Groupoid $C^*$-algebras and their canonical diagonal subalgebras

Efren Ruiz
Work in progress with Toke Carlsen, Aidan Sims, and Mark Tomforde

University of Hawai‘i at Hilo

APPLICATIONS OF MODEL THEORY TO OPERATOR ALGEBRAS
Objects of interest

(A, D)

- A and D are separable C*-algebras
- D is a commutative C*-subalgebra of A
Objects of interest

\[(A, D)\]

- \(A\) and \(D\) are separable \(C^*\)-algebras
- \(D\) is a commutative \(C^*\)-subalgebra of \(A\)

Main Example

\[A = C^*(\mathcal{G}) \quad \text{and} \quad D = C_0(\mathcal{G}^{(0)})\]

- \(\mathcal{G}\) is a second-countable, locally compact, Hausdorff, étale groupoid
  - \(s : \gamma \mapsto \gamma^{-1}\gamma\) and \(r : \gamma \mapsto \gamma\gamma^{-1}\)
  - are local homeomorphisms.
$C_r^*(\mathcal{G})$

$C_c(\mathcal{G})$:

$$(f \ast g)(\gamma) = \sum_{\lambda \beta = \gamma} f(\lambda)g(\beta)$$

$$f^*(\gamma) = f(\gamma^{-1})$$

$\pi^u_\lambda : C_c(\mathcal{G}) \to B(\ell^2(s^{-1}(u)))$, where

$$(\pi^u_\lambda (f) \xi)(\gamma) = \sum_{\lambda \beta = \gamma} f(\lambda)\xi(\beta)$$

$$\|f\|_r := \sup \left\{ \|\pi^u_\lambda (f)\| : u \in \mathcal{G}^{(0)} \right\}$$

$C_r^*(\mathcal{G}) := \frac{C_c(\mathcal{G})}{\|\cdot\|_r}$
Motivating Examples

Theorem (Tomiyama)

Let $X$ and $Y$ be compact, Hausdorff spaces and let \((X, \sigma)\) and \((Y, \tau)\) be topologically free dynamical systems. Then the following are equivalent:

1. \(\mathcal{C}(X) \rtimes \sigma Z \cong \mathcal{C}(Y) \rtimes \tau Z\)
2. \((X, \sigma)\) and \((Y, \tau)\) are continuous orbit equivalent, i.e., there exist a homeomorphism \(h : X \to Y\) and continuous functions \(m, n : X \to Z\) such that
   \[
   h(\sigma(x)) = \tau m(x) \quad \text{and} \quad \tau(h(x)) = h(\sigma n(x)).
   \]
Motivating Examples

Theorem (Tomiyama)

Let $X$ and $Y$ be compact, Hausdorff spaces and let $(X, \sigma)$ and $(Y, \tau)$ be topologically free dynamical systems. Then the following are equivalent:

1. $(C(X) \rtimes_\sigma \mathbb{Z}, C(X)) \cong (C(Y) \rtimes_\tau \mathbb{Z}, C(Y))$ and

2. $(X, \sigma)$ and $(Y, \tau)$ are continuous orbit equivalent, i.e., there exist a homeomorphism $h: X \to Y$ and continuous functions $m, n: X \to \mathbb{Z}$ such that

$$h(\sigma(x)) = \tau^{m(x)}(h(x)) \quad \text{and} \quad \tau(h(x)) = h(\sigma^{n(x)}(x)).$$
**Transformation groupoid:** $X \rtimes_{\sigma} \mathbb{Z}$

1. $X \rtimes_{\sigma} \mathbb{Z} = X \times \mathbb{Z}$ (Product topology)
2. $(x, n)(y, m) = (x, n + m)$ if and only if $\sigma^n(x) = y$
3. $(n, x)^{-1} = (\sigma^n(x), -n)$
4. $(X \rtimes_{\sigma} \mathbb{Z})^{(0)} = X \times \{0\} \cong X.$
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4. $(X \rtimes_\sigma \mathbb{Z})^{(0)} = X \times \{0\} \cong X.$

Theorem

$$(C(X) \rtimes_\sigma \mathbb{Z}, C(X)) \cong (C^*_r(X \rtimes_\sigma \mathbb{Z}), C((X \rtimes_\sigma \mathbb{Z})^{(0)}))$$
Theorem (Tomiyama and Renault)

