SUB-ACTIONS FOR YOUNG TOWERS

A Dissertation
Presented to
the Faculty of the Department of Mathematics
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

By
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August 2008
UMI Number: 3328476

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Acknowledgements

First, this work would not have been possible without the guidance of my advisor, Dr. Matthew Nicol. Thank you for the opportunity to work with you and for your patience while I have learned. I would also like to express my gratitude to the NSF for funding my research over the past three years. To the members of my committee, thank you for your time and input.

I would also like to thank my friends from Troy University, where I learned how much I loved mathematics. Thanks, first of all, to my mentor, Dr. Patrick Rossi, for making me believe that I could succeed in graduate school. I will always be grateful for your wise guidance. To Robert Sheppard for being my friend and confidant during a very difficult time, and helping me to imagine a new life for myself. You are forever my friend. To Diane Porter, Dr. Ken Roblee, Dr. Govind Menon, Dr. Sergei Belyi, and all the others at Troy that have taught me so much, I hope that you know how much everything you did has meant to me during the last four years.

I would like to thank my family for supporting me in my dreams. To my parents, Karen and Shelburn Branton, thank you for your encouragement and love and willingness to watch me follow my own path, even when it has put miles between us.

Lastly, to Richard, who has solved many a typesetting bug for me during the writing of this dissertation. Your love and daily support has gotten me through some very stressful times, your sense of humor has kept me on the right side of sanity, and I will be forever grateful for your companionship.
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Abstract

Let $T : X \to X$ be a dynamical system and $\phi : X \to \mathbb{R}$ a function on $X$. A function $\theta : X \to \mathbb{R}$ is called a sub-action for $\phi$ if $\theta$ satisfies the equation

$$\phi \leq \theta \circ T - \theta + m(\phi, T)$$

where

$$m(\phi, T) = \sup \left\{ \int \phi \, d\mu : \mu \text{ is an invariant probability measure for } T \right\}.$$ 

The existence and regularity of sub-actions are important for the study of optimizing measures. We prove the existence of Hölder sub-actions for Lipschitz functions on certain classes of Manneville-Pomeau type maps. We also construct locally Hölder sub-actions for Lipschitz functions on Young Towers. In some settings (uniform hyperbolicity and Manneville-Pomeau maps) this implies Hölder sub-actions for the underlying system modeled by the tower.
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Chapter 1

Introduction

As we begin, let us recall some basic definitions from dynamics and give the setting under which we will work. Let $T : X \to X$ be a mapping of the topological space $X$, and consider the dynamical system $(X, T)$, with $\phi : X \to \mathbb{R}$ an observable (function) on $X$. Let $\mathcal{M}_T$ be the set of Borel probability measures on $X$ which are invariant under $T$, that is

$$\mathcal{M}_T = \{ \mu \mid \mu(T^{-1}(A)) = \mu(A) \text{ for all } A \subseteq X, \mu \text{ Borel probability measure on } X \}$$

and define

$$m(\phi, T) = \sup \left\{ \int \phi \, d\mu : \mu \in \mathcal{M}_T \right\}$$

the maximum ergodic average of $T$.

A function $\theta : X \to \mathbb{R}$ is called a sub-action if it satisfies the equation

$$\phi \leq \theta \circ T - \theta + m(\phi, T)$$

which is called the sub-cohomology equation.
For future reference, we recall the following definition: a function \( \theta \) is said to be \( \alpha \)-Hölder, where \( 0 < \alpha \leq 1 \), if for all \( x, y \in X \)

\[
|\theta(x) - \theta(y)| \leq Cd(x, y)\alpha
\]

for some constant \( C > 0 \). We will denote the set of \( \alpha \)-Hölder funtions by \( C^\alpha \), and note that when \( \alpha = 1 \) the set \( C^\alpha \) corresponds to the set of Lipschitz functions. Also, we define the Hölder constant \( C = \text{Höld}_\alpha(\theta) \), given that a function \( \theta \) is \( \alpha \)-Hölder, by

\[
\text{Höld}_\alpha(\theta) = \sup_{d(x, y) > 0} \left\{ \frac{|\theta(x) - \theta(y)|}{d(x, y)^\alpha} \right\}
\]

and we define the Hölder norm

\[
\|\theta\|_\alpha = |\theta|_\infty + \text{Höld}_\alpha(\theta)
\]

where \( |\theta|_\infty = \sup\{\theta(x)|x \in X\} \). We are now ready to begin with a review of some previous results concerning the existence of sub-actions.

1 Motivation for Constructing Sub-Actions

Much of the motivation for constructing sub-action functions has arisen from the study of ‘ergodic optimization,’ and, in particular, the search for those \( T \)-invariant measures \( m \) which maximize or minimize the mean, \( \int \phi \, dm \), of \( \phi \). In other words, we want to find those \( \mu \in \mathcal{M}_T \) for which \( \int \phi \, d\mu = m(\phi, T) \). In an influential paper, *Optimal Orbits of Hyperbolic Systems*, Hunt and Yuan [10] present the following conjectures.

**Conjecture 4.1** For an Axiom A or uniformly expanding system \( T \) and a (topologically) generic smooth function \( f \), there exists an optimal periodic orbit.
Here an optimal periodic orbit is an orbit
\[ \{T^i x\}_{i=0}^{n-1} \]
for which the Dirac measure, \( \delta \), supported on the periodic orbit is maximizing, so that
\[
\frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x) = m(f, T).
\]
Also, we note that by specifying a topologically generic function it is meant that there exists a set of functions which contains a dense \( G_\delta \), or residual set, for which there exists an optimal periodic orbit.

**Conjecture 4.2** For an Axiom A or uniformly expanding system \( T \) and a smooth \( (C^1) \) function \( f \), the optimal average \( m(f, T) \) can be exponentially well approximated by averages over periodic orbits. To be precise, there exist constants \( C > 0 \) and \( \alpha > 0 \), and a sequence of periodic orbits
\[ \{T^i y_n\}_{i=0}^{p_n-1} \]
of period \( p_n \), such that

(i) the sequence \( \{p_n\} \) satisfies
\[
\lim_{n \to \infty} p_n = \infty,
\]
and

(ii) the inequality
\[
m(f, T) - \frac{1}{p_n} \sum_{i=0}^{p_n-1} f \circ T^i(y_n) < Ce^{-\alpha p_n}
\]
is satisfied.
In short, Yuan and Hunt conjectured that for uniformly hyperbolic systems for an open dense set of observables (in the Hölder topology) such an optimal measure is supported on a periodic orbit, typically of low period. They showed that such orbits are stable to perturbation of $\phi$ but that if a maximizing measure is not supported on a periodic orbit, then it is not stable under perturbation of $\phi$.

The construction of continuous sub-actions has proven useful in obtaining such results, and the technique is used in various settings, see [11, 14] for example. We will provide a brief synopsis of the general strategy here. The setting is a dynamical system, $(X, T)$ with suitably regular observable $\phi : X \to \mathbb{R}$. First, the sub-cohomology equation may be rewritten to obtain the equation

$$\phi = \theta \circ T - \theta + m(\phi, T) - r$$

so that $r : X \to \mathbb{R}$ is continuous and non-negative.

Then for every $\nu \in M_T$ we have

$$\int \phi \, d\nu = m(\phi, T) - \int r \, d\nu$$

with the conclusion that $\nu$ is maximizing if and only if $\int r \, d\nu = 0$; and, due to the fact that $r \geq 0$, we may conclude that the set of maximizing measures of $\phi$ is the same as the set

$$\{\mu \in M_T | \text{supp} (\mu) \subset r^{-1} (0) \}.$$ 

This fact will lead to a conclusion concerning the existence and/or uniqueness of maximizing or minimizing measures. For a more specific instance of the use of this strategy, see [8].
2 Sub-Actions for Uniformly Hyperbolic Systems

There have been many results establishing the existence of Hölder sub-actions for uniformly hyperbolic systems. These sub-actions are constructed given a specific hyperbolic dynamical system, \((X, T)\), and an observable \(\phi : X \to \mathbb{R}\) which satisfies some regularity condition, in most cases that \(\phi\) is Lipschitz continuous.

One of the classical results on the existence of sub-actions was published in 2001, and is found in *Le Poisson n'a pas d'aretes*, or "The fish has no bones" [7], where Thierry Bousch obtained a Hölder sub-action for Lipschitz observables in the case of the doubling map of the circle, \(T(x) = 2x \mod 1\).

Also, in 2001, in *Lyapunov minimizing measures for expanding maps of the circle*, [8], Contreras, Lopes, and Thieullen consider maps, \(f : S^1 \to S^1\), of the circle which are in the set

\[ \mathcal{F}_{\alpha^+} = \bigcup_{\beta > \alpha} C^{1+\beta} \]

where \(C^{1+\beta}\) is the set of maps on \(S^1\) whose first derivatives are in \(C^\beta\). They add the assumptions that the map \(f\) is a covering map of degree \(D\), expanding, orientation preserving, and is such that

\[ \min \{|f'(x)| x \in S^1\} > 1. \]

Their result is stated below.

**Theorem 2.1** Let \(f \in \mathcal{F}_{\alpha^+}\) be as above, and \(A \in C^\alpha\). Then there exists \(V \in C^\alpha\) such that \(A \leq V \circ f - V + m(A, f)\). In particular, \(A\) is cohomologous to \(m(A, f)\) on the support of any maximizing measure \(\mu \in \mathcal{M}(A, f)\), that is, \(A = V \circ f - V + m(A, f)\) on the support of \(\mu\).
There are a few interesting notes to be made concerning this result. First, note that the $\alpha$-Hölder regularity of the observable $A$ is preserved in the proof, so that the sub-action $V$ is $\alpha$-Hölder as well. Second, the result goes further than the sub-cohomology equation, and asserts that, indeed, on the support of each maximizing measure, $\mu$, the function $V$ actually satisfies the cohomology equation. These additional properties are used to prove several results in the area of Ergodic Optimization, which we will discuss in due time.

In 2003, in *Sub-Actions for Anosov Diffeomorphisms*, [14], authors A. Lopes and Ph. Thieullen consider $C^2$ Anosov diffeomorphisms $f$ on a compact manifold $M$ of dimension $d \geq 2$ with $\alpha$-Hölder observable $A$. They prove the existence of a Hölder sub-action in this case. However, unlike Theorem 1.1 above, the $\alpha$-Hölder regularity of the observable is not preserved.

**Theorem 2.2** Let $f : M \to M$ be a $C^2$ transitive Anosov diffeomorphism on a compact manifold, $M$, without boundary. For any given $\alpha$-Hölder observable (function), $A : M \to \mathbb{R}$, there exists a $\beta$-Hölder function, $V : M \to \mathbb{R}$, that we call a sub-action, such that:

$$A \leq V \circ f - V + m(A, f)$$

where $m(A, f) = \sup \{ \int f \, d\mu | \mu \in \mathcal{M}(f) \}$, $\mathcal{M}(f)$ is the set of $f$-invariant probability measures and

$$\beta = \alpha \frac{\ln(1/\gamma_s)}{\ln(\gamma_u/\gamma_s)}, \quad \text{Höld}_\beta(V) \leq \frac{C(M)}{\min(1 - \gamma_u^{-\alpha}, 1 - \gamma_s^{\alpha})} \text{Höld}_{\alpha}(A)$$

This result gives a $\beta$-Hölder sub-action for the $\alpha$-Hölder observable, and, in fact, $\beta < \alpha$ meaning that the regularity is somewhat weakened by the construction.

