Stable finiteness and pure infiniteness of the C^*-algebras of higher-rank graphs

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University of Houston, July 31 2017

Overview

Let E be a directed graph such that the graph C^*-algebra $C^*(E)$ is simple ($\iff$ E is cofinal and every cycle has an entry).

Dichotomy (Kumjian-Pask-Raeburn, 1998)

$C^*(E)$ is either AF or purely infinite.

This dichotomy fails for simple C^*-algebras of k-graphs (Pask-Raeburn-Rørdam-Sims, 2006).

Conjecture

C^*-algebras of k-graphs are either stably finite or purely infinite.

I will:

- describe a class of rank-2 graphs whose C^*-algebras are A$\mathbb{T}$ algebras (hence are neither AF nor purely infinite);
- outline some results towards proving the conjecture.

k-graphs

The path category $\mathcal{P}(E)$ of a directed graph E:

has objects $\mathcal{P}(E)^0$ the set of vertices E^0, morphisms $\mathcal{P}(E)^*$ the set E^* of finite paths in E, $\lambda \in E^*$ has domain $s(\lambda)$ and codomain $r(\lambda)$, the composition of $\lambda, \eta \in E^*$ is defined when $s(\lambda) = r(\eta)$ and is $\lambda\eta = \lambda_1 \cdots \lambda_{|\lambda|}\eta_1 \cdots \eta_{|\eta|}$, and the identity morphism on $v \in E^0$ is the path v of length 0.

Crucial observation: each path λ of length $|\lambda| = m + n$ has a unique factorisation $\lambda = \mu\nu$ where $|\mu| = m$ and $|\nu| = n$.

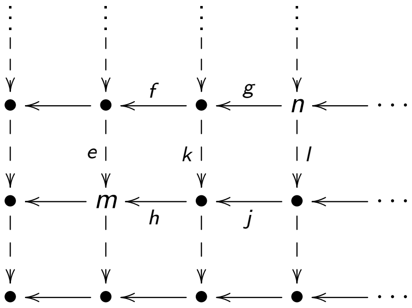
Defn: (Kumjian-Pask, 2000)

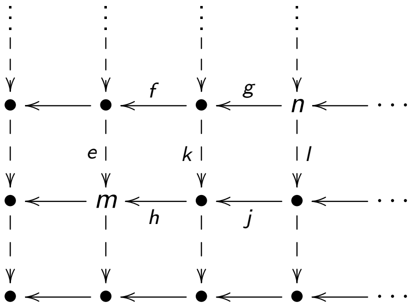
A k-graph is a countable category $\Lambda = (\Lambda^0, \Lambda^*, r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$, called the degree map, satisfying the following factorisation property: if $\lambda \in \Lambda^*$ and $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then there are unique $\mu, \nu \in \Lambda^*$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$.

Example: With $d : E^* \to \mathbb{N}$ by $\lambda \mapsto |\lambda|$, $\mathcal{P}(E)$ is a 1-graph.
From now on $k = 2$.

If $m = (m_1, m_2), n \in \mathbb{N}^2$, then $m \le n$ iff $m_i \le n_i$ for $i = 1, 2$.
Example: Let $\Omega^0 := \mathbb{N}^2$ and $\Omega^* := \{(m, n) \in \mathbb{N}^2 \times \mathbb{N}^2 : m \le n\}$.
Define $r, s : \Omega^* \to \Omega^0$ by $r(m, n) := m$ and $s(m, n) := n$, composition by $(m, n)(n, p) = (m, p)$. Define $d : \Omega^* \to \mathbb{N}^2$ by $d(m, n) := n - m$. Then (Ω, d) is a 2-graph.