Let $X$ and $Y$ be second-countable, compact, Hausdorff spaces and let $(X, \sigma)$ and $(Y, \tau)$ be topologically free dynamical systems. Then the following are equivalent:

1. $(C(X) \rtimes_{\sigma} \mathbb{Z}, C(X)) \cong (C(Y) \rtimes_{\tau} \mathbb{Z}, C(Y))$,

2. $(X, \sigma)$ and $(Y, \tau)$ are continuous orbit equivalent, i.e., there exist a homeomorphism $h: X \to Y$ and continuous functions $m, n: X \to \mathbb{Z}$ such that

$$h(\sigma(x)) = \tau^{m(x)}(h(x)) \quad \text{and} \quad \tau(h(x)) = h(\sigma^{n(x)}(x)),$$

and

3. $X \rtimes_{\sigma} \mathbb{Z} \cong Y \rtimes_{\tau} \mathbb{Z}$. 
Cuntz-Krieger algebras

One-sided shift space

Let $A \in M_N(\{0,1\})$.

1. $X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, 2, \ldots, N\}^\mathbb{N} : A(x_n, x_{n+1}) = 1\}$

2. $\sigma_A : X_A \rightarrow X_A$, $[\sigma_A((x_n)_{n \in \mathbb{N}})]_n = x_{n+1}$
Theorem (Matsumoto-Matui, Brownlowe-Carlsen-Whittaker, Arklint-Eilers-R (Carlsen-Winger))

Let $A \in M_N(\{0, 1\})$ and let $B \in M_{N'}(\{0, 1\})$. Then the following are equivalent:

1. $(\mathcal{O}_A, C(X_A)) \cong (\mathcal{O}_B, C(X_B))$ and

2. $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are continuous orbit equivalent, i.e., there exist a homeomorphism $h: X_A \rightarrow X_B$, and continuous functions $k, l: X_A \rightarrow \mathbb{N}$ and $k', l': X_B \rightarrow \mathbb{N}$ such that

   \[
   \sigma_B^{k(x)}(h(\sigma_A(x))) = \sigma_A^{l(x)}(h(x))
   \]

   \[
   \sigma_A^{k'(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l'(y)}(h^{-1}(y)).
   \]
Theorem (Matsumoto-Matui, Carlsen-Eilers-Ortega-Restorff)

Let $A \in M_N(\{0, 1\})$ and let $B \in M_{N'}(\{0, 1\})$. Then the following are equivalent:

1. $(\mathcal{O}_A \otimes \mathbb{K}, C(X_A) \otimes c_0(\mathbb{N})) \cong (\mathcal{O}_B \otimes \mathbb{K}, C(X_B) \otimes c_0(\mathbb{N}))$ and

2. the two-sided shift spaces $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are flow equivalent.
The groupoid of a one-sided shift space

Let $A \in M_N(\{0, 1\})$.

1. \[ G_A = \{ (x, n - m, y) : x, y \in X_A, n, m \in \mathbb{Z}_{>0}, \sigma_A^n(x) = \sigma_A^m(y) \} \]
2. \((x, n - m, y)(x', n' - m', y') = (x, n + n' - m - m', y')\) if and only if $y = x'$
3. \((x, n - m, y)^{-1} = (y, m - n, x)\)
4. \[ G_A^{(0)} = \{ (x, 0, x) : x \in X_A \} \cong X_A \]
5. \[ \mathcal{Z}(U, n, m, V) = \{ (x, n - m, y) : x \in U, y \in V, \sigma_A^n(x) = \sigma_A^m(y) \}, \]
   $U, V$ are open subets of $X_A$
Theorem

\((\mathcal{O}_A, C(X_A)) \cong (C^*(G_A), C(G^{(0)}_A))\)
Theorem (Matsumoto-Matui, Brownlowe-Carlsen-Whittaker, Arkling-Eilers-R (Carlsen-Winger))

Let $A \in M_N(\{0, 1\})$ and let $B \in M_{N'}(\{0, 1\})$. Then the following are equivalent:

1. $(\mathcal{O}_A, C(X_A)) \cong (\mathcal{O}_B, C(X_B))$,

2. there exists a continuous orbit equivalence between $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$, and

3. $G_A \cong G_B$. 