In 2007, in *Ergodic Optimization for Non-compact Dynamical Systems* [13], Jenkinson, Mauldin, and Urbanski work with non-compact dynamical systems and find
sufficient conditions which will guarantee the existence of a sub-action. These results will be discussed in detail as an avenue to formalize cohomology classes in Chapter 3. For now, let us give some results for systems where the requirements of uniform hyperbolicity are relaxed.

3 Sub-Actions for Nonuniformly Hyperbolic Systems

For the precise definition of nonuniformly hyperbolic and nonuniformly expanding systems, see Chapter 2. For now we will think of them as having "some hyperbolicity" or being "mostly expanding".

In the setting of nonuniformly expanding systems, some work has been done to achieve results similar to those for the expanding systems of the previous section. In 2003, in Sub-Actions for Weakly Hyperbolic One-Dimensional Systems [17], under the assumption that the observable is monotone in some neighborhood of the origin, Rafael R. Souza obtains a Hölder sub-action for a map \( f : [0,1] \rightarrow [0,1] \) which meets the following requirements:

- \( f \) is expanding of degree 2,
- \( f \) is continuous on \([0,c]\) and \((c,1]\) for some \( c \in (0,1) \),
- \( f(0) = 0, f(c) = 1, f(1) = 1 \),
- \( f \) approaches 0 from the right at \( c \),

\[
\lim_{x \to c^+} f(x) = 0,
\]
• \( f \) is of class \( C^1 \) in \([0, c)\), and in \((c, 1]\), with \( f'(x) > 1 \) for all \( x \in (0, c) \cup (c, 1) \),

• \( f \) has one-sided derivatives at 0, \( c \), and 1 given by \( f_+'(0) = 1 \), \( f_-'(c) > 1 \), \( f_+'(c) > 1 \), and \( f_-'(1) > 1 \).

Note that these conditions give a map with an indifferent fixed point at \( x = 0 \), i.e. \( f_+'(0) = 1 \), and for any choice of \( \beta > 0 \) the derivative \( f'(x) > 1 \) for all \( x \in [\beta, 1) \), \( x \neq c \). The indifferent fixed point distinguishes this case from those uniformly expanding systems previously studied. The result given by Souza is stated below.

**Theorem 3.1** Let \( f : [0,1] \rightarrow [0,1] \) be a map satisfying the above conditions, and let the observable \( A : [0,1] \rightarrow \mathbb{R} \) be \( \alpha \)-Hölder in each of the intervals \([0, c)\) and \((c, 1]\). Further, assume that \( A \) is monotone (increasing or decreasing) in a small neighborhood of \( x = 0 \), and that \( A(0) < \text{m}(A, f) \) and \( A(1) < \text{m}(A, f) \). Then there exists a function \( S : [0,1] \rightarrow \mathbb{R} \) such that \( A \leq S \circ f - S + \text{m}(A, f) \). Also, \( S \) is \( \alpha \)-Hölder on \([0,1]\).

Here the requirements that \( A(0) < \text{m}(A, f) \) and \( A(1) < \text{m}(A, f) \) are added in order to prevent the degenerate case where the measures supported on the fixed points are maximizing.

One thing should be noted here for future reference. The construction in the proof of the theorem above requires the assumption that the observable \( A \) is monotone in a small neighborhood of the origin. We will show in this work that such an assumption is unnecessary when the observable is Lipschitz continuous.

There are some very interesting questions pertaining to nonuniformly expanding maps.

1. Under what conditions do nonuniformly expanding or nonuniformly hyperbolic maps have a continuous sub-action?
(2) Is Hölder continuity of the observable enough to guarantee the existence of a continuous sub-action?

(3) If Hölder continuity is insufficient, will Lipschitz continuity guarantee the existence of a continuous sub-action?

The answer to (2) is known to be: not always. In recent work [16], Ian Morris has given interesting counterexamples, by proving the non-existence of continuous sub-actions for Hölder observables possessing small Hölder exponent over certain maps with indifferent fixed point. He has also independently obtained the existence of sub-actions for observables of a sufficiently high Hölder exponent [16]. The map he works with is a map $T : [0, 1] \rightarrow [0, 1]$ which satisfies the following:

- $T$ is piecewise $C^1$,
- $T$ is piecewise surjective,
- $T(0) = 0$,
- $T'(0) = 1$,
- $T'(x) > 1$ for all $x \in (0, 1]$ for which $T'(x)$ exists,
- $T''(x)$ is defined in the region $(0, \delta)$, for some $\delta > 0$, and
  \[
  \lim_{x \to 0} T''(x)x^{1-\alpha}
  \]
  exists and is nonzero.

Such a map is said to be of Manneville-Pomeau type $\alpha$. Some conditions under which this map has a continuous sub-action are given by the following theorem.
Theorem 3.2 (Morris) Let \( T : [0, 1] \rightarrow [0, 1] \) be of Manneville-Pomeau type \( \alpha > 0 \) and suppose that \( \alpha < \gamma \leq 1 \). Then for every \( \gamma \)-Hölder function \( f : [0, 1] \rightarrow \mathbb{R} \) there exists a function \( g : [0, 1] \rightarrow \mathbb{R} \) which is \((\gamma - \alpha)\)-Hölder such that \( f \leq g \circ T - g + m(f, T) \).

This theorem establishes the existence of a Hölder sub-action of exponent \((\gamma - \alpha)\) provided that the observable is Hölder with exponent \( \gamma > \alpha \).

The next result comes by provision of direct counter-example, and provides instances where there will be no continuous sub-action even though the observable is Hölder. As a result, we know that for some systems, even though seemingly strict regularity conditions are satisfied, there may be no sub-action with any “desirable” regularity.

Theorem 3.3 (Morris) Let \( T : [0, 1] \rightarrow [0, 1] \) be of Manneville-Pomeau type \( \alpha > 0 \). Then the following hold:

(a) If \( 0 < \gamma < \alpha/(1 + \alpha) \), then there exists a \( \gamma \)-Hölder function \( f \) such that if the relation \( f \leq g \circ T - g + m(f, T) \) holds for \( g : [0, 1] \rightarrow \mathbb{R} \), then \( g \) is unbounded.

(b) If \( 0 < \gamma < \alpha/(2 + 2\alpha) \), then there exists a \( \gamma \)-Hölder function, \( f \), such that the relation \( f \leq g \circ T - g + m(f, T) \) is satisfied for some upper semi-continuous function \( g : [0, 1] \rightarrow \mathbb{R} \), but it is not satisfied for any continuous function \( g : [0, 1] \rightarrow \mathbb{R} \).
4 A Different Approach to Sub-Actions: The Young Tower

For certain nonuniformly expanding and nonuniformly hyperbolic systems, L. S. Young has constructed a Markov Extension, or Young Tower, which she uses to study the statistical properties of such systems, [18, 19]. This construction exploits a common structure to many nonuniformly hyperbolic systems and allows a detailed analysis of their statistical properties, such as rate of decay of correlations for Hölder observables, central limit theorems, and so on. This approach has been successfully applied to a variety of dynamical systems, for example: Lozi maps and certain piece-wise hyperbolic maps; a class of Hénon maps; Poincaré maps of billiards with convex scatterers and certain $C^2$ unimodal maps. For a more complete list see [1, Section 4.3].

Our goal was to use the axiomatic structure of Young to construct sub-actions for large classes of nonuniformly hyperbolic systems, in the same way that the regularity of solutions to the coboundary equation over nonuniformly hyperbolic systems could be studied by considering the related equations on a Young Tower [6]. The motivation was to bypass the case-by-case analysis of previous research and prove general results concerning sub-actions and optimal measures or orbits.

The first, and major, difficulty of our technique is that regularity may be lost when projecting a sub-action constructed on a tower to the underlying manifold (whereas lifting a function from the underlying manifold to the tower preserves or improves regularity). Furthermore, whereas there is a good correspondence between the statistical properties of the tower and the statistical properties of the underlying
system with respect to its natural invariant measure, there is not such a good correspondence between topological properties. In fact, optimal orbits or measures for $T$ may be discarded in a Young Tower construction.

However, we were able to carry out this program for certain systems modeled by Young Towers. In Chapter 5 we will construct locally Hölder sub-actions for Young Towers of one dimensional maps, obtaining as our main result Hölder sub-actions in the setting of intermittent (Liverani-Saussol-Vaienti (LSV)) maps \cite{19} as well as in the more general settings of systems such as Axiom A. In particular, we extend the earlier work obtained by Souza in \cite{17} in that we do not assume that the observable is monotone in a neighborhood of the origin. We also obtain, via a unifying proof, some known results on the existence of Hölder sub-actions for Lipschitz observables on uniformly hyperbolic systems.

In Chapter 4 we describe the construction of the Young Tower, $\Delta$, the partitioning of the base, the map $F : \Delta \to \Delta$ and how it arises from the map $T : X \to X$. We then discuss the properties of the tower, and the results that we will need, namely backwards contraction and bounded distortion. In Chapter 5 we construct a sub-action for the Young Tower of a one dimensional nonuniformly expanding map. More precisely, suppose $X$ is a one dimensional manifold (with Euclidean metric $d$) having compact closure. Suppose $T : X \to X$ is a nonuniformly expanding map and the observable $\phi : X \to \mathbb{R}$ is Lipschitz. Suppose also that the system admits a Young Tower, $(F, \Delta)$, where $F : \Delta \to \Delta$ has observable $\phi : \Delta \to \mathbb{R}$ lifted from $X$ in the usual way: $\phi(x, l) = \phi(T^l x)$. Further, suppose that the partition elements of the base of the tower, $\Lambda$, are intervals. This is the case for many one-dimensional maps, including the beta transformation of the unit interval, $x \to \beta x \mod 1$, logistic maps...
from the family

\[ \{ f_a | f_a(x) = 1 - ax^2, \ a \text{ in a positive measure set in } (0, 2) \} \]

as in [1], and intermittent type maps, such as the Liverani-Saussol-Vaienti map.

Under these assumptions, we show that there is a measurable sub-action function, \( \theta : \Delta \rightarrow \mathbb{R} \), such that \( \phi \leq \theta \circ F - \theta + m(\phi, F) \). Furthermore, we show that \( \theta \) is bounded above and Hölder continuous with Hölder exponent \( \beta < 1 \) on the base of the tower and all iterates of it under \( F \). The Hölder constant is bounded on the base of the tower and all iterates of it under \( F \), but this bound may increase under iteration of \( F \).

Further, we ask if the sub-action \( \theta \) which we constructed in section 2 may be extended to a sub-action \( \tilde{\theta} \) on the manifold \( X \). We will use the natural projection \( \pi : \Delta \rightarrow X \), given by \( \pi(x, l) = T^l x \) which intertwines \( F \) and \( T \) by the relation \( T \circ \pi = \pi \circ F \). The image, \( \pi(\Delta) \), is \( T \)-invariant, but may or may not be closed. In addition, the projection is not necessarily onto \( X \). This projection may be infinite-to-one, as is the case for a Young Tower for the logistic map. However, in some cases the projection is finite to one on a large enough set, and some of the regularity of \( \theta \) may be recaptured. In the case of uniformly hyperbolic systems such as Axiom A and intermittent type maps, \( \pi(\Delta) = X \), and for the LSV map \( \pi \) projects \( \Delta \) onto \((0, 1]\).

