Groupoid C^*-algebras and their canonical diagonal subalgebras

Efren Ruiz
Work in progress with Toke Carlsen, Aidan Sims, and Mark Tomforde

University of Hawai'i at Hilo

APPLICATIONS OF MODEL THEORY TO OPERATOR ALGEBRAS

Objects of interest

$$(A, D)$$

- A and D are separable C^*-algebras
- D is a commutative C^*-subalgebra of A

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Main Example

$$A = C^*(\mathcal{G}) \quad \text{and} \quad D = C_0(\mathcal{G}^{(0)})$$

- $\mathcal{G}$ is a second-countable, locally compact, Hausdorff, étale groupoid

$$s : \gamma \mapsto \gamma^{-1}\gamma \quad \text{and} \quad r : \gamma \mapsto \gamma\gamma^{-1}$$

are local homeomorphisms.

$C_r^*(\mathcal{G})$

$C_c(\mathcal{G})$:

$$(f \star g)(\gamma) = \sum_{\lambda\beta=\gamma} f(\lambda)g(\beta)$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}$$

$\pi_\lambda^u \colon C_c(\mathcal{G}) \to B(\ell^2(s^{-1}(u)))$,

$$(\pi_\lambda^u(f)\xi)(\gamma) = \sum_{\lambda\beta=\gamma} f(\lambda)\xi(\beta)$$

$$\|f\|_r := \sup\left\{\|\pi_\lambda^u(f)\| \ : \ u \in \mathcal{G}^{(0)}\right\}$$

$$C_r^*(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_r}$$

Motivating Examples

Motivating Examples

Theorem (Tomiyama)

Let X and Y be compact, Hausdorff spaces and let (X, σ) and (Y, τ) be topologically free dynamical systems. Then the following are equivalent:

1. $(C(X) \rtimes_\sigma \mathbb{Z}, C(X)) \cong (C(Y) \rtimes_\tau \mathbb{Z}, C(Y))$ *and*
2. (X, σ) *and* (Y, τ) *are continuous orbit equivalent, i.e., there exist a homeomorphism* $h\colon X \to Y$ *and continuous functions* $m, n\colon X \to \mathbb{Z}$ *such that*

$$h(\sigma(x)) = \tau^{m(x)}(h(x)) \quad \text{and} \quad \tau(h(x)) = h(\sigma^{n(x)}(x)).$$

Transformation groupoid: $X \rtimes_\sigma \mathbb{Z}$

1. $X \rtimes_\sigma \mathbb{Z} = X \times \mathbb{Z}$ (Product topology)
2. $(x, n)(y, m) = (x, n + m)$ if and only if $\sigma^n(x) = y$
3. $(n, x)^{-1} = (\sigma^n(x), -n)$
4. $(X \rtimes_\sigma \mathbb{Z})^{(0)} = X \times \{0\} \cong X$.

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Theorem

$$(C(X) \rtimes_\sigma \mathbb{Z}, C(X)) \cong (C_r^*(X \rtimes_\sigma \mathbb{Z}), C((X \rtimes_\sigma \mathbb{Z})^{(0)}))$$

Theorem (Tomiyama and Renault)

Let X and Y be second-countable, compact, Hausdorff spaces and let (X, σ) and (Y, τ) be topologically free dynamical systems. Then the following are equivalent:

1. $(C(X) \rtimes_\sigma \mathbb{Z}, C(X)) \cong (C(Y) \rtimes_\tau \mathbb{Z}, C(Y))$,
2. (X, σ) *and* (Y, τ) *are continuous orbit equivalent, i.e., there exist a homeomorphism* $h\colon X \to Y$ *and continuous functions* $m, n\colon X \to \mathbb{Z}$ *such that*
$$h(\sigma(x)) = \tau^{m(x)}(h(x)) \quad \text{and} \quad \tau(h(x)) = h(\sigma^{n(x)}(x)),$$
and
3. $X \rtimes_\sigma \mathbb{Z} \cong Y \rtimes_\tau \mathbb{Z}$.

Cuntz-Krieger algebras

One-sided shift space

Let $\mathsf{A} \in \mathsf{M}_N(\{0,1\})$.