\[
(\mathcal{O}_A, C(X_A)) \cong (\mathcal{O}_B, C(X_B)) 
= (C^*_r(G_A), C(G_A^{(0)}))
\]
Theorem (Renault, Brownlowe-Carlsen-Whittaker)

Let $\mathcal{G}, \mathcal{H}$ be second-countable, locally compact, Hausdorff, étale groupoids. Then the following are equivalent:

1. $(C^*_r(\mathcal{G}), C_0(\mathcal{G}^{(0)})) \cong (C^*_r(\mathcal{H}), C_0(\mathcal{H}^{(0)}))$

2. $\mathcal{G} \cong \mathcal{H}$

whenever $\mathcal{G}, \mathcal{H}$ are topologically principal groupoids or $\mathcal{G}, \mathcal{H}$ are groupoids associated to one-sided shift spaces.
Theorem (Renault, Brownlowe-Carlsen-Whittaker)

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whenever $\mathcal{G}, \mathcal{H}$ are topologically principal groupoids or $\mathcal{G}, \mathcal{H}$ are groupoids associated to one-sided shift spaces.

Key Idea

Construct a groupoid

$$\mathcal{H}(C^*_r(\mathcal{G}), C_0(\mathcal{G}^0))$$

such that

$$\mathcal{H}(C^*_r(\mathcal{G}), C_0(\mathcal{G}^0)) \cong \mathcal{G}.$$
Definition

A **semidiagonal pair** of $C^*$-algebras is a pair $(A, D)$ consisting of a separable $C^*$-algebra $A$ and a subalgebra $D$ of $A$ such that

1. $D$ is abelian,

2. $D$ contains an approximate identity for $A$,

3. for each $\phi \in \hat{D}$, the quotient $D'/J_\phi$ of $D'$ by the ideal $J_\phi := \ker(\phi)D'$ is a unital $C^*$-algebra, and

4. for each $\phi \in \hat{D}$, there exist $d \in D$ and an open neighbourhood $U$ of $\phi$ such that $d + J_\psi = 1_{D'/J_\psi}$ for all $\psi \in U$. 
Definition

Let $A$ be a $C^*$-algebra and $D$ be a $C^*$-subalgebra of $A$. A
*
ormalizer of $D$ is an element $n \in A$ such that

$$nDn^* \cup n^*Dn \subseteq D.$$
**Definition**

Let $A$ be a $C^*$-algebra and $D$ be a $C^*$-subalgebra of $A$. A **normalizer of** $D$ is an element $n \in A$ such that

$$nDn^* \cup n^*Dn \subseteq D.$$ 

**Theorem (Kumjian, Renault)**

Let $A$ be a $C^*$-algebra and $D$ an abelian $C^*$-subalgebra of $A$ that contains an approximate unit for $A$. Suppose that $n$ is a normalizer of $D$. Then there is a homeomorphism

$$\alpha_n : \{ u \in \hat{D} : u(n^*n) > 0 \} \rightarrow \{ u \in \hat{D} : u(nn^*) > 0 \}$$

such that $u(n^*n)\alpha_n(u)(d) = u(n^*dn)$ for all $d \in D$. 
**Lemma**

Let \((A, D)\) be a semidiagonal pair, \(n, m\) be normalizers of \(D\), and \(\phi \in \hat{D}\). Suppose there exists an open neighborhood \(U\) of \(\phi\) such that

\[
U \subseteq \text{supp}(n^* n) \cap \text{supp}(m^* m).
\]

Then for any \(d \in D\) with \(\text{supp}(d) \subseteq U\) and \(\phi(d) = 1\), we have that

\[
\phi(m^* nn^* m)^{-\frac{1}{2}} dn^* md
\]

is in \(D'\) and

\[
\phi(m^* nn^* m)^{-\frac{1}{2}} dn^* md + J_{\phi}
\]

is a unitary in \(D' / J_{\phi}\) that is independent of the choices of \(U\) and \(d\).
\[ S(A, D) = \left\{(n, \phi) \in \mathcal{N}(D) \times \hat{D} : \phi(n^*n) > 0 \right\} \]

\((n, \phi) \sim (m, \psi)\) if and only if

1. \(\phi = \psi\),

2. there exists an open neighborhood of \(\phi\) such that \(\alpha_n|_U = \alpha_m|_U\), and