To visualise Ω, we draw its 1-skeleton, the coloured directed graph with paths (edges) of degree $e_1 := (1, 0)$ drawn in blue (solid) and those of degree $e_2 := (0, 1)$ in red (dashed):

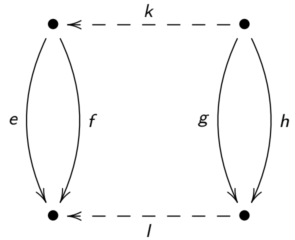




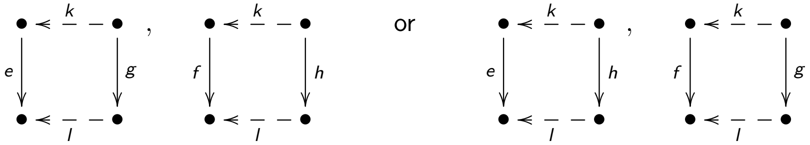
- The path (m, n) with source n and range m is the 2×1 rectangle in the top left.
- The different routes efg, hkg, hjl from n to m represent the different factorisations of (m, n).
- Composition of morphisms involves taking the convex hull of the corresponding rectangles.
- Can factor a path $\lambda = \alpha\nu$ where α is a blue and ν is a red path.

- In other 2-graphs Λ, a path of degree $(2, 1)$ is a copy of this rectangle in Ω wrapped around the 1-skeleton of Λ in a colour-preserving way.
- The 1-skeleton alone need not determine the k-graph.

Example: If the 1-skeleton contains



then we must specify how the blue-red paths *ek* and *fk* factor as red-blue paths. The paths of degree $(1,1)$ could be either



- To make a 2-coloured graph into a 2-graph, it suffices to find a collection of squares in which each red-blue path and each blue-red path occur exactly once. (It's more complicated for $k \geq 3$.)

Notation: write Λ^m for the paths of degree $m \in \mathbb{N}^2$.

The C^*-algebra of a k-graph

Let Λ be a row-finite 2-graph: $r^{-1}(v) \cap \Lambda^m$ is finite for every $v \in \Lambda^0$ and $m \in \mathbb{N}^2$. A vertex v is a source if there exists $i \in \{1, 2\}$ such that $r^{-1}(v) \cap \Lambda^{e_i} = \emptyset$.

A Cuntz-Krieger Λ-family consists of partial isometries $\{S_\lambda : \lambda \in \Lambda^*\}$ such that

1. $\{S_v : v \in \Lambda^0 \subset \Lambda^*\}$ are mutually orthogonal projns;
2. $S_\lambda S_\mu = S_{\lambda\mu}$;
3. $S_\lambda^* S_\lambda = S_{s(\lambda)}$;
4. for $v \in \Lambda^0$ and i such that $r^{-1}(v) \cap \Lambda^{e_i} \neq \emptyset$, we have $S_v = \sum_{\lambda \in r^{-1}(v) \cap \Lambda^{e_i}} S_\lambda S_\lambda^*$.

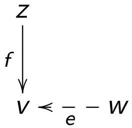
$C^*(\Lambda)$ is universal for Cuntz-Krieger Λ-families.

Key lemma

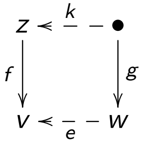
If Λ has no sources, i.e. $r^{-1}(v) \cap \Lambda^{e_i} \neq \emptyset$ for $i = 1, 2$ and all $v \in \Lambda^0$, then $C^*(S_\lambda) = \overline{\operatorname{span}}\{S_\lambda S_\mu^*\}$.

Idea: relation (4) implies that for every $m \in \mathbb{N}^2$ we have $S_v = \sum_{\lambda \in \Lambda^m, r(\lambda) = v} S_\lambda S_\lambda^*$.

But: the key lemma does not hold for all 2-graphs, e.g.,



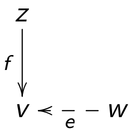
Relation (4) at v says that $S_e S_e^* = S_v = S_f S_f^*$. It follows that $S_e^* S_f$ is a partial isometry with range and source projns S_w and S_z. So $S_e^* S_f$ cannot be written as a sum of $S_\mu S_\lambda^*$.
This doesn't happen for



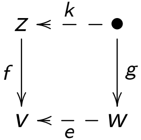
Here $fk = eg$; relation (4) at $z = s(f)$ with degree $(0,1)$ gives

$$S_e^* S_f = S_e^* S_f S_{s(f)} = S_e^* S_f S_k S_k^* = S_e^* S_{fk} S_k^* = S_e^* S_{eg} S_k^* = S_g S_k^*.$$

Roughly speaking, Λ is locally convex if



implies there exists k, g such that



Theorem (Raeburn-Sims-Yeend, 2003)

If Λ is locally convex and row-finite, then each $s_v \in C^*(\Lambda)$ is non-zero and and $C^*(\Lambda) = \overline{\mathrm{span}}\{s_\lambda s_\mu^*\}$.