1. $X_\mathsf{A} = \{(x_n)_{n\in\mathbb{N}} \in \{1,2,\ldots,N\}^{\mathbb{N}} \ : \ \mathsf{A}(x_n, x_{n+1}) = 1\}$
2. $\sigma_\mathsf{A} \colon X_\mathsf{A} \to X_\mathsf{A}$, $[\sigma_\mathsf{A}((x_n)_{n\in\mathbb{N}})]_n = x_{n+1}$

Theorem (Matsumoto-Matui, Brownlowe-Carlsen-Whittaker, Arklint-Eilers-R (Carlsen-Winger))

Let $\mathsf{A} \in \mathsf{M}_N(\{0,1\})$ *and let* $\mathsf{B} \in \mathsf{M}_{N'}(\{0,1\})$. *Then the following are equivalent:*

1. $(\mathcal{O}_{\mathsf{A}}, C(X_{\mathsf{A}})) \cong (\mathcal{O}_{\mathsf{B}}, C(X_{\mathsf{B}}))$ *and*
2. $(X_{\mathsf{A}}, \sigma_{\mathsf{A}})$ *and* $(X_{\mathsf{B}}, \sigma_{\mathsf{B}})$ *are continuous orbit equivalent, i.e., there exist a homeomorphism* $h\colon X_{\mathsf{A}} \to X_{\mathsf{B}}$, *and continuous functions* $k, l\colon X_{\mathsf{A}} \to \mathbb{N}$ *and* $k', l'\colon X_{\mathsf{B}} \to \mathbb{N}$ *such that*

$$\sigma_{\mathsf{B}}^{k(x)}(h(\sigma_{\mathsf{A}}(x))) = \sigma_{\mathsf{A}}^{l(x)}(h(x))$$

$$\sigma_{\mathsf{A}}^{k'(y)}(h^{-1}(\sigma_{\mathsf{B}}(y))) = \sigma_{\mathsf{A}}^{l'(y)}(h^{-1}(y)).$$

Theorem (Matsumoto-Matui, Carlsen-Eilers-Ortega-Restorff)

Let $\mathsf{A} \in \mathsf{M}_N(\{0,1\})$ *and let* $\mathsf{B} \in \mathsf{M}_{N'}(\{0,1\})$. *Then the following are equivalent:*

1. $(\mathcal{O}_\mathsf{A} \otimes \mathbb{K}, C(X_\mathsf{A}) \otimes c_0(\mathbb{N})) \cong (\mathcal{O}_\mathsf{B} \otimes \mathbb{K}, C(X_\mathsf{B}) \otimes c_0(\mathbb{N}))$ *and*
2. *the two-sided shift spaces* $(\overline{X}_\mathsf{A}, \overline{\sigma}_\mathsf{A})$ *and* $(\overline{X}_\mathsf{B}, \overline{\sigma}_\mathsf{B})$ *are flow equivalent.*

The groupoid of a one-sided shift space

Let $\mathrm{A} \in \mathrm{M}_N(\{0,1\})$.

1. $G_{\mathrm{A}} = \{(x, n-m, y) \colon x, y \in X_{\mathrm{A}}, n, m \in \mathbb{Z}_{>0}, \sigma_{\mathrm{A}}^n(x) = \sigma_{\mathrm{A}}^m(y)\}$
2. $(x, n-m, y)(x', n'-m', y') = (x, n+n'-m-m', y')$ if and only if $y = x'$
3. $(x, n-m, y)^{-1} = (y, m-n, x)$
4. $G_{\mathrm{A}}^{(0)} = \{(x, 0, x) \ : \ x \in X_{\mathrm{A}}\} \cong X_{\mathrm{A}}$
5. $\mathcal{Z}(U, n, m, V) =$ $\{(x, n-m, y) \ : \ x \in U, y \in V, \sigma_{\mathrm{A}}^n(x) = \sigma_{\mathrm{A}}^m(y)\}$,

   U, V are open subets of X_{A}

Theorem

$$(\mathcal{O}_{\mathsf{A}}, C(X_{\mathsf{A}})) \cong (C_r^*(G_{\mathsf{A}}), C(G_{\mathsf{A}}^{(0)}))$$

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1. $(\mathcal{O}_{\mathsf{A}}, C(X_{\mathsf{A}})) \cong (\mathcal{O}_{\mathsf{B}}, C(X_{\mathsf{B}}))$,
2. *there exists a continuous orbit equivalence between* $(X_{\mathsf{A}}, \sigma_{\mathsf{A}})$ *and* $(X_{\mathsf{B}}, \sigma_{\mathsf{B}})$, *and*
3. $G_{\mathsf{A}} \cong G_{\mathsf{B}}$.