3. \(\phi(m^*nn^*m)^{-\frac{1}{2}}dn^*md + J_\phi \in \mathcal{U}_0(D'/J_\phi)\).
The groupoid $\mathcal{H}(A, D)$

$$\mathcal{H}(A, D) = \{ [(n, \phi)] : (n, \phi) \in S(A, D) \}$$
The groupoid $\mathcal{H}(A, D)$

$$\mathcal{H}(A, D) = \{[(n, \phi)] : (n, \phi) \in S(A, D)\}$$

1. $[(n, \phi)][(m, \psi)] = [(nm, \psi)]$ if and only if $\phi = \alpha_m(\psi)$

2. $[(n, \phi)]^{-1} = [(n^*, \alpha_n(\phi))]$

3. $\mathcal{Z}(n, U) = \{[(n, \phi)] : \phi \in U \text{ and } \phi(n^*n) > 0\}$

$U$ open subset of $\hat{D}$. 
Main result

Let \( \mathcal{G} \) be a groupoid.

\[
\text{Iso}(\mathcal{G}) = \left\{ g \in \mathcal{G} : g^{-1}g = gg^{-1} \right\}
\]

and for each \( x \in \mathcal{G}^{(0)} \)

\[
\mathcal{G}^x_x = \left\{ g \in \mathcal{G} : g^{-1}g = gg^{-1} = x \right\}.
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\]

**Theorem (Carlsen-R-Sims-Tomforde)**

Let \( \mathcal{G} \) be a second-countable, locally compact, Hausdorff, étale groupoid with \( \text{Iso}(\mathcal{G})^\circ \cap \mathcal{G}^x_x \) a torsion free abelian group for all \( x \in \mathcal{G}^{(0)} \). Then

\[
\mathcal{H}(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)})) \cong \mathcal{G}.
\]
Theorem (Carlsen-R-Sims-Tomforde)

Suppose $G$ and $H$ are second-countable, locally compact, Hausdorff, étale groupoids with

$$\text{Iso}(G)^{\circ} \cap G_x^x \quad \text{and} \quad \text{Iso}(H)^{\circ} \cap H_y^y$$

torsion free abelian groups for all $x \in G^{(0)}$ and for all $y \in H^{(0)}$. Then the following are equivalent:

1. $G \cong H$ and
2. $(C^*_r(G), C_0(G^{(0)})) \cong (C^*_r(H), C_0(H^{(0)}))$. 
Examples

1. **Topologically principal groupoids:** $G$ be a second-countable, locally compact, Hausdorff, étale groupoid such that

$$\left\{ x \in G^{(0)} : G^x_x \text{ is trivial} \right\}$$

is dense in $G^{(0)}$. 
Examples

1. **Topologically principal groupoids:** Let $\mathcal{G}$ be a second-countable, locally compact, Hausdorff, étale groupoid such that

$$\left\{ x \in \mathcal{G}^{(0)} : \mathcal{G}_x^x \text{ is trivial} \right\}$$

is dense in $\mathcal{G}^{(0)}$.

$$\operatorname{Iso}(\mathcal{G})^\circ \cap \mathcal{G}_x^x = \mathcal{G}^{(0)} \cap \mathcal{G}_x^x = \{x\}$$
Examples

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$$\text{Iso}(G)^\circ \cap G^x_x = G^{(0)} \cap G^x_x = \{x\}$$

2. **Transformation Groupoid:** $X \curvearrowright G$ where $G$ is a countable, discrete, torsion free, abelian group and $X$ is a second-countable, locally compact, Hausdorff space

$$X \rtimes G$$
Examples

1. **Topologically principal groupoids:** \( G \) be a second-countable, locally compact, Hausdorff, étale groupoid such that

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\left\{ x \in G^{(0)} : G^x_x \text{ is trivial} \right\}
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is dense in \( G^{(0)} \).

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\text{Iso}(G)^\circ \cap G^x_x = G^{(0)} \cap G^x_x = \{x\}
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2. **Transformation Groupoid:** \( X \ltimes G \) where \( G \) is a countable, discrete, torsion free, abelian group and \( X \) is a second-countable, locally compact, Hausdorff space

\[
X \rtimes G
\]

\[
(X \rtimes G)^x_x \subseteq G
\]
Why the condition on $\text{Iso}(\mathcal{G})^\circ$?