We consider uniformly hyperbolic systems and the LSV map in Chapter 6, using our construction in Chapter 5 to obtain a Hölder sub-action. This result strengthens the result obtained by R. Souza in [17] in the case of a Lipschitz observable in that we do not require that \( \phi \) be monotone in a neighborhood of the origin.

In Chapter 7 we adapt our construction from chapter 5 in order to obtain a locally Hölder sub-action for the Young Tower of a two dimensional nonuniformly hyperbolic
map (which has a uniformly contracting stable foliation). In this case we assume $X$ is a two dimensional compact manifold with Euclidean metric $d$, $T : X \to X$ is a $C^{1+\epsilon}$ diffeomorphism on $X$ having a stable foliation, and $\phi : X \to \mathbb{R}$ is Lipschitz. Suppose, also, that $(T, X)$ admits Young Tower $(F, \Delta)$, and $\phi : \Delta \to \mathbb{R}$ is lifted to $\Delta$ in the natural way. Under these assumptions, we show that there is a sub-action function $\tilde{\theta} : \Delta \to \mathbb{R}$ such that $\phi \leq \tilde{\theta} \circ F - \tilde{\theta} + m(\phi, F)$. Furthermore, we show that $\tilde{\theta}$ is bounded above and Hölder on the base $\Delta_0$ of the Young Tower, and all iterates of it under $F$. Again, the Hölder constant is bounded on the base and all iterates of it under $F$, although this bound may increase under iteration by $F$. The Hölder exponent is $(\gamma)/(1 + \gamma)$, where

$$\gamma = \frac{\log \lambda^s_{max}}{\log \lambda^u_{max}}$$

with $\lambda^s_{max}, \lambda^u_{max}$ the greatest contraction and least expansion rates, respectively.
Chapter 2

Nonuniformly Expanding and Nonuniformly Hyperbolic Dynamical Systems

When the strict requirements of expansion and hyperbolicity are relaxed, it is possible to study a wider array of dynamical systems. In particular, we may study maps such as the Manneville-Pomeau type $\alpha$ map in [16], which is the indifferent fixed point map studied by Souza in [17]. We present, below, the defining qualities we will assume for a nonuniformly expanding map, as well as the axioms we will embrace for a nonuniformly hyperbolic map.

1 Definition of Nonuniformly Expanding System

Let $(X, d)$ be a locally compact, separable, and bounded metric space and suppose $T : X \to X$ is a mapping such that $T$ is nonsingular, and that there is an ergodic,
Borel probability measure \( \eta \). Note, the mapping \( T : X \to X \) is nonsingular if 
\( \eta(T^{-1}A) = 0 \) whenever \( \eta(A) = 0 \).

Let \( \Lambda \subset X \) be a measurable subset with \( \eta(\Lambda) > 0 \). Suppose that there is an at most countable partition \( \{\Lambda_i\} \) of \( \Lambda \) with \( \eta(\Lambda_i) > 0 \) for all \( i \), and that there exist integers \( R_i \geq 1 \), and constants \( \lambda > 1; C, D > 0 \) and \( \gamma \in (0,1) \) such that for all \( j \),

1. the induced map \( F = T^{R_j} \) is a measure-theoretic bijection,

2. \( d(Fx, Fy) \geq \lambda d(x, y) \), for all \( x, y \in \Lambda \),

3. \( d(T^k x, T^k y) \leq Cd(Fx, Fy) \), for all \( x, y \in \Lambda_j, k < R_j \),

4. the function

\[
g_j = \frac{d(\eta|\Lambda_j \circ F^{-1})}{d\eta|\Lambda}
\]

satisfies

\[
|\log g_j(x) - \log g_j(y)| \leq Dd(x, y)\gamma
\]

for all \( x, y \in \Lambda \), and

5. the sum

\[
\sum_{j=0}^{\infty} R_j \eta(\Lambda_j)
\]

is finite.

Then, we say that the mapping \( T : X \to X \) is a nonuniformly expanding map, and the pair \( (X, T) \) is said to be a nonuniformly expanding dynamical system.
The Axioms for Nonuniformly Hyperbolic Systems

The definition of a nonuniformly hyperbolic map is adapted from that of Ian Melbourne and Matthew Nicol in [15] in which the assumptions made by L. S. Young in [18] are modified so that the differential structure is “bypassed” and some of Young’s results are stated as assumptions.

Let $T : M \rightarrow M$ be a diffeomorphism, possibly with singularities, defined on a Riemannian manifold $(M, d)$. Let $\Lambda \subset M$ be given, along with a family, $\{W^s\}$, of subsets of $M$ that cover $\Lambda$. We will call these sets stable disks, and whenever $x \in W^s$ for some $s$ we label it $W^s(x)$. We say that the system is nonuniformly hyperbolic if the assumptions (A1)-(A4) below hold.

(A1) There exists a partitioning $\{\Lambda_j\}$ of $\Lambda$ and integers $R_j \geq 1$ such that for all $x \in \Lambda_j$ we have $T^{R_j} (W^s(x)) \subset W^s (T^{R_j}(x))$.

Using the integers $\{R_j\}$, we define the return time function $R : \Lambda \rightarrow \mathbb{Z}^+$ by

$$R|_{\Lambda_j} = R_j$$

and the induced map $f : \Lambda \rightarrow \Lambda$ by

$$f(x) = T^{R(x)}(x).$$

We also define the suspension

$$\Delta = \{(x, l) \in \Lambda \times \mathbb{N} : 0 \leq l \leq R(x)\} / \sim$$
where \((x, R(x)) \sim (f(x), 0)\), and we go on to define the discrete suspension map \(F : \Delta \to \Delta\) by

\[
F(x, l) = \begin{cases} 
(x, l + 1), & \text{if } l < R(x) - 1, \\
(f(x), 0), & \text{if } l = R(x) - 1.
\end{cases}
\]

Define a separation time \(s : \Lambda \times \Lambda \to \mathbb{N}\) by \(s(x, x') = \) the largest integer \(n\) for which \(f^k(x)\) and \(f^k(y)\) lie in the same partition element of \(\Lambda\) for \(k = 0, 1, \ldots, n\), where \(s(x, x') = 0\) if \(x\) and \(x'\) do not lie in the same partition element of \(\Lambda\). Now, to extend this to a separation time defined on \(\Delta\) we make the following assignments for points \(p = (x, l), q = (x', l') \in \Delta:\)

\[
s(p, q) = \begin{cases} 
s(x, x'), & \text{if } l = l', \\
0, & \text{otherwise.}
\end{cases}
\]

This gives a separation time \(s : \Delta \times \Delta \to \mathbb{N}\). Note that this separation time, which is used in [15], differs from the separation time \(s\) which may be found in [18]. We will use the definition given above, and the definition given by L. S. Young in [18] will not appear in this work.

Now we will continue with our statement of the axioms for nonuniformly hyperbolic dynamical systems.

\((A2)\) There is a distinguished subset \(W^u \subset \Lambda\), which we will call an unstable leaf such that each stable disk intersects \(W^u\) in precisely one point, and there exist constants \(C \geq 1\) and \(\alpha \in (0, 1)\) such that

\((i)\) for all \(y \in W^s(x)\) and all \(n \geq 0,\)

\[
d(T^nx, T^ny) \leq C\alpha^n,
\]

and
(ii) for all $x, y \in W^u$ and all $0 \leq n < R$,

$$d(T^n x, T^n y) \leq C \alpha^{s(x,y)}.$$ 

For the assumption $(A3)$ below, we will need the following definitions. Define $\Lambda = \Lambda/ \sim$ where $x \sim x'$ if and only if $x \in W^s(x')$. We extend this definition to obtain the following:

- A partition $\{\Lambda_j\}$ of $\Lambda$,
- A return time function $R : \Lambda \to \mathbb{Z}^+$,
- An induced map $f_0 : \Lambda \to \Lambda$,
- The corresponding suspension map $F : \Delta \to \Lambda$, and
- The natural projection $\pi : \Delta \to \Lambda$.

Using this structure, we now introduce the final axioms.

$(A3)$ The map $f : \Lambda \to \Lambda$ and the partition $\{\Lambda_j\}$ separate points in $\Lambda$.

$(A4)$ There exist $F$-invariant probability measures $\bar{m}$ on $\Delta$ and $\overline{m}$ on $\Lambda$ such that the following are true

1. The maps, $\pi : \Delta \to M$ and $\overline{\pi} : \Delta \to \Lambda$, are measure preserving, and
2. $F : \Lambda \to \Lambda$ is a Young Tower.

Though we have not yet given the explicit definition of the Young Tower, we will do so in Chapter 4. For now, we assume for the axiom $(A4)$ that the map $F : \Lambda \to \Lambda$ satisfies the properties we will state explicitly later.
Chapter 3

Cohomology Classes

We will give here the theoretical setting into which our idea of a sub-action function fits. This will allow a broader understanding of the sub-cohomological equation as well as the sub-action functions themselves. Most of the definitions are from [13], and we have stated them here for reference purposes. We also include the motivation for the study of sub-actions as solutions to the sub-cohomology equation.

1 Continuous Coboundaries and Cohomologous Functions

Let \( T : X \rightarrow X \) be a continuous map on the topological space \( X \), and let \( \phi : X \rightarrow \mathbb{R} \) be a continuous and bounded function on \( X \). Then the function \( \phi \circ T - \phi \) is called a \emph{continuous coboundary}. It is useful to note that this definition differs slightly from that presented in [13] in that they call \( \phi - \phi \circ T \) the coboundary. These definitions are equivalent, and we use the former simply because it is more convenient with respect to our statement of the sub-cohomology equation. Also, for our purposes, we...
will refer to $\phi \circ T - \phi$, where $\phi$ is continuous and bounded, from this point onward, simply as a coboundary.

Further, we say that two functions $f, g$ are cohomologous if there exists a coboundary function $\phi \circ T - \phi$ such that

$$f = \phi \circ T - \phi + g$$

or, equivalently, $f$ and $g$ are said to be cohomologous if they differ by a coboundary function.

## 2 The Equivalence Relation

We define an equivalence relation on $C(X)$, the space of continuous functions on $X$, as follows: two functions $f, g \in C(X)$ are equivalent if and only if they are cohomologous. Further, we write $f \sim g$ when $f$ and $g$ are equivalent. The resulting equivalence classes of functions are said to be cohomology classes.

One important note to be made here is that several important objects which emerge in the study of ergodic optimization are independent of the representative from the cohomology class. These are

- the maximum ergodic average, $m(f, T)$,
- maximizing measures, and
- maximizing periodic orbits.

We provide some explanation of why this is true. Note that $f \sim g$ implies that $f = \phi \circ T - \phi + g$ for a continuous and bounded function $\phi$. Then, for all $T$-invariant
measures \( \mu \in \mathcal{M}_T \), the integral

\[
\int (\phi \circ T - \phi) \, d\mu = 0.
\]

Thus \( m(f, T) = m(g, T) \) for all cohomologous functions \( f, g \). This then gives that if \( \mu \in \mathcal{M}_T \) is maximizing for \( f \), we have

\[
m(g, T) = m(f, T) = \int f \, d\mu = \int (\phi \circ T - \phi + g) \, d\mu = \int g \, d\mu
\]

so that \( \mu \) is maximizing for \( g \) as well.