Theorem (Renault, Brownlowe-Carlsen-Whittaker)

Let $\mathcal{G}, \mathcal{H}$ be second-countable, locally compact, Hausdorff, étale groupoids. Then the following are equivalent:

1. $(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)})) \cong (C_r^*(\mathcal{H}), C_0(\mathcal{H}^{(0)}))$
2. $\mathcal{G} \cong \mathcal{H}$

whenever $\mathcal{G}, \mathcal{H}$ are topologically principal groupoids or $\mathcal{G}, \mathcal{H}$ are groupoids associated to one-sided shift spaces.

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Key Idea

Construct a groupoid

$$\mathcal{H}(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}))$$

such that

$$\mathcal{H}(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)})) \cong \mathcal{G}.$$

Definition

A ***semidiagonal pair*** of C^*-algebras is a pair (A, D) consisting of a separable C^*-algebra A and a subalgebra D of A such that

1. D is abelian,
2. D contains an approximate identity for A,
3. for each $\phi \in \widehat{D}$, the quotient D'/J_ϕ of D' by the ideal $J_\phi := \ker(\phi)D'$ is a unital C^*-algebra, and
4. for each $\phi \in \widehat{D}$, there exist $d \in D$ and an open neighbourhood U of ϕ such that $d + J_\psi = 1_{D'/J_\psi}$ for all $\psi \in U$.

Definition

Let A be a C^*-algebra and D be a C^*-subalgebra of A. A ***normalizer of** D* is an element $n \in$ A such that

$$nDn^* \cup n^* Dn \subseteq D.$$

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Let A be a C^*-algebra and D be a C^*-subalgebra of A. A ***normalizer of*** D is an element $n \in$ A such that

$$nDn^* \cup n^*Dn \subseteq D.$$

Theorem (Kumjian, Renault)

Let A be a C^*-algebra and D an abelian C^*-subalgebra of A that contains an approximate unit for A. Suppose that n is a normalizer of D. Then there is a homeomorphism

$$\alpha_n : \{u \in \widehat{D} : u(n^*n) > 0\} \to \{u \in \widehat{D} : u(nn^*) > 0\}$$

such that $u(n^*n)\alpha_n(u)(d) = u(n^*dn)$ for all $d \in D$.

Lemma

Let (A, D) be a semidiagonal pair, n, m be normalizers of D, and $\phi \in \widehat{D}$. Suppose there exists an open neighborhood U of ϕ such that

$$U \subseteq \operatorname{supp}(n^*n) \cap \operatorname{supp}(m^*m).$$

Then for any $d \in D$ with $\operatorname{supp}(d) \subseteq U$ and $\phi(d) = 1$, we have that

$$\phi(m^*nn^*m)^{-\frac{1}{2}} dn^*md$$

is in D' and

$$\phi(m^*nn^*m)^{-\frac{1}{2}} dn^*md + J_\phi$$

is a unitary in D'/J_ϕ that is independent of the choices of U and d.

$$S(A, D) = \left\{(n, \phi) \in \mathcal{N}(D) \times \widehat{D} \,:\, \phi(n^* n) > 0\right\}$$

$(n, \phi) \sim (m, \psi)$ if and only if

1. $\phi = \psi$,
2. there exists an open neighborhood of ϕ such that $\alpha_n|_U = \alpha_m|_U$, and
3. $\phi(m^* n n^* m)^{-\frac{1}{2}} d n^* m d + J_\phi \in \mathcal{U}_0(D'/J_\phi)$.

The groupoid $\mathcal{H}(A, D)$

$$\mathcal{H}(A, D) = \{[(n, \phi)] \ : \ (n, \phi) \in S(A, D)\}$$

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1. $[(n, \phi)][(m, \psi)] = [(nm, \psi)]$ if and only if $\phi = \alpha_m(\psi)$
2. $[(n, \phi)]^{-1} = [(n^*, \alpha_n(\phi))]$
3. $\mathcal{Z}(n, U) = \{[(n, \phi)] \ : \ \phi \in U \text{ and } \phi(n^* n) > 0\}$

   U open subset of $\widehat{D}$.