1. $\text{Iso}(\mathcal{G})^\circ \cap G_x^\times$ is abelian implies

$$C_0(\mathcal{G}^{(0)})' \cong C_r^*(\text{Iso}(\mathcal{G})^\circ),$$

$$C_0(\mathcal{G}^{(0)})' / J_u \cong C_r^*(\text{Iso}(\mathcal{G})_u^\circ),$$

and $(C_r^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}))$ is a semidiagonal pair.
Why the condition on $\text{Iso}(\mathcal{G})^\circ$?

1. $\text{Iso}(\mathcal{G})^\circ \cap G^\times \chi$ is abelian implies

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C_0(\mathcal{G}^{(0)})' \cong C^*_r(\text{Iso}(\mathcal{G})^\circ),
\]

\[
C_0(\mathcal{G}^{(0)})'/J_u \cong C^*_r(\text{Iso}(\mathcal{G})^\circ)_u, \text{ and } (C^*_r(\mathcal{G}), C_0(\mathcal{G}^{(0)})) \text{ is a semidiagonal pair.}
\]

2. If $G$ is an abelian and torsion free group, then the map

\[
\gamma \in G \mapsto [U_\gamma] \in \mathcal{U}(C^*_r(G))/\mathcal{U}_0(C^*_r(G))
\]

is an isomorphism from $G$ to $\mathcal{U}(C^*_r(G))/\mathcal{U}_0(C^*_r(G))$. 
Let $X$ be a locally compact Hausdorff space and let

$$
\sigma : \text{dom}(\sigma) \to \text{ran}(\sigma)
$$

be a local homeomorphism from an open subset $\text{dom}(\sigma)$ of $X$ to an open subset $\text{ran}(\sigma)$ of $X$. 
Rank-one Deaconu-Renault systems

Let $X$ be a locally compact Hausdorff space and let

$$\sigma : \text{dom}(\sigma) \to \text{ran}(\sigma)$$

be a local homeomorphism from an open subset $\text{dom}(\sigma)$ of $X$ to an open subset $\text{ran}(\sigma)$ of $X$. Inductively define

$$D_n = \text{dom}(\sigma^n) := \sigma^{-1}(\text{dom}(\sigma^{n-1}(x))) \quad \text{and} \quad \text{ran}(\sigma^n) := \sigma^n(\text{dom}(\sigma^n)).$$
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$$D_n = \text{dom}(\sigma^n) := \sigma^{-1}(\text{dom}(\sigma^{n-1}(x))) \quad \text{and} \quad \text{ran}(\sigma^n) := \sigma^n(\text{dom}(\sigma^n)).$$

Then 

$$\sigma^n : \text{dom}(\sigma^n) \to \text{ran}(\sigma^n)$$

is a local homeomorphism and $\sigma^m \circ \sigma^n = \sigma^{m+n}$.
Deaconu-Renault Groupoid

\[ G(X, \sigma) = \bigcup_{n, m \in \mathbb{N}} \{(x, n - m, y) : \sigma^n(x) = \sigma^m(y)\} \]
Deaconu-Renault Groupoid

\[ G(X, \sigma) = \bigcup_{n,m \in \mathbb{N}} \{(x, n - m, y) : \sigma^n(x) = \sigma^m(x)\} \]

1. \((x, n - m, y)(x', n' - m', y') = (x, n + n' - m - m', y')\)
   if and only if \(y = x'\)

2. \((x, n - m, y)^{-1} = (y, m - n, x)\)

3. \(\mathcal{Z}(U, n, m, V) = \{(x, n - m, y) : x \in U, y \in V, \sigma^n(x) = \sigma^m(y)\}\)

\(U\) open subset of \(D_n\), \(V\) open subset of \(D_m\), and \(\sigma^n|_U\) and \(\sigma^m|_V\) are homeomorphisms
Theorem (Carlsen-R-Sims-Tomforde)

Let \((X, \sigma)\) and \((Y, \tau)\) be Deaconu–Renault systems, and suppose that \(h : X \rightarrow Y\) is a homeomorphism. Then the following are equivalent:

1. there is an isomorphism \(\phi : C^*(G(X, \sigma)) \rightarrow C^*(G(Y, \tau))\) such that \(\phi(C_0(X)) = C_0(Y)\) with \(\phi(f) = f \circ h^{-1}\) for \(f \in C_0(Y)\) and