The theory for the C^*-algebras of locally convex graphs is well developed: there are gauge-invariant and Cuntz-Krieger uniqueness theorems (RSY), criteria for simplicity (Robertson-Sims 2009), gauge-invariant ideals are known, etc.

Definition (Pask-Raeburn-Rørdam-Sims, 2006)

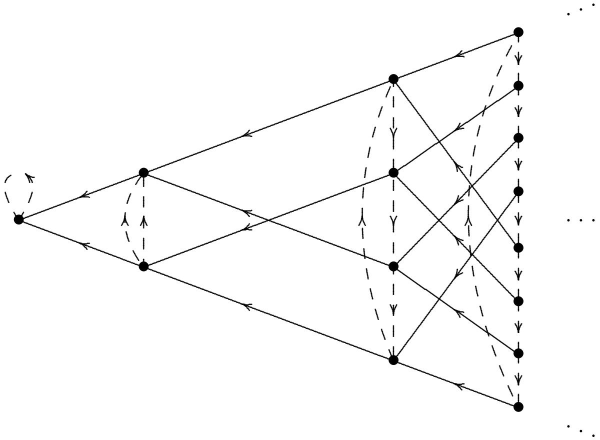
A rank-2 Bratteli diagram of depth $N \in \mathbb{N} \cup \{\infty\}$ is a row-finite 2-graph Λ such that $\Lambda^0 = \bigsqcup_{n=0}^{N} V_n$ of non-empty finite sets which satisfy:

- for every blue edge e, there exists n such that $r(e) \in V_n$ and $s(e) \in V_{n+1}$;
- all vertices which are sinks in the blue graph belong to V_0, and all vertices which are sources in the blue graph belong to V_N;
- every v in Λ^0 lies on an isolated cycle in the red graph, and for each red edge f there exists n such that $r(f), s(f) \in V_n$.

Lemma

Rank-2 Bratteli diagrams are locally convex.

Example



What do the C^*-algebras look like?

Spse Λ is a rank-2 BD of depth $N < \infty$ and that the vertices in V_N lie on a single cycle. $C^*(\Lambda) = \overline{\text{span}}\{s_\lambda s_\mu^* : s(\lambda) = s(\mu)\}$, and using the blue CK relation, we may assume that $s(\lambda) \in V_N$. Using the factorisation property, write $\lambda = \alpha\nu$ where α is blue with $s(\alpha) \in V_N$ and ν is red. Then

$$s_\lambda s_\mu^* = s_\alpha s_\nu s_{\nu'}^* s_\beta^*$$

Now consider a red path which goes once around the cycle at the Nth level. The red CK relations say that $s_\nu^* s_\nu = s_{s(\nu)} = s_\nu s_\nu^*$.

Propn

Let $Y = \{\text{blue paths } \lambda : s(\lambda) \in V_N\}$. Then $C^*(\Lambda) \cong M_Y(C(\mathbb{T}))$.

Idea: Fix a red edge e at the Nth level. For each $\alpha, \beta \in Y$, let $\nu(\alpha, \beta)$ be the part of the cycle that joins $s(\alpha)$ and $s(\beta)$ not containing e. Let $\theta(\alpha, \beta) = \begin{cases} s_\alpha s_{\nu(\alpha,\beta)} s_\beta^* \\ s_\alpha s_{\nu(\alpha,\beta)}^* s_\beta^*. \end{cases}$ For each $\alpha \in Y$, let $\lambda(\alpha)$ be the cycle based at $s(\alpha)$, and $U = \sum_{\alpha \in Y} s_\alpha s_{\lambda(\alpha)} s_\alpha^*$.

Theorem (Pask-Raeburn-Rørdam-Sims, 2006)

Let Λ be an infinite rank-2 BD. Then $C^*(\Lambda)$ is an A$\mathbb{T}$ algebra.