Now suppose that

\[
\{T^i(x)\}_{i=0}^{n-1}
\]

is a maximizing periodic orbit for \( f \). Then using the fact that \( T^n x = x \) and

\[
m(f, T) = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)
\]
we have

\[ m(g, T) = m(f, T) \]

\[ = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) \]

\[ = \frac{1}{n} \sum_{i=0}^{n-1} \left[ \phi \circ T^i (T^i x) - \phi (T^i x) + g(T^i x) \right] \]

\[ = \phi \circ T^n (x) - \phi (x) + \frac{1}{n} \sum_{i=0}^{n-1} g(T^i x) \]

\[ = \frac{1}{n} \sum_{i=0}^{n-1} g(T^i x) \]

which gives that

\[ \{T^i(x)\}_{i=0}^{n-1} \]

is a maximizing periodic orbit for \( g \) as well. Thus, solutions to the cohomology equation are extremely useful in the area of ergodic optimization. However, solutions to the sub-cohomology equation have also proven useful in many of the same ways under appropriate modifications and slightly more complicated arguments. This motivates the study of sub-actions as a means of finding some results on maximizing measures and maximum periodic orbits in the case where no sufficiently regular solution to the cohomology equation can be found.

3 Cohomology and Sub-Cohomology

The purpose of the introduction here of cohomology classes has been to gain a better understanding of the setting of the sub-action function in hopes that we may give some interpretation of its meaning as well as predict when such a function may exist.
Let \( T : X \to X \) be a dynamical system, and \( \phi : X \to \mathbb{R} \) a function on \( X \). Then we say that a function \( \theta : X \to \mathbb{R} \) is an action on (or for) the dynamical system \((X, T)\) if \( \theta \) satisfies the cohomology equation

\[
\phi = \theta \circ T - \theta + m(\phi, T)
\]

so that the observable \( \phi \) is a coboundary. The existence of an action implies that the \( \phi \) is cohomologous to a constant, namely the maximum ergodic average of \( \phi \) in \((X, T)\), i.e. \( \phi \) and \( m(\phi, T) \) belong to the same cohomology class of functions on \( X \).

Further, a function \( \theta : X \to \mathbb{R} \) is said to be a sub-action function if it satisfies the sub-cohomology equation,

\[
\phi \leq \theta \circ T - \theta + m(\phi, T),
\]

and we extend the previous terminology to include this case by saying that here \( \phi \) and \( m(X, T) \) are sub-cohomologous with the addition of the note that we will not speak of “sub-cohomology classes” as they are not equivalence classes and do not preserve those ergodic optimization objects which the cohomology classes were found to preserve so readily. However, for our purposes, we will keep this notation when speaking of these functions.
Chapter 4

Young Towers

Though many of the components of the definition of a Young Tower have been previously defined in Chapter 2, we will slightly alter the notation in parts to suit our needs. Thus, we present the definition in its entirety here, without regard to repetition.

1 Our Assumptions

Certain nonuniformly expanding or nonuniformly hyperbolic systems can be shown to admit a Young Tower as described below. We work with these maps, and we refer to Young’s original papers [18, 19] and Baladi’s book [1] for more details.

Let $T : X \to X$ be a $C^{1+\epsilon}$ diffeomorphism, where $X$ is a compact manifold with metric $d$. We say that a subset $\Lambda \subset X$ has hyperbolic product structure if there exist a continuous family of unstable disks $\Gamma^u = \{\gamma^u\}$ and a continuous family of stable disks $\Gamma^s = \{\gamma^s\}$ such that

(i) $\dim \gamma^u + \dim \gamma^s = \dim X$, 

...
(ii) the $\gamma^u$-disks intersect the $\gamma^s$-disks transversally, and the angles between the intersecting disks are bounded away from 0,

(iii) each $\gamma^u$-disk meets each $\gamma^s$-disk in exactly one point, and

(iv) $\Lambda = (\cup \gamma^u) \cap (\cup \gamma^s)$.

We adopt the following assumptions, found in [18, Section 1].

(P1) There exists $\Lambda \subset X$ with a hyperbolic product structure [18, Definition 1]:

$$\Lambda = \{(\cup \gamma^u) \cap (\cup \gamma^s) : \gamma^u \in \Gamma^u, \gamma^s \in \Gamma^s\}$$

where $\Gamma^u, \Gamma^s$ are two families of $C^1$ disks in $X$. Here we note as in [18] that a subset $\Lambda_0 \subset \Lambda$ is called an $s$-subset if $\Lambda_0$ also has a hyperbolic structure and its defining families can be chosen to be $\Gamma^u$ and $\Gamma^s \subset \Gamma^s$. $U$-subsets are defined analogously.

(P2) There are pairwise disjoint $s$-subsets $\Lambda_1, \Lambda_2, \ldots \subset \Lambda$, with the properties that

(i) on each $\gamma^u$-disk

$$\mu_{\gamma^u} \{(\Lambda - \cup \Lambda_i) \cap \gamma^u\} = 0,$$

(ii) for each $i$, there exists $R_i \in \mathbb{Z}^+$ such that $T^{R_i} \Lambda_i$ is a $u$-subset of $\Lambda$, and for all $x \in \Lambda_i$,

$$T^{R_i} (\gamma^s(x)) \subset \gamma^s (T^{R_i} x)$$

and

$$T^{R_i} (\gamma^u(x)) \supset \gamma^u (T^{R_i} x),$$

(iii) for each $n$, there are at most finitely many $i$'s with $R_i = n$, and
(iv) \( \min R_1 \geq R_0 > 1 \), for some \( R_0 \) depending only on \( f \).

(P3) There exists \( 0 < \alpha < 1 \) such that for all \( y \in \gamma^s(x) \),

\[
d(F^n x, F^n y) \leq C \alpha^n,
\]

for all \( n \geq 0 \), where \( F \) is the map on the tower as discussed below.

Now, for each pair \( x, y \in A \), define the separation time to be \( s(x, y) = \) the smallest integer \( n \) such that \( F^n x \) and \( F^n y \) lie in different elements of the partition of the tower.

(P4) For \( y \in \gamma^u(x) \) and \( 0 < k < n \leq s(x, y) \), we have

\[
(i) \quad d(T^n x, T^n y) \leq C \alpha^{s(x, y) - n},
\]

\[
(ii) \quad \log \left( \prod_{i=k}^{n} \frac{\det DT^u(T^ix)}{\det DT^u(T^iy)} \right) \leq C \alpha^{s(x, y) - n}.
\]

Finally, we state assumption (P5) below.

(P5) For all \( n \geq 0 \),

\[
\log \left( \prod_{i=n}^{\infty} \frac{\det DT^u(T^ix)}{\det DT^u(T^iy)} \right) \leq C \alpha^{s(x, y)}.
\]

Under these assumptions, Young constructs a Markov extension (Young Tower) \( (F, \Delta) \) over \( T : X \to X \) with base \( A \). The set \( A \) is decomposed as \( A = \cup_j A_j \) and there is a return function \( R : A \to \mathbb{N} \), with constant value \( R_j \) on each \( A_j \). Define \( T^R(x) = T^{R(x)}(x) \) and define the tower by

\[
\Delta := \{(x, l) : x \in A; \ l = 0, 1, \ldots, R(x) - 1 \}.
\]
where \( \Delta_0 = \{(x, 0) : x \in \Lambda\} = \Lambda \) in a natural identification.

There is a countable partition \( P_0 \) of \( \Lambda = \Delta_0 \) into elements \( \{\Delta_{j,0}\} \) together with a return function \( R : \Lambda \to \mathbb{N} \), with \( R|_{\Delta_{i,0}} = R_i \). The Tower Map \( F : \Delta \to \Delta \) is defined by

\[
F(x, i) = \begin{cases} 
(x, i + 1), & \text{if } x \in \Lambda_j \text{ and } i < R_j - 1, \\
(T^{R_j}x, 0), & \text{if } x \in \Lambda_j \text{ and } i = R_j - 1.
\end{cases}
\]

The projection \( \pi : \Delta \to X \), given by \( \pi(x, l) = T^lx \), intertwines \( F \) and \( T \) by the relation \( T \circ \pi = \pi \circ F \). Moreover, for each \( i \), \( F^{R_i} \) maps \( \Delta_{i,0} \) bijectively onto \( \Delta_0 \) [19, Section 1.1].

For \( l < R(i) \) we define

\[
\Delta_{i,l} = \{(x, l) : (x, 0) \in \Delta_{i,0}\}.
\]

Thus, \( \{\Delta_{i,l}\} \) forms a partition, call it \( P_{\Delta} \), of \( \Delta \).

The induced map \( F^R : \Lambda \to \Lambda \) of the base has a Lebesgue equivalent measure \( \mu_{\Lambda} \), which is lifted to an \( F \)-invariant measure \( \mu_{\Delta} \) on the tower and \( \mu = \pi \circ \mu_{\Delta} \) is an absolutely continuous invariant measure for \( T \) [18].

## 2 Contraction and Expansion Properties from the Tower

The following contraction/expansion properties hold:

\[ (\tilde{A}1) \] (Backwards contraction and bounded distortion) There exists \( C > 0 \) which will be used throughout for this uniform constant, \( 0 < \alpha < 1 \) such that if \( y \in \gamma^\mu(x) \cap \Lambda \):

\[
d(T^jx, T^jy) \leq C\alpha^{s(x,y) - j}
\]
and
\[ \frac{1}{C} \leq \frac{\|D^u(T^jx)\|}{\|D^u(T^jy)\|} \leq C, \]
for all \( j = 0, \ldots, s(x,y), \) [19, P4(b)]. Here \( D^u \) denotes the derivative along unstable leaves \( \gamma^u \), and \( s(x,y) \) is the first time \( n \) for which \( F^n(x,0) \) and \( F^n(y,0) \) lie in different elements of \( P_\Delta \).

\( (\tilde{A}2) \) [19, P3] There exists \( \lambda_s < 1 \) such that for all \( \gamma^s \in \Gamma^s \) and every \( x, y \) in the same stable leaf, \( \gamma^s \),
\[ d(T^jx, T^jy) \leq C\lambda_s^j \]
where \( C > 0 \) is a uniform constant.

We define
\[ \lambda^s_{\text{max}} := \max_{x,y} \{ \lambda : d(Tx,Ty) \geq \lambda d(x,y), \ y \in W^s(x) \} \]
and
\[ \lambda^u_{\text{max}} := \max_{x,y} \{ \lambda : d(Tx,Ty) \geq \lambda d(x,y), \ y \in W^u(x) \} \]
as the least contraction and greatest expansion rates, respectively.