2. there is a groupoid isomorphism \(\Theta : G(X, \sigma) \rightarrow G(Y, \tau)\) such that \(\Theta|_X = h\).
Two Deaconu–Renault systems, \((X, \sigma)\) and \((Y, \tau)\), is said to be \textit{continuous orbit equivalent} if there exist a homeomorphism \(h : X \to Y\) and continuous maps \(k, l : \text{dom}(\sigma) \to \mathbb{N}\) and \(k', l' : \text{dom}(\tau) \to \mathbb{N}\) such that

\[
\tau^{l(x)}(h(x)) = \tau^{k(x)}(h(\sigma(x)))
\]

and

\[
\sigma^{l'(y)}(h^{-1}(y)) = \sigma^{k'(y)}(h^{-1}(\tau(y)))
\]

for all \(x \in \text{dom}(\sigma)\) and \(y \in \text{dom}(\tau)\).
\[ P(x) = \{ m - n : m, n \in \mathbb{N}, x \in D_m \cap D_n, \text{ and } \sigma^n(x) = \sigma^m(x) \} \]

\[ mp(x) := \begin{cases} 
\min(\mathbb{Z}_+ \cap P(x)) & \text{if } \mathbb{Z}_+ \cap P(x) \neq \emptyset \\
\infty & \text{otherwise}
\end{cases} \]
\[ P(x) = \{ m - n : m, n \in \mathbb{N}, x \in D_m \cap D_n, \text{ and } \sigma^n(x) = \sigma^m(x) \} \]

\[ mp(x) := \begin{cases} 
\min(\mathbb{Z}_+ \cap P(x)) & \text{if } \mathbb{Z}_+ \cap P(x) \neq \emptyset \\
\infty & \text{otherwise}
\end{cases} \]

We say that a continuous orbit equivalence \((h, l, k, l', k')\) **preserves periodicity** if \(mp(h(x)) < \infty \iff mp(x) < \infty\), and

\[
\begin{align*}
\left| \sum_{n=0}^{mp(x)-1} l(\sigma^n(x)) - k(\sigma^n(x)) \right| &= mp(h(x)) \quad \text{and} \\
\left| \sum_{n=0}^{mp(y)-1} l'(\tau^n(y)) - k'(\tau^n(y)) \right| &= mp(h^{-1}(y))
\end{align*}
\]

whenever \(mp(x), mp(y) < \infty\), \(\sigma^{mp(x)}(x) = x\), and \(\tau^{mp(y)}(y) = y\)
Theorem (Carlsen-R-Sims-Tomforde)

Let \((X, \sigma)\) and \((Y, \tau)\) be Deaconu–Renault systems, and suppose that \(h : X \to Y\) is a homeomorphism. Then the following are equivalent:

1. there is an isomorphism \(\phi : C^* (G(X, \sigma)) \to C^* (G(Y, \tau))\) such that \(\phi(C_0(X)) = C_0(Y)\) with \(\phi(f) = f \circ h^{-1}\) for \(f \in C_0(Y)\);

2. there is a groupoid isomorphism \(\Theta : G(X, \sigma) \to G(Y, \tau)\) such that \(\Theta|_X = h\); and

3. there is a periodicity-preserving continuous orbit equivalence from \((X, \sigma)\) to \((Y, \tau)\) with underlying homeomorphism \(h\).
Theorem (Carlsen-R-Sims-Tomforde)

Let $X$ and $Y$ be second-countable, compact, Hausdorff spaces and $(X, \sigma)$ and $(Y, \tau)$ be dynamical systems. Then the following are equivalent:

1. $X \rtimes_\sigma \mathbb{Z} \cong Y \rtimes_\tau \mathbb{Z}$,

2. $G(X, \sigma) \cong G(Y, \tau)$,

3. $(C(X) \times_\sigma \mathbb{Z}, C(X)) \cong (C(Y) \times_\tau \mathbb{Z}, C(Y))$,

4. there is a periodicity preserving continuous orbit equivalence between $(X, \sigma)$ and $(Y, \tau)$, and

5. there exist decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $\sigma|X_1$ is conjugate to $\tau|Y_1$ and $\sigma|X_2$ is conjugate to $\tau^{-1}|Y_2$. 
Let $X \curvearrowleft G$ and $Y \curvearrowleft H$ group actions, where $X$ and $Y$ are second-countable, locally compact, Hausdorff spaces, $G$ and $H$ are countable, torsion free, abelian discrete groups. Then the following are equivalent:

1. $(C_0(X) \rtimes_r G, C_0(X)) \cong (C_0(Y) \rtimes_r H, C_0(Y))$ and
2. $X \rtimes G \cong Y \rtimes H$. 