Idea: Let Λ_N be the BD of depth N obtained by chopping. Then Λ_N is locally convex, $\{s_\lambda : \lambda \in \Lambda_N^*\}$ is a CK Λ_N-family in $C^*(\Lambda)$. Check $C^*(\Lambda_N)$ embeds in $C^*(\Lambda)$. Then $C^*(\Lambda) = \overline{\bigcup_N C^*(\Lambda_N)}$. Each $C^*(\Lambda_N)$ is isomorphic to a direct sum with summands of the form $M_Y(C(\mathbb{T}))$.

Criteria for simplicity of $C^*(\Lambda)$ is cofinality plus, for example, the length of the red cycles increasing with N.

So the AF/purely infinite dichotomy fails for simple $C^*(\Lambda)$ (when Λ^0 is infinite). Is there a stably finite/purely infinite dichotomy?

Theorem (Clark-aH-Sims, 2016)

Let Λ be a row-finite 2-graph with no sources such that $C^*(\Lambda)$ is simple. For $i = 1, 2$, let
$A_i(v, w) = \#$paths of degree e_i from w to v. TFAE:

1. $C^*(\Lambda)$ is AF-embeddable.
2. $C^*(\Lambda)$ is quasidiagonal.
3. $C^*(\Lambda)$ is stably finite.
4. $\Big(\mathrm{image}(1 - A_1^t) + \mathrm{image}(1 - A_2^t)\Big) \cap \mathbb{N}\Lambda^0 = \{0\}$.
5. Λ admits a faithful graph trace.

Remarks

- (4) is a K-theoretic condition, and is independent of the factorisation property of the graph. To find it we were motivated by a theorem of N. Brown from 1998: if A is AF and $\alpha \in \mathrm{Aut} A$, then TFAE: 1) $A \rtimes \mathbb{Z}$ is AF-embeddable, 2) quasidiagonal, 3) stably finite, 4) $\alpha_* : K_0(A) \to K_0(A)$ "compresses no elements".

- $g : \Lambda^0 \to [0, \infty)$ is a graph trace on Λ if

$$g(v) = \sum_{\lambda \in v\Lambda^n} g(s(\lambda))$$

  for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. A faithful graph trace induces a faithful semi-finite trace on $C^*(\Lambda)$ (Pask-Rennie-Sims, 2008), and then recent results of Tikussis-Winter-White give (5) $\Longrightarrow$ (2).

Recent work: (Pask-Sierakowski-Sims, May 2017)

- Show the C-aH-S theorem holds for twisted C^*-algebras of higher-rank graphs.
- If a certain semigroup associated to Λ is almost unperforated, then $C^*(\Lambda)$ is either stably finite or purely infinite.
  - The use of the semigroup is motivated by work by Rainone on crossed products.

When is a simple $C^*(\Lambda)$ purely infinite?

We don't have a result of the form: $C^*(\Lambda)$ is purely infinite if and only some condition on the k-graph. Partial results:

1. (Anantharaman-Delaroche, 1997) If the graph groupoid G_Λ is locally contracting, then $C^*(\Lambda)$ is purely infinite.
2. (Sims, 2006) If every vertex can be reached from a cycle with an entrance, then $C^*(\Lambda)$ is purely infinite.
3. A pair (μ, ν) with $s(\mu) = s(\nu)$ and $r(\mu) = r(\nu)$ is a generalised cycle if the cylinder sets satisfy $Z(\mu) \subseteq Z(\nu)$. It has an entrance if the containment is strict.
   - (Evans-Sims, 2012) If Λ contains a generalised cycle with an entrance, then $C^*(\Lambda)$ has an infinite projection.
   - (J. Brown-Clark-aH, 2017) If every vertex can be reached from a generalised cycle with an entrance, then G_Λ is locally contracting.
4. (J. Brown-Clark-Sierakowski, 2015) $C^*(\Lambda)$ is purely infinite if and only if s_v is infinite for every vertex v.

5. (Bönicke-Li, July 2017)
   - Suppose that G is an ample groupoid which is essentially principal and inner exact. Let $\mathcal{B}$ be a basis for $G^{(0)}$ consisting of compact open sets. If each element of $\mathcal{B}$ is paradoxical in a technical sense, then $C_r^*(G)$ is purely infinite.
   - Suppose that G is an ample groupoid such that $C^*(G)$ is simple, and that its unit space is compact. If a certain semigroup is almost unperforated, then $C_r^*(G)$ is either stably finite or purely infinite.