Main result

Let $\mathcal{G}$ be a groupoid.

$$\mathrm{Iso}(\mathcal{G}) = \left\{ g \in \mathcal{G} \ : \ g^{-1}g = gg^{-1} \right\}$$

and for each $x \in \mathcal{G}^{(0)}$

$$G_x^x = \left\{ g \in \mathcal{G} \ : \ g^{-1}g = gg^{-1} = x \right\}.$$

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Theorem (Carlsen-R-Sims-Tomforde)

Let $\mathcal{G}$ be a second-countable, locally compact, Hausdorff, étale groupoid with $\mathrm{Iso}(\mathcal{G})^{\circ} \cap G_x^x$ a torsion free abelian group for all $x \in \mathcal{G}^{(0)}$. Then

$$\mathcal{H}(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)})) \cong \mathcal{G}.$$

Theorem (Carlsen-R-Sims-Tomforde)

Suppose $\mathcal{G}$ and $\mathcal{H}$ are second-countable, locally compact, Hausdorff, étale groupoids with

$$\mathrm{Iso}(\mathcal{G})^\circ \cap G_x^x \quad \textit{and} \quad \mathrm{Iso}(\mathcal{H})^\circ \cap H_y^y$$

torsion free abelian groups for all $x \in \mathcal{G}^{(0)}$ and for all $y \in \mathcal{H}^{(0)}$. Then the following are equivalent:

1. $\mathcal{G} \cong \mathcal{H}$ *and*
2. $(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)})) \cong (C_r^*(\mathcal{H}), C_0(\mathcal{H}^{(0)}))$.

Examples

1. **Topologically principal groupoids:** $\mathcal{G}$ be a second-countable, locally compact, Hausdorff, étale groupoid such that
$$\left\{x \in \mathcal{G}^{(0)} \;:\; \mathcal{G}_x^x \text{ is trivial}\right\}$$
is dense in $\mathcal{G}^{(0)}$.

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2. **Transformation Groupoid:** $X \curvearrowleft G$ where G is a countable, discrete, torsion free, abelian group and X is a second-countable, locally compact, Hausdorff space

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$$(X \rtimes G)_x^x \trianglelefteq G$$

Why the condition on $\mathrm{Iso}(\mathcal{G})^\circ$?

1. $\mathrm{Iso}(\mathcal{G})^\circ \cap G_x^x$ is abelian implies

$$C_0(\mathcal{G}^{(0)})' \cong C_r^*(\mathrm{Iso}(\mathcal{G})^\circ),$$

$C_0(\mathcal{G}^{(0)})'/J_u \cong C_r^*(\mathrm{Iso}(\mathcal{G})_u^\circ)$, and $(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}))$ is a semidiagonal pair.

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   $C_0(\mathcal{G}^{(0)})'/J_u \cong C_r^*(\mathrm{Iso}(\mathcal{G})_u^\circ)$, and $(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}))$ is a semidiagonal pair.

2. If G is an abelian and torsion free group, then the map

$$\gamma \in G \mapsto [U_\gamma] \in \mathcal{U}(C_r^*(G))/\mathcal{U}_0(C_r^*(G))$$

   is an isomorphism from G to $\mathcal{U}(C_r^*(G))/\mathcal{U}_0(C_r^*(G))$.

Rank-one Deaconu-Renault systems

Let X be a locally compact Hausdroff space and let

$$\sigma : \mathrm{dom}(\sigma) \to \mathrm{ran}(\sigma)$$

be a local homeomorphism from an open subset $\mathrm{dom}(\sigma)$ of X to an open subset $\mathrm{ran}(\sigma)$ of X.

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$$D_n = \operatorname{dom}(\sigma^n) := \sigma^{-1}(\operatorname{dom}(\sigma^{n-1}(x)) \quad \text{and} \quad \operatorname{ran}(\sigma^n) := \sigma^n(\operatorname{dom}(\sigma^n)).$$

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Then

$$\sigma^n\colon \mathrm{dom}(\sigma^n) \to \mathrm{ran}(\sigma^n)$$

is a local homeomorphism and $\sigma^m \circ \sigma^n = \sigma^{m+n}$.