Theorem (Carlsen-R-Sims-Tomforde)
Theorem

The following are equivalent:

1. \( X \rtimes G \cong Y \rtimes H \) and

2. there exist homeomorphism \( h: X \to Y \), continuous functions \( \phi: X \times G \to H \) and \( \eta: Y \times H \to G \) such that
   
   \begin{align*}
   (a) & \quad h(x\gamma) = h(x)\phi(x, \gamma), \\
   (b) & \quad h^{-1}(y) = h^{-1}(y)\eta(y, \lambda), \\
   (c) & \quad \phi(x, \gamma_1\gamma_2) = \phi(x, \gamma_1)\phi(x\gamma_1, \gamma_2) \text{ or} \\
   & \quad \eta(y, \lambda_1\lambda_2) = \eta(x, \lambda_1)\eta(y\lambda_1, \lambda_2), \text{ and} \\
   (d) & \quad \gamma \mapsto \phi(x, \gamma) \text{ is a bijection}
   \end{align*}

\[
G_x = \{ \gamma \in G : x\gamma = x \} \to H_{h(x)} = \{ \lambda \in H : h(x)\lambda = h(x) \}
\]

and \( \lambda \mapsto \eta(y, \lambda) \) is a bijection from

\[
H_y \to G_{h^{-1}(y)}.
\]
If $A, B$ are $C^*$-algebras, then an $A\text{-}B$-imprimitivity bimodule is an $A\text{-}B$ bimodule equipped with inner products $\langle \cdot, \cdot \rangle_B$ and $\langle \cdot, \cdot \rangle_A$ satisfying $x \cdot \langle y, z \rangle_B = A\langle x, y \rangle \cdot z$ for all $x, y, z$, and such that $X$ is complete in the norm given by the right inner product.
If $A, B$ are $C^*$-algebras, then an $A–B$-imprimitivity bimodule is an $A–B$ bimodule equipped with inner products $\langle \cdot, \cdot \rangle_B$ and $A\langle \cdot, \cdot \rangle$ satisfying $x \cdot \langle y, z \rangle_B = A\langle x, y \rangle \cdot z$ for all $x, y, z$, and such that $X$ is complete in the norm given by the right inner product.

Let $(A_1, D_1)$ and $(A_2, D_2)$ be pairs of $C^*$-algebras such that $D_i$ is an abelian subalgebra of $A_i$ containing an approximate identity for $A_i$. Let $X$ be an $A_1–A_2$-imprimitivity bimodule. We say that $X$ is an $(A_1, D_1)–(A_2, D_2)$-imprimitivity bimodule if

$$X = \text{span}\{x \in X : \langle D_1 \cdot x, x \rangle_{A_2} \subseteq D_2 \text{ and } A_1 \langle x, x \cdot D_2 \rangle \subseteq D_1\}.$$
Theorem (Carlsen-R-Sims-Tomforde)

Suppose \( \mathcal{G} \) and \( \mathcal{H} \) are second-countable, locally compact, Hausdorff, étale groupoids with

\[ \text{Iso}(\mathcal{G})^\circ \cap G_x^x \quad \text{and} \quad \text{Iso}(\mathcal{H})^\circ \cap H_y^y \]

torsion free abelian groups for all \( x \in \mathcal{G}^{(0)} \) and for all \( y \in \mathcal{H}^{(0)} \).

Then the following are equivalent:

1. \( \mathcal{G} \) and \( \mathcal{H} \) are equivalent;
2. there exists an \( (C^*_r(\mathcal{G}), C_0(\mathcal{G}^{(0)})) - (C^*_r(\mathcal{H}), C_0(\mathcal{H}^{(0)})) \)-imprimitivity bimodule;
3. \( (C^*_r(\mathcal{G}) \otimes \mathbb{K}, C_0(\mathcal{G}^{(0)}) \otimes c_0(\mathbb{N})) \) and \( (C^*_r(\mathcal{H}) \otimes \mathbb{K}, C_0(\mathcal{H}^{(0)}) \otimes c_0(\mathbb{N})) \) are isomorphic.