$, $[P_s]_0 \gg [P_r]_0$ when $s > r$ are in S. So $K_0(\mathfrak{A}) \supseteq \mathbb{Z}^S$.
Apply the downward Löwenheim-Skolem to get a separable subalgebra of $\mathfrak{A}$ which is elementarily equivalent to $\mathfrak{A}$ and contains these projections.

$\square$

Future Goals

- Characterize elementary equivalence of pseudomatricial algebras.
- Find axiomatizations or characterizations for the pseudocompact and pseudomatricial C*-algebras.
- Determine if infinite-dimensional pseudomatricial C*-algebras can be exact or quasidiagonal.

Thank you!