## 3 Gouzél’s Assumptions

In addition to these contraction and expansion properties, we adopt the assumption of Gouézel in [9] that there exists \( C > 0 \) such that for all \( l \) and \( k < R_l \), for all \( x, y \in \Delta_{k,l} \), we have
\[ d(x, y) \leq Cd(T^{R_l-k}x, T^{R_l-k}y). \]
Though not entirely equivalent to the assumption of L. S. Young in [18], it is true for all the systems with which we will work. Namely uniformly hyperbolic, Axiom A, and LSV-type maps.
Chapter 5

Sub-Actions for Young Towers

First, suppose \( T : X \to X \) is a nonuniformly expanding map, so that \( T : X \to X \) has a Young Tower but has no stable foliation, and \( X \) is a one dimensional manifold with compact closure and Euclidean metric \( d \), and \( \phi : X \to \mathbb{R} \) is a Lipschitz function. Suppose, also, that \( (F, \Delta) \) is a Young Tower for the system with function \( \phi : \Delta \to \mathbb{R} \) lifted to the tower in the usual way, by \( \tilde{\phi}(x, l) = \phi(T^l(x)) \).

1 The Main Theorem

Theorem 1.1 Suppose \( T : X \to X \) is a nonuniformly expanding system, where \( X \) is a one dimensional manifold with Euclidean metric \( d \), and has a Lipschitz observable \( \phi : X \to \mathbb{R} \). Suppose \( (F, \Delta) \) is a Young Tower for the system and \( \phi : \Delta \to \mathbb{R} \) is lifted to the tower by \( \tilde{\phi}(x, l) = \phi(T^l(x)) \), and we write \( \phi \) for \( \tilde{\phi} \) for simplicity. Then there exists a function \( \theta \) on \( \Delta \), which is bounded above on each level of the tower, such that \( \phi(p) \leq m(\phi, F) + \theta(Fp) - \theta(p) \), where \( m(\phi, F) = \sup \{ \int \phi d\nu : \nu \text{ is } F \text{ invariant} \} \). The function \( \theta \) is Hölder with respect to \( d \) on the base \( \Delta_0 \) of the Young Tower and
all iterates $F^k \Delta_0$ under $F$ (the Hölder exponent $\beta \in (0, 1)$ may be taken arbitrarily close to 1 uniformly in $k$, but the Hölder constant, $\text{Hold}_\beta(\theta)$, may increase with $k$).

2 Proof of the Main Theorem

Recall $F(x, l) = (x, l + 1)$ if $l < R(x) - 1$ and $F(x, R(x) - 1) = (T^{R(x)}x, 0)$.

Without loss of generality we may assume that $r(\phi, F) = \sup \{ \int \phi \, d\nu : \nu \text{ is } F \text{ invariant} \} = 0$. This is because we could equivalently define $\phi' = \phi - m(\phi, F)$ and prove our result for $\phi'$.

With $p = (x, l)$ we define a function $\theta : \Delta \to \mathbb{R}$, which will be a candidate for a sub-action on the tower, by

$$\theta(p) = \sup \left\{ \sum_{j=0}^{n-1} \phi F^j z, : n \geq 1, F^n z = p, z \in \Delta_0 \right\}$$

and to prove the conclusion of the main theorem, we will show that $\theta$ satisfies the following:

(a) $\theta$ is well-defined,

(b) $\phi(p) \leq \theta \circ F(p) - \theta(p)$ for all $p \in \Delta$, and

(c) $\theta$ is Hölder.

(a) The Function $\theta$ is Well-Defined

Proof

Fix $i, l \in \mathbb{N}$, and consider first the element $\Delta_{i,l} \subset \Delta$ of the tower. We show that the function

$$\theta(p) := \sup \left\{ \sum_{j=0}^{n-1} \phi \circ F^j z, : F^n z = p, z \in \Delta_0 \right\}$$
is well-defined for all \( p \in \Delta_{i,t} \).

Let \( p = (x, l) \in \Delta_{i,t} \) be given. We adopt the notation \( z_p = (x, 0) \in \Delta_{i,0} \) so that \( z_p \) is the "inverse image under \( F^m \)" of \( p \) in the base of the tower. Note that

\[
\sum_{j=0}^{l} \phi \circ F^j z_p \leq (l + 1) \max_{j \in \{0, 1, \ldots, l\}} \phi(x, j)
\]

where \( F^l z_p = p, z_p \in \Delta_{i,0} \). Then, since \( \phi \) is bounded, the sum above is uniformly bounded in \( p \) above and below on \( \Delta_{i,t} \).

Now, since we are taking a supremum we need only show that \( \theta(p) \) is bounded above on \( \Delta_{i,t} \) (in fact we will show that there is a bound above on \( \Delta_{i,t} \) which is uniform in \( l \)).

Define a sequence \( \{z_k\} \) in an "inverse branch" of \( p \) such that \( z_k \in \Delta_0, F^{n_k} z_k = z_{k-1} \), where \( n_k = R(z_k) \). Let us note that \( z_1 = z_p \) given before, so that \( F^l z_1 = p \) and \( n_1 = l \).

Define \( \tau_k = R(z_k) + R(z_{k-1}) + \ldots + R(z_2) \). Recall \( P_0 \) is the partition of \( \Delta_0 \) into \( \Delta_{i,0} \). Let \( Q(z_k) \) be the partition element of \( \bigvee F^{-\tau_k} P_0 \) of points which are not separated from \( z_k \) by the partition \( P_\Delta \) for \( \tau_k \) iterates of \( F \). It is important to note that we will assume \( Q(z_k) \) to be an interval. Note that by construction of the tower \( F^{\tau_k} Q(z_k) = \Delta_0 \) and in particular \( F^{\tau_k} Q(z_k) \supset Q(z_k) \). Hence, due to the assumption that \( Q(z_k) \) is an interval, there exists at least one periodic point in \( Q(z_k) \) of period \( \tau_k \). Call this periodic point \( q_k \) so that we have a sequence of points

\[
\{q_k\}_{k=1}^{\infty}
\]

such that \( F^{\tau_k} q_k = q_k \).

Now by the property of backwards contraction in \( (\tilde{A}1) \), we have

\[
\left| \sum_{j=0}^{\tau_k} \phi F^j z_k - \sum_{j=0}^{\tau_k} \phi F^j q_k \right| \leq \sum_{j=0}^{\tau_k} C \|\phi\|_{\text{Lip}} \alpha^{\tau_k-j}
\]
and, furthermore
\[ \sum_{j=0}^{\tau_k} \alpha^{\tau_k-j} \leq \frac{1}{1-\alpha} \]
since it is a geometric series. Combining these facts gives
\[ \left| \sum_{j=0}^{\tau_k} \phi F^j z_k - \sum_{j=0}^{\tau_k} \phi F^j q_k \right| \leq C \|\phi\|_{Lip} (1-\alpha)^{-1} \]

Now,
\[ \sum_{j=0}^{\tau_k+l-1} \phi F^j z_k = \sum_{j=0}^{\tau_k-1} \left( \phi F^j z_k - \phi F^j q_k \right) + \sum_{j=0}^{\tau_k-1} \phi F^j q_k + \sum_{j=\tau_k}^{\tau_k+l-1} \phi F^j z_k \]
\[ \leq \sum_{j=0}^{\tau_k-1} \phi F^j q_k + \sum_{j=\tau_k}^{\tau_k+l-1} \phi F^j z_k + \sum_{j=0}^{\tau_k} \left| \phi F^j z_k - \phi F^j q_k \right| \]
\[ \leq l \|\phi\|_{\infty} + \sum_{j=0}^{\tau_k} \left| \phi F^j z_k - \phi F^j q_k \right| \]
where we have used the following:

- Since \( q_k \) is a periodic point, the Dirac measure supported on the orbit of \( q_k \) is an invariant Borel probability measure, and as such, it yields an ergodic average
  \[ \frac{1}{\tau_k} \sum_{j=0}^{\tau_k-1} \phi \circ F^j (q_k) \leq m(\phi, F) \]
  and \( m(\phi, F) = 0 \) by assumption. Thus, the first sum is non-positive.

- Since \( \phi \) is bounded, and the second term has \( l \) terms, we may bound it by
  \[ l \|\phi\|_{\infty} \]

Now since we have shown above that
\[ \sum_{j=0}^{\tau_k-1} \left| \phi F^j z_k - \phi F^j q_k \right| \]
is bounded, we have that
\[ \sum_{j=0}^{n-l-1} \phi F_j z_k \]
is bounded. Since the inverse branch was chosen arbitrarily, this bound holds for each inverse branch. Hence, \( \theta \) is the supremum of a bounded set, and thus is well-defined as a function on \( \Delta \). Note that this bound is uniform in \( l \) over all \( \Delta_{i,l} \).

(b) The Function \( \theta \) Satisfies the Sub-cohomology Equation

Proof

We now show that \( \theta(p) \) satisfies
\[ \theta(p) + \phi(p) \leq \theta(Fp). \]

There are two cases to consider:

Case 1: \( p = (x, l) \) with \( l < R(x) - 1 \), and
Case 2: \( p = (x, R(x) - 1) \).

Suppose we are in case 1. Then
\[ \theta(Fp) = \sup \left\{ \sum_{j=0}^{n-1} \phi F_j z, \ : F^n z = F_p, z \in \Delta_0 \right\} \]

Note that
\[
\theta(Fp) = \sup \left\{ \sum_{j=0}^{n-1} \phi F_j y + \phi(p) \ : F^n y = p, y \in \Delta_0 \right\}
\]
\[ = \sup \left\{ \sum_{j=0}^{n-1} \phi F_j y, \ : F^n y = p, y \in \Delta_0 \right\} + \phi(p)
\]
\[ = \theta(p) + \phi(p), \]

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so that we actually have equality in the sub-cohomology equation.

Now suppose we are in case 2. Here we have

\[\theta(Fp) = \sup \left\{ \sum_{j=0}^{n-1} \phi F^j y, \, : \, F^n y = Fp, \, y \in \Delta_0 \right\} \]

\[\geq \sup \left\{ \sum_{j=0}^{n-1} \phi F^j y, \, : \, F^n y = p, \, y \in \Delta_0 \right\} + \phi(p).\]

We have inequality rather than equality in this case since in the later sum we are restricting to inverse sequences, \( \{y_n\} \), such that the last return to \( \Delta_0 \) lies in the same partition element of \( \Delta_0 \) as \( (x, 0) \). Thus, in case 2 we have

\[\theta(Fp) \geq \theta(p) + \phi(p).\]

Note that we have equality \( \theta(Fp) = \theta(p) + \phi(p) \) unless \( p = (x, R(x) - 1) \), that is, unless we are at the highest "level" of the tower and will be returning to the base on the next iteration of \( F \).

(c) The Function \( \theta \) is Hölder with Respect to the Euclidean Metric: Two Cases

Proof

Choose \( r^* \in \mathbb{N} \) such that

\[\log(r^*) + \frac{r^*}{2} \log(\alpha) < 0\]

so that we have

\[r \alpha^{\frac{r^*}{2}} < 1\]
for all $r > r^*$ and suppose $p, q \in \Lambda$, not necessarily in the same element of the partition, such that $d(p, q) = \alpha^s$ where $\gamma \leq s < \gamma + 2$, for an even natural number $\gamma > r^*$.

Now choose $p_j, q_j \in \Lambda_j$ such that $F^{R(j)}p_j = p$ and $F^{R(j)}q_j = q$. In other words, $p_j$ and $q_j$ are in the base of the tower (in the same partition element) and map onto $p$ and $q$, respectively, (also in the base, but not necessarily in the same partition element) after just one trip up the tower. For the purposes of this proof, we will consider two cases:

- Case I: $R(j) > \frac{\gamma}{2}$
- Case II: $R(j) \leq \frac{\gamma}{2}$

and we will write $R_j$ for $R(j)$.

Case I: Suppose $R_j > \frac{\gamma}{2}$.

Define $M = R_j - \frac{\gamma}{2}$ so that $R_j - M = \frac{\gamma}{2}$. Then we have

$$\sum_{k=0}^{R_j-1} |\phi(F^k p_j) - \phi(F^k q_j)| \leq \sum_{k=0}^{R_j-1} \|\phi\|_{Lip} d(F^k p_j, F^k q_j)$$

$$= \sum_{k=0}^{M} \|\phi\|_{Lip} d(F^k p_j, F^k q_j) + \sum_{k=M+1}^{R_j-1} \|\phi\|_{Lip} d(F^k p_j, F^k q_j)$$

$$\leq \sum_{k=0}^{M} C_1 \alpha^{R_j-k} + \sum_{k=M+1}^{R_j-1} C_2 d(p, q)$$

where we have used the property of backwards contraction to simplify the first term, and we have used the fact that $d(F^k p_j, F^k q_j)$ remains constant as $p_j$ and $q_j$ are traveling up the tower under iteration of $F$ to simplify the second term and combine the two constants to create a new constant $C_2$. 37
Now we simplify this expression further by using the fact that the first summation is part of a geometric series. This will give us

\[
\sum_{k=0}^{M} C_1 \alpha^{R_j-k} + \sum_{k=M+1}^{R_j-1} C_2 d(p, q) \leq \frac{C_1 \alpha^{R_j-M}}{1 - \alpha} + C_2 (R_j - M - 1) \alpha^\gamma
\]

\[
= \frac{C_1}{1 - \alpha} \alpha^\frac{\gamma}{2} + C_2 \frac{\gamma}{2} \alpha^\gamma
\]

\[
\leq \max \left\{ \frac{C_1}{1 - \alpha} C_2 \left( \alpha^\frac{\gamma}{2} + \frac{\gamma}{2} \alpha^\gamma \right) \right\}
\]

\[
\leq \max \left\{ \frac{C_1}{1 - \alpha} C_2 \left[ \alpha^\frac{\gamma}{2} \left( 1 + \gamma \alpha^\frac{\gamma}{2} \right) \right] \right\}
\]

Now, since $\gamma > r^*$ we have, by our choice of $r^*$, that $\gamma \alpha^\frac{\gamma}{2} \leq 1$. By redefining new constants, and using the fact that $d(p, q) = \alpha^s \leq \alpha^\gamma$, we obtain the following inequality from our simplification:

\[
\sum_{k=0}^{R_j-1} |\phi (F^k p_j) - \phi (F^k q_j)| \leq \max \left\{ \frac{C_1}{1 - \alpha} C_2 \right\} \left( 2 \alpha^\frac{\gamma}{2} \right)
\]

\[
= C_3 \alpha^\frac{\gamma}{2}
\]

\[
\leq C d(p, q)^\frac{1}{2}
\]

Therefore, in Case I, we have found that $\theta$ is Hölder with exponent $1/2$ on the base of the tower.

Case II: Suppose $R_j \leq \gamma/2$. Then

\[
\sum_{k=0}^{R_j-1} |\phi (F^k p_j) - \phi (F^k q_j)| \leq \sum_{k=0}^{R_j-1} \|\phi\|_{Lip} d(F^k p_j, F^k q_j)
\]

\[
\leq \sum_{k=0}^{R_j-1} D_1 d(p, q)
\]

where we use, as before, the fact that $d(F^k p_j, F^k q_j)$ is unchanging as $k$ goes from 0 to $R_j - 1$. Again we use the fact that $d(p, q) = \alpha^s$ and $s \geq \gamma$ to further simplify and

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obtain

\[
\sum_{k=0}^{R_j-1} \left| \phi(F^k p_j) - \phi(F^k q_j) \right| \leq \sum_{k=0}^{R_j-1} D_1 d(p, q) \\
= (R_j D_1) \alpha^s \\
\leq D \alpha^s \frac{3}{2} \\
\leq D d(p, q)^{\frac{3}{2}}.
\]

Therefore, in Case II, we see that \( \theta \) is Hölder with exponent 1/2 on the base of the tower by making the note that \( \Lambda \subset X \), and the closure of \( X \) is compact, hence there exists

\[
\{B_i\}_{i=1}^{N}
\]

balls of radius \( \alpha^{r/2} \) which cover \( \Lambda \); thus, the assumption that \( d(p, q) < \alpha^s \) may be extended for all \( p, q \in \Lambda \). Hence, we can combine the two cases to conclude that \( \theta \) is Hölder on \( \Lambda \).

Also, observe that we have chosen \( r^* \) and \( M \) so that \( \theta \) has Hölder exponent 1/2. This was arbitrary, and the proof will function just as easily to show that \( \theta \) has Hölder exponent \( \beta \in \mathbb{R} \), as long as \( \beta < 1 \). Thus, \( \theta \) is Hölder on all of \( \Lambda_0 \) with any exponent \( \beta < 1 \).

To complete the proof, note that each level of the tower acts as a “copy” of its representatives in the base. Thus, the proof to show that \( \theta \) is Hölder with exponent \( \beta \) on each level of the tower will go through just as above, perhaps with an increase in the Hölder constant, but with no change to \( \beta \).
3 Further Remarks

(a) A Minimal Non-Negative Sub-Action

We note that in the proof we may modify the definition of $\theta$ so that $\theta$ is non-negative. In fact, if we define

$$\theta(p) = \max \left\{ 0, \sup \left\{ \sum_{j=0}^{n-1} \phi F^j z, \quad n \geq 1, F^nz = p, z \in \Delta_0 \right\} \right\}$$

then it is straightforward to check that all properties of $\theta$ are the same, but now $\theta$ is non-negative. We will show that $\theta$ is the minimal non-negative function which satisfies the sub-cohomology equation.

Suppose that $\theta$ is the non-negative sub-action for $(\phi, F)$ as defined above, so that $\theta(Fp) \geq \theta(p) + \phi(p)$, and suppose that $V(p)$ is another non-negative function such that $V(Fp) \geq V(p) + \phi(p)$. Take $p \in \Delta_{t,t}$ and an arbitrary branch $z_{n_k} \in \Delta_0$ such that $F^{n_k}z_{n_k} = p$, then

$$V(p) \geq \sum_{j=0}^{n_k-1} \phi (F^j z_{n_k}) + V(z_{n_k})$$

Since $V(z_{n_k}) \geq 0$, we see

$$V(p) \geq \sup_{z_{n_k}} \left\{ \sum_{j=0}^{n_k-1} \phi (F^j z_{n_k}) \right\} = \theta(p).$$

Hence, $\theta$ is minimal amongst non-negative functions $V$ with the property that $V(Fp) \geq V(p) + \phi(p)$.
(b) **Remark on Coboundaries**

Note that if $\phi$ is a coboundary, that is if there exists a measurable function $\psi$ which satisfies the cohomology equation

$$\phi(p) = \psi(Fp) - \psi(p)$$

then $\theta = \psi + C$ for $\mu_\Delta$ almost every $p \in \Delta$ for some constant $C \in \mathbb{R}$.

To see this note that

$$(\theta - \psi)(p) \leq (\theta - \psi)(Fp)$$

and hence, by the ergodicity of $F$, the set $\{\theta - \psi \geq K\}$ has measure zero or one for all $K \in \mathbb{R}$. Therefore, there exists a $C \in \mathbb{R}$ such that

$$\{p \in \Delta | \theta(p) = \psi(p) + C\}$$

has measure one.
Chapter 6

Applications

We now ask if the sub-action $\theta$ on the tower may be extended to a sub-action on the manifold, $X$, and, if so, which degree of regularity may be expected. Recall that there is a natural projection, $\pi : \Delta \to X$, given by $\pi(x, t) = T^t x$, which intertwines $F$ and $T$ by the relation $T \circ \pi = \pi \circ F$. The image $\pi(\Delta)$ is a $T$-invariant subset of $X$, but may or may not be closed.

In general $m(\phi, T) \geq m(\phi, F)$; however, in the case of Axiom A maps, Anosov diffeomorphisms, and the Liverani-Saussol Vaienti map discussed below, we have $m(\phi, T) = m(\phi, F)$ since the measures supported on periodic orbits which are present in the tower are dense in the weak-star topology and the supremum is realized as the supremum over periodic orbits.

We will assume, for the statement of the next result, that $m(\phi, T) = m(\phi, F)$ as the maps we plan to work with in this chapter have this property. From here we wish to construct a version of $\theta$ which is a Hölder function on $X$ and which satisfies the sub-cohomology equation. We have some successes and some troubles, which we will describe in detail in this chapter.
1 The Extension of $\theta$ to $X$

Note that $\theta$ is Hölder; thus, the extension

$$\tilde{\theta}(z) := \sup \{\theta(x, l) : \pi(x, l) = z\}$$

will be measurable if it is finite $\mu_\Delta$ almost everywhere. Furthermore, $\tilde{\theta}$ is a sub-action for $X$ as follows,

$$\tilde{\theta}(Tz) = \sup \{\theta(x, l) : T^l x = Tz\}$$

$$\geq \sup \{\theta(x, l) : T^{l-1} x = z\}$$

$$\geq \sup \left\{\sup \left\{\sum_{k=0}^{l-2} F^k x : F^l x = z\right\} : T^{l-1} x = z\right\} + \phi(z)$$

$$\geq \tilde{\theta}(z) + \phi(z).$$

If the projection is infinite-to-one almost everywhere, as is the case for certain logistic maps, it is difficult to show any regularity. The extra structure in the case of the LSV map below, as well as Axiom A maps, allows us to circumvent this problem. We are able to establish stronger regularity for $\tilde{\theta}$ if the projection is finite to one since it is easy to show that the supremum of a finite collection of Hölder maps (with uniform Hölder exponent) is Hölder. In particular if the system is uniformly hyperbolic, Axiom A or intermittent, $\tilde{\theta}$ will be Hölder since the Young Tower may be constructed to have a bounded height (i.e. we may require that the return function $R \leq M$ for some $M \in \mathbb{N}$) and $\pi(\Delta)$ is an open dense subset of $X$. This gives an interesting corollary.

**Corollary:** Assume $m(\phi, T) = m(\phi, F)$, the map $T : X \to X$ is uniformly hyperbolic, and the observable $\phi : X \to \mathbb{R}$ is Lipschitz. Then there is a Hölder sub-action function, $\theta : X \to \mathbb{R}$. 

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To see that $\tilde{\theta}$ is Hölder, it is important to note that points which are the images of boundaries of $\Delta_i, f$ will also be continuity points of $\tilde{\theta}$ by the continuity of the map $T$.

Although, in general, the projection may not be Hölder, and the best we might hope for is measurability, in some nonuniformly hyperbolic systems it is possible to extend $\theta$ to a Hölder sub-action on $X$.

2 Theorem: The Hölder Sub-Action for the LSV Map

The following example strengthens the result obtained by Souza in [17] for a Lipschitz observable in that we do not require that $\phi$ be monotone in a small neighborhood of the origin.