Deaconu-Renault Groupoid

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1. $(x, n-m, y)(x', n'-m', y') = (x, n+n'-m-m', y')$ if and only if $y = x'$
2. $(x, n-m, y)^{-1} = (y, m-n, x)$
3. $\mathcal{Z}(U, n, m, V) =$ $\{(x, n-m, y) \,:\, x \in U, y \in V\,, \sigma^n(x) = \sigma^m(y)\}$

   U open subset of D_n, V open subset of D_m, and $\sigma^n|_U$ and $\sigma^m|_V$ are homeomorphisms

Theorem (Carlsen-R-Sims-Tomforde)

Let (X, σ) and (Y, τ) be Deaconu–Renault systems, and suppose that $h : X \to Y$ is a homeomorphism. Then the following are equivalent:

1. *there is an isomorphism $\phi : C^*(G(X, \sigma)) \to C^*(G(Y, \tau))$ such that $\phi(C_0(X)) = C_0(Y)$ with $\phi(f) = f \circ h^{-1}$ for $f \in C_0(Y)$ and*
2. *there is a groupoid isomorphism $\Theta : G(X, \sigma) \to G(Y, \tau)$ such that $\Theta|_X = h$.*

Two Deaconu–Renault systems, (X, σ) and (Y, τ), is said to be ***continuous orbit equivalent*** if there exist a homeomorphism $h : X \to Y$ and continuous maps $k, l : \mathrm{dom}(\sigma) \to \mathbb{N}$ and $k', l' : \mathrm{dom}(\tau) \to \mathbb{N}$ such that

$$\tau^{l(x)}(h(x)) = \tau^{k(x)}(h(\sigma(x)))$$

and

$$\sigma^{l'(y)}(h^{-1}(y)) = \sigma^{k'(y)}(h^{-1}(\tau(y)))$$

for all $x \in \mathrm{dom}(\sigma)$ and $y \in \mathrm{dom}(\tau)$.

$$\mathsf{P}(x) = \{m - n \;:\; m, n \in \mathbb{N},\; x \in D_m \cap D_n,\; \text{and } \sigma^n(x) = \sigma^m(x)\}$$

$$\mathrm{mp}(x) := \begin{cases} \min(\mathbb{Z}_+ \cap \mathsf{P}(x)) & \text{if } \mathbb{Z}_+ \cap \mathsf{P}(x) \neq \emptyset \\ \infty & \text{otherwise} \end{cases}$$

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We say that a continuous orbit equivalence (h, l, k, l', k') ***preserves periodicity*** if $\mathrm{mp}(h(x)) < \infty \iff \mathrm{mp}(x) < \infty$, and

$$\left| \sum_{n=0}^{\mathrm{mp}(x)-1} l(\sigma^n(x)) - k(\sigma^n(x)) \right| = \mathrm{mp}(h(x)) \text{ and}$$

$$\left| \sum_{n=0}^{\mathrm{mp}(y)-1} l'(\tau^n(y)) - k'(\tau^n(y)) \right| = \mathrm{mp}(h^{-1}(y))$$

whenever $\mathrm{mp}(x), \mathrm{mp}(y) < \infty$, $\sigma^{\mathrm{mp}(x)}(x) = x$, and $\tau^{\mathrm{mp}(y)}(y) = y$

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2. *there is a groupoid isomorphism $\Theta : G(X, \sigma) \to G(Y, \tau)$ such that $\Theta|_X = h$; and*
3. *there is a periodicity-preserving continuous orbit equivalence from (X, σ) to (Y, τ) with underlying homeomorphism h.*

Theorem (Carlsen-R-Sims-Tomforde)

Let X and Y be second-countable, compact, Hausdorff spaces and (X, σ) and (Y, τ) be dynamical systems. Then the following are equivalent:

1. $X \rtimes_\sigma \mathbb{Z} \cong Y \rtimes_\tau \mathbb{Z}$,
2. $G(X, \sigma) \cong G(Y, \tau)$,
3. $(C(X) \times_\sigma \mathbb{Z}, C(X)) \cong (C(Y) \times_\tau \mathbb{Z}, C(Y))$,
4. *there is a periodicity preserving continuous orbit equivalence between (X, σ) and (Y, τ), and*
5. *there exist decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $\sigma|X_1$ is conjugate to $\tau|Y_1$ and $\sigma|X_2$ is conjugate to $\tau^{-1}|Y_2$.*

Theorem (Carlsen-R-Sims-Tomforde)

Let $X \curvearrowleft G$ and $Y \curvearrowleft H$ group actions, where X and Y are second-countable, locally compact, Hausdorff spaces, G and H are countable, torsion free, abelian discrete groups. Then the following are equivalent:

1. $(C_0(X) \rtimes_r G, C_0(X)) \cong (C_0(Y) \rtimes_r H, C_0(Y))$ *and*
2. $X \rtimes G \cong Y \rtimes H$.

Theorem

The following are equivalent:

1. $X \rtimes G \cong Y \rtimes H$ *and*
2. *there exist homeomorphism* $h\colon X \to Y$*, continuous functions* $\phi\colon X \times G \to H$ *and* $\eta\colon Y \times H \to G$ *such that*
   - (a) $h(x\gamma) = h(x)\phi(x,\gamma)$,
   - (b) $h^{-1}(y) = h^{-1}(y)\eta(y,\lambda)$,
   - (c) $\phi(x,\gamma_1\gamma_2) = \phi(x,\gamma_1)\phi(x\gamma_1,\gamma_2)$ *or* $\eta(y,\lambda_1\lambda_2) = \eta(x,\lambda_1)\eta(y\lambda_1,\lambda_2)$, *and*
   - (d) $\gamma \mapsto \phi(x,\gamma)$ *is a bijection*

$$G_x = \{\gamma \in G : x\gamma = x\} \to H_{h(x)} = \{\lambda \in H : h(x)\lambda = h(x)\}$$

and $\lambda \mapsto \eta(y,\lambda)$ *is a bijection from*

$$H_y \to G_{h^{-1}(y)}.$$

If A, B are C^*-algebras, then an ***A–B-imprimitivity bimodule*** is an A–B bimodule equipped with inner products $\langle \cdot, \cdot \rangle_B$ and ${}_A\langle \cdot, \cdot \rangle$ satisfying $x \cdot \langle y, z \rangle_B = {}_A\langle x, y \rangle \cdot z$ for all x, y, z, and such that X is complete in the norm given by the right inner product.

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Let (A_1, D_1) and (A_2, D_2) be pairs of C^*-algebras such that D_i is an abelian subalgebra of A_i containing an approximate identity for A_i. Let X be an A_1–A_2-imprimitivity bimodule. We say that X is an ***(A_1, D_1)–(A_2, D_2)-imprimitivity bimodule*** if

$$X = \overline{\operatorname{span}\{x \in X : \langle D_1 \cdot x, x \rangle_{A_2} \subseteq D_2 \text{ and } {}_{A_1}\langle x, x \cdot D_2 \rangle \subseteq D_1\}}.$$

Theorem (Carlsen-R-Sims-Tomforde)

Suppose $\mathcal{G}$ and $\mathcal{H}$ are second-countable, locally compact, Hausdorff, étale groupoids with

$$\mathrm{Iso}(\mathcal{G})^\circ \cap G_x^x \quad \textit{and} \quad \mathrm{Iso}(\mathcal{H})^\circ \cap H_y^y$$

torsion free abelian groups for all $x \in \mathcal{G}^{(0)}$ and for all $y \in \mathcal{H}^{(0)}$. Then the following are equivalent:

(1) *$\mathcal{G}$ and $\mathcal{H}$ are equivalent;*

(2) *there exists an $(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}))$–$(C_r^*(\mathcal{H}), C_0(\mathcal{H}^{(0)}))$-imprimitivity bimodule;*

(3) *$(C_r^*(\mathcal{G}) \otimes \mathbb{K}, C_0(\mathcal{G}^{(0)}) \otimes c_0(\mathbb{N}))$ and $(C_r^*(\mathcal{H}) \otimes \mathbb{K}, C_0(\mathcal{H}^{(0)}) \otimes c_0(\mathbb{N}))$ are isomorphic.*