**Theorem 2.1** If $f : [0,1] \rightarrow [0,1]$ is the Liverani-Saussol-Vaienti map

$$f(x) = \begin{cases} x (1 + 2^\gamma x^\gamma), & x \in [0, \frac{1}{2}) \\ 2x - 1, & x \in [\frac{1}{2}, 1] \end{cases}$$

where $\gamma \in (0, 1)$, and $\phi : [0,1] \rightarrow \mathbb{R}$ is Lipschitz, then there is a function $\tilde{\theta} : [0,1] \rightarrow \mathbb{R}$ such that

$$\phi(p) \leq m(\phi, F) + \tilde{\theta}(Fp) - \tilde{\theta}(p).$$

Furthermore, $\tilde{\theta}$ is Hölder with exponent $\beta \leq 1 - \gamma$.

We refer to Figure 6.1 on page 45 for an example of the type of LSV map which we will consider.
Figure 6.1: The Liverani-Saussol-Vaienti Map when $\gamma = 0.9$
Remark: The sub-action obtained for this map assumes that $\phi$ is Lipschitz. As such, this strengthens the result obtained in [17] only in the case of Lipschitz observable. It would be a simple modification of our proof to extend this to the case where $\phi$ is Hölder with sufficiently high Hölder exponent as well. We also note that there are further results for Hölder observables in the paper by I. Morris, [16], which we have discussed in Chapter 1.

3 The Young Tower for the LSV Map

Consider the LSV map $f : [0, 1] \to [0, 1]$. We construct a Young Tower for the map as follows. Define $\{w_n\}_{n \in \mathbb{N}} \subset [0, 1/2]$ recursively as follows:

\[
\begin{align*}
    w_0 & := \frac{1}{2}, \\
    w_n & := f^{-1} w_{n-1} \cap \left[0, \frac{1}{2}\right].
\end{align*}
\]

See Figure 6.2 on page 47 for the placement of the sequence.

Define $\alpha_n := [w_{n+1}, w_n)$. See Figure 6.3 on page 48 for the placement of these intervals.

We choose $\Lambda = [1/2, 1]$ as the base of the tower. Note that $f(\Lambda) = [0, 1]$ and $f([3/4, 1]) = \Lambda$. Hence, for each $i \in \mathbb{N} \cup \{0\}$ there is an interval in $\Lambda$ which maps onto $\alpha_i$. Furthermore, for $i \neq j$, these intervals are disjoint since $f$ maps $\Lambda$ one-to-one onto $[0, 1]$. Define $\Lambda_i$ so that $f(\Lambda_i) = \alpha_i$ for all $i \in \mathbb{N}$. Then

$$\Lambda = \bigcup_{i=0}^{\infty} \Lambda_i.$$ 

See Figure 6.4 on page 49 for the partitioning of the base.
Figure 6.2: The sequence \( \{w_n\} \)
Figure 6.3: The collection \( \{a_n\} \) of intervals
Figure 6.4: The partition \( \{A_n\} \) of the base
Now, to calculate the return time function, we simply observe that
\[
\begin{align*}
    f_1(\Lambda_i) &= \alpha_i \\
    f_2(\Lambda_i) &= \alpha_{i-1} \\
    &\quad \vdots \\
    f_{i+1}(\Lambda_i) &= \alpha_0 \\
    f_{i+2}(\Lambda_i) &= \left[\frac{1}{2}, 1\right].
\end{align*}
\]

Then, \(R_i = i + 1\) on \(\Lambda_i\) (i.e., for all \(x \in \Lambda_i\), \(R(x) = i + 1\)). Also, note that
\(f([3/4, 1]) = [1/2, 1]\) so that
\(R_{[3/4, 1]} = 0\).

Now, for each \(i \in \mathbb{N} \cup \{0\}\), and for each \(j \in \{0, 1, \ldots, i + 1\}\), we define the set
\[
\Delta_{i,j} = \{(x, j) \mid x \in \Lambda_i\}
\]
which will create the partition of the tower. Here \(i\) gives the “horizontal location,”
and \(j\) gives the “vertical location,” or “level,” of the partition element \(\Delta_{i,j}\). Finally,
we define the Young Tower for \([0, 1], f\) by
\[
\Delta = \bigcup_{i=0}^{\infty} \left[\bigcup_{j=0}^{i+1} \Delta_{i,j}\right]
\]
and we think of the tower as an object which looks like the collection of disjoint
intervals in which each base element is iterated up onto a copy of itself in the next
level of the tower until its own top level, where it expands to cover the base. We
reference Figure 6.5 on page 51 to see this partitioning.
Figure 6.5: The Young Tower partitioning
4 Preliminaries for the Construction of the Sub-Action $\tilde{\theta}$

Now, the previous theorem gives a sub-action $\theta : \Delta \to \mathbb{R}$ so that $\theta|_{\Lambda}$ is Hölder. We wish to find a Hölder sub-action, $\tilde{\theta} : [0,1] \to \mathbb{R}$.

(a) The Sequence of Points $\{z_j\}$

Fix $i \in \mathbb{N}$ and choose $x \in \alpha_i$. Note that for each natural number $j \in \mathbb{N}$ there is a unique point $z_j \in \Lambda_{i+j}$ such that $f^j(z_j) = x$. See Figure 6.6 on page 53 for the location of the point $z_j$ in the unit interval.

The properties of the point $z$ which make it unique for each $j$ are

1. it is in a particular partition element of the base, and
2. it maps onto $x$ before it returns to $\Lambda_{i+j}$

Thus, there is a sequence

$$\{z_j\}_{j=0}^{\infty}$$

of points converging to $1/2$ which all map to $x$ under sufficient iteration of $f$.

(b) The Sequence of Functions $\{\theta_j\}$

We define a sequence $\{\theta_j\}_{j=1}^{\infty}$ by

$$\theta_j(x) = \theta(z_j) + \sum_{k=1}^{j-1} \phi(f^k z_j)$$

where $\theta$ is the sub-action given in section 2. As our sequence $\{z_j\}$ is defined only for points $x \in [0,1/2)$, we define $\theta_j(x) = \theta(x)$ for all $j$ whenever $x \in \Lambda$. We will use this
Figure 6.6: The selection of $z_j$ for the definition of $\theta_j$
sequence of functions to obtain a sub-action, $\tilde{\theta}$, on the whole interval $[0,1]$ which we will define later.

(c) $\theta_j$ is Hölder on $[0,1]$ Uniformly in $j$

Proof

Before we construct $\tilde{\theta}$ we will show that $\theta_j$ is Hölder. First we show regularity on $\alpha_i$ for fixed $i$. Fix $i, j \in \mathbb{N}$, and $x, y \in \alpha_i$. Then choose $z_j, w_j \in \Lambda_{i+j}$ as before such that $f^j z = x$, $f^j w = y$ and write simply $z, w$ for simplicity. Now, there exist unique sequences $\{x_n\}, \{y_n\} \subset \Lambda = [0,1/2]$ such that

- $x_n, y_n \in \Lambda_n$, for all $n$,
- $f(x_i) = x$, $f(y_i) = y$,
- $f^{n+1}(x_n) = f^{i+1}(x_i) = x_0 \in \alpha_0$, $f^{n+1}(y_n) = f^{i+1}(y_i) = y_0 \in \alpha_0$.

Now for each $n \in \mathbb{N}$, choose $\gamma_n \in (0,1)$ so that $l(\alpha_n) = \gamma_n l(\alpha_0)$ where $l$ denotes length in the Euclidean metric. Since

$$\sum_{n=1}^{\infty} l(\alpha_n) = \frac{1}{2}$$

we find that

$$\sum_{n=1}^{\infty} \gamma_n = \sum_{n=1}^{\infty} \frac{l(\alpha_n)}{l(\alpha_0)} = \frac{1}{l(\alpha_0)} \sum_{n=1}^{\infty} l(\alpha_n) < +\infty.$$
By bounded distortion there exists $M$ such that

\[
\frac{1}{M} \leq \frac{|x_n - y_n|}{|x_0 - y_0|} l(\alpha_n) \leq M.
\]

Therefore, we have

\[
\frac{1}{M} \leq \frac{|x_n - y_n|}{|x_0 - y_0|} \gamma_n \leq M
\]

and

\[
|x_n - y_n| \leq M \gamma_n |x_0 - y_0|.
\]

Next, we use the fact that $\theta$ is Hölder and the fact that

\[
|\theta_j(x) - \theta_j(y)| = \left| (\theta(z) - \theta(w)) + \left( \sum_{k=1}^{j-1} \phi(f^k z) - \sum_{k=1}^{j-1} \phi(f^k w) \right) \right|
\]

and conclude that it is sufficient to show that

\[
\sum_{k=1}^{j-1} \phi \circ f
\]

is Hölder. This calculation follows:
\[ \left| \sum_{k=1}^{j-1} (\phi (f^k z) - \phi (f^k w)) \right| \leq \sum_{k=1}^{j-1} \left| \phi (f^k z) - \phi (f^k w) \right| \]
\[ \leq \| \phi \|_{Lip} \sum_{k=1}^{j-1} |f^k(z) - f^k(w)| \]
\[ = \| \phi \|_{Lip} \sum_{k=1}^{j-1} |x_{i+j-k} - y_{i+j-k}| \]
\[ \leq \| \phi \|_{Lip} |x_i - y_i| M^2 \sum_{k=1}^{j-1} \gamma_{i+j-k} \]
\[ \leq \| \phi \|_{Lip} |x - y|^\beta M^2 \frac{1}{\gamma_i} \sum_{k=1}^{j-1} \gamma_{i+j-k} \]
\[ \leq \| \phi \|_{Lip} |x_0 - y_0|^{\beta-1} |x - y|\beta M^2 \frac{|0 - w_{i+1}|}{|w_{i+1} - w_i|\beta} \]
\[ \leq \| \phi \|_{Lip} |x_0 - y_0|^{\beta-1} |x - y|\beta M^2 \frac{|w_{i+1}|}{|w_{i+1}|(1+\gamma)\beta} \]
\[ \leq M^2 \| \phi \|_{Lip} |x_0 - y_0|^{\beta-1} |x - y|\beta \]

Hence, provided that \( \beta \leq 1/(1 + \gamma) \), the function \( \theta_j \) is Hölder on \([0,1]\) with Hölder exponent \( \beta \leq 1/(1 + \gamma) \).

5 The Sub-Action \( \tilde{\theta} \)

Now that we have the preliminary work necessary, we are ready to define the function \( \tilde{\theta} : [0,1] \rightarrow \mathbb{R} \) by

\[ \tilde{\theta}(x) = \sup_{j \in \mathbb{N}} \theta_j(x). \]
Our ultimate goal is to show that the function $\tilde{\theta}$ is uniformly Hölder on $[0, 1]$. We will have several intermediate steps.

(a) **Continuity at the Point** $x = 1/2$

It is important to note, first of all, that $\tilde{\theta} = \sup_j \theta_j$ is continuous at $x = 1/2$. This is not at all obvious since the function $f$ is discontinuous at the point $x = 1/2$.

**Proof**

To see this, fix $\epsilon > 0$ and choose $z \in \Lambda, n \in \mathbb{N}$. Then we have that $f^nz = 1/2$ and also that

$$\left| \tilde{\theta} \left( \frac{1}{2} \right) - \sum_{j=0}^{n-1} \phi \left( f^j z \right) \right| < \frac{\epsilon}{6}$$

and choose $\delta > 0$ such that we have for all $y \in (z, z + \delta)$ we have $f^ny - 1/2 < \delta$ and

$$\sum_{k=0}^{n-1} \left| \phi \left( f^k z \right) - \phi \left( f^k y \right) \right| < \frac{\epsilon}{6}$$

Note here that if $w \in (1/2, 1/2 + \delta)$ then there is a unique $\tilde{w} \in (z, z + \delta)$ such that $f^n(\tilde{w}) = w$.

Lastly, for all $w \in (1/2, 1/2 + \delta)$ choose $w_0 \in \Lambda$ and $m$ the smallest natural number such that $f^m(w_0) = w$ and

$$\left| \tilde{\theta}(w) - \sum_{l=0}^{m-1} \phi \left( f^l w_0 \right) \right| < \frac{\epsilon}{6}$$

We consider the special case where $w_0 = \tilde{w}$, i.e., when the value of $\tilde{\theta}(w)$ is approximated within $\epsilon/6$ by one trip up the tower. Figure 6.7 on page 58 shows the case where $n = 5$ with the orbits of $z$ denoted by $\bullet$ and the orbit of $\tilde{w}$ denoted by $\times$.

Then for all $w \in (1/2, 1/2 + \delta)$ we have
Figure 6.7: The orbits of \( z \) and \( \hat{w} \)
so that in this case $\tilde{\vartheta}$ is continuous at $x = 1/2$.

Now for the case where two trips up the tower are required to approximate $\tilde{\vartheta}(w)$ we consider a refinement of the partitioning of the base. Note that there is an interval inside the partition element, $\Lambda_k$, containing $\tilde{w}$ which maps onto $\Lambda_k$ after the first trip up the tower. Then, the left endpoint of this interval, call it $\tilde{z}$, maps onto $z$ after one trip up the tower. Then, by backward contraction, we must have

$$\sum_{k=0}^{n-1} |\phi(f^k \tilde{z}) - \phi(f^k z)| < \frac{\epsilon}{3}.$$  

Then we have

$$\left| \tilde{\vartheta} \left( \frac{1}{2} \right) - \tilde{\vartheta}(w) \right| \leq \left| \tilde{\vartheta} \left( \frac{1}{2} \right) - \sum_{j=0}^{n-1} \phi(f^j z) \right| + \left| \sum_{j=0}^{n-1} \phi(f^j z) - \sum_{k=0}^{n-1} \phi(f^k \tilde{z}) \right|$$

$$+ \left| \sum_{k=0}^{n-1} \phi(f^k \tilde{z}) - \tilde{\vartheta}(w) \right| < \epsilon.$$

Note that this proof could be modified for $w_0$ which requires any finite number of trips up the tower to approximate $\tilde{\vartheta}(w)$.

Thus, $\tilde{\vartheta}$ is continuous at the point $x = 1/2$. 

\[\Box\]
(b) Regularity Near the Origin

Using our results for the function $\theta_j$, we may now conclude that $\tilde{\theta}$ is uniformly Hölder on $\alpha$, with exponent $\beta$ on $[0, 1/2]$, and with any exponent less than one on $[1/2, 1]$. However, we need to show that $\tilde{\theta}$ is Hölder near the point $x = 0$.

**Proof**

We let $\tilde{\theta} : [0, 1] \to \mathbb{R}$ be given by $\tilde{\theta}(x) = \sup \theta_j(x)$ as before, and we recall the sequence

$$\{w_n\}_{n \in \mathbb{N}}$$

in $[0, 1/2]$ given by $w_0 := 1/2, w_n := f^{-1}w_{n-1} \cap [0, 1/2]$ . We show in a neighborhood of zero, the function $\tilde{\theta}$ is Hölder using this sequence. Fix $a \in \mathbb{N}$, and note that $w_n \to 0$ as $n \to \infty$. Then we have

$$|\tilde{\theta}(w_{n+a}) - \tilde{\theta}(w_n)| \leq \sum_{j=n}^{n+a} C |w_{j+1} - w_j|^\beta$$

$$\leq \sum_{j=1}^{\infty} C \left(j^{-1-\frac{1}{\gamma}}\right)^{\frac{1}{1+\gamma}}$$

$$= n^{-\frac{1}{1+\gamma}} \left(-\frac{1}{\gamma+1} + 1\right)$$

$$= n^{-\frac{1-\gamma}{\gamma}}$$

$$\leq d(w_n, 0)^{1-\gamma}.$$ 

Therefore, we conclude that in a neighborhood of zero, $\tilde{\theta}$ is Hölder with exponent $1 - \gamma < 1/(1 + \gamma)$. Thus, $\tilde{\theta}$ is Hölder on $[0, 1]$ with exponent $\beta \leq 1 - \gamma$. 

\[\blacksquare\]
(c) The Sub-Cohomology Equation

For every point $p \in [0, 1/2]$ we have that $\tilde{\theta}(p) = \sup\{\theta(x, l) : \pi(x, l) = p\}$, so that $\tilde{\theta}$ satisfies the sub-cohomology equation, and is indeed a sub-action on $[0, 1]$. The reader may observe that we have excluded the point $x = 0$ in our construction. This will not cause a problem for us since, in the case of the Liverani-Saussol-Vaienti map, the Dirac measure supported on the fixed point $x = 0$ is in the closure of the set of measures supported on periodic orbits. Thus, $m(\phi, F) = m(\phi, f)$.

6 Generalization of the Proof

Remark 6.1 We may generalize our result for the LSV map by considering the nonuniformly hyperbolic map for which we assume the following:

- $f$ is a piecewise smooth two to one map, with discontinuity at $x_0 \in (0, 1)$,
- the limits at $x_0$ satisfy
  \[ \lim_{x \to x_0^+} f(x) = 1 \]
  and
  \[ \lim_{x \to x_0^-} f(x) = 0, \]
- $f'(x) > 1$ for all $0 < x \leq 1, x \neq x_0$, and $f'(0) = 1$, and
- the behavior of $f$ near zero is the same as $x + x^{1+\gamma}$, i.e.
  \[ \lim_{x \to 0^+} \frac{f(x)}{x + x^{1+\gamma}} = 1. \]
For this dynamical system, a proof similar to the one presented in this chapter may be constructed using the critical assumption that the behavior of the system near the origin is the same as that of the map $x + x^{1+\gamma}$.

In this more general case, we modify the construction of the proof by first defining a new base for the tower. The sequence of points

$$\{w_n\}_{n=0}^{\infty}$$

and the collection

$$\{\alpha_n\}_{n=0}^{\infty}$$

are given by

$$w_0 := x_0,$$

$$w_n := f^{-1}w_{n-1} \cap [0, x_0], \text{ for all } n \in \mathbb{N}, \text{ and}$$

$$\alpha_n := [w_{n+1}, w_n].$$

Then, we define a base for the tower,

$$\Lambda = \{\Lambda_i\}_{i=1}^{\infty} \cup \left[ f^{-1}(x_0) \cap [x_0, 1], 1 \right]$$
so that for all $i \in \mathbb{N}$ we have

\[
\begin{align*}
f(A_i) &= \alpha_i \\
f^2(A_i) &= \alpha_{i-1} \\
& \quad \vdots \\
f^i(A_i) &= \alpha_1 \\
f^{i+1}(A_i) &= \alpha_0 \\
f^{i+2}(A_i) &= [x_0, 1].
\end{align*}
\]

Now the tower is given in the usual way:

\[
\Delta = \bigcup_{i=1}^{\infty} \left[ [R|_{A_i}] \right]
\]

where $\Delta_{i,j} = \{(x,j) | x \in A_i\}$.

Using these modifications, and the fact that the behavior near the origin is as $x + x^{1+\gamma}$, the remainder of the proof will go through without problems.
Chapter 7

Stable Foliation

Now suppose that $T : X \rightarrow X$ is a $C^{1+\epsilon}$ diffeomorphism on the compact manifold $X$ with Euclidean metric $d$ where $\Lambda$ has hyperbolic product structure. In this case, the main theorem may be extended to include the stable foliation according to the following proposition.

1 Theorem: Sub-Actions for Young Towers of Nonuniformly Hyperbolic Systems with Stable Foliations

Theorem 1.1 If $T : X \rightarrow X$ is a $C^{1+\epsilon}$ diffeomorphism, $X$ a compact manifold with Euclidean metric $d$, and $(F, \Delta)$ is a Young Tower for the system, with Lipschitz observable $\phi$, then there exists a function $\tilde{\theta}$ on $\Delta$, which is bounded above on each level of the tower, such that $\phi(p) \leq m(\phi, F) + \tilde{\theta}(Fp) - \tilde{\theta}(p)$. The function $\tilde{\theta}$ is Hölder on the base $\Delta_0$ of the Young Tower, and on $F^j \Delta_0$ for all $j \in \mathbb{N}$. The Hölder exponent
may be taken as $\gamma/(1 + \gamma)$, where

$$\gamma = \frac{\log \lambda_{\text{max}}^s}{\log \lambda_{\text{max}}^u}.$$ 

2 The Proof Modified for Stable Foliation

Proof

We will begin by defining a new candidate for our sub-action. For each $p = (x, l) \in \Delta$ we let $(\overline{x}, l)$ be the unique point such that the corresponding base point $(\overline{x}, 0)$ lies in the distinguished leaf $W^u \subset \Delta_0$ with $\overline{x} \in W^s(x)$. More precisely, if $(x, l) \in \Delta$ we consider its representative $(x, 0)$ in $\Delta_0$ and the point $(\overline{x}, 0) \in W^u$ such that $x \in W^s(\overline{x})$. We denote $(\overline{x}, l)$ by $\bar{p}$.

Define $\psi : \Delta \to \mathbb{R}$ by

$$\psi(p) = \sum_{j \geq 0} [\phi \circ F^j(\bar{p}) - \phi \circ F^j(p)],$$

and $\kappa : \Delta \to \mathbb{R}$ by

$$\kappa(p) = \sup \left\{ \sum_{j=0}^{n-1} \phi F^j z, : F^n z = \bar{p}, z \in \Delta_0 \right\}.$$ 

Finally, we define $\tilde{\theta} : \Delta \to \mathbb{R}$ by

$$\tilde{\theta}(p) = \kappa(p) + \psi(p).$$

Note that

$$\overline{FP} = (x, l + 1) = (\overline{x}, l + 1) = F\bar{p}.$$ 

Thus, a similar technique to the one used in the proof of Theorem 2.1 shows that $\tilde{\theta}$ satisfies the sub-action equation.
Also, $\kappa$ can be shown, using a similar technique, to be well-defined, bounded, and Hölder with exponent less than 1.

We also have that $\psi$ is well-defined and bounded because of uniform contraction on stable leaves (Condition A2). It remains for us to show that the function $\psi$ is Hölder.

To see that $\psi$ is Hölder as a function of $p$ with exponent $\gamma/(1 + \gamma)$, where

$$\gamma = -\frac{\log \lambda_{\text{max}}^u}{\log \lambda_{\text{max}}},$$

suppose $d(p, q) = (\lambda_{\text{max}}^u)^{-\beta N}$ where $\beta > 1$ is to be determined later. We then have the following estimate

$$|\psi(p) - \psi(q)| \leq \left| \sum_{j=1}^N (\phi F^j(p) - \phi F^j(q)) \right| + \sum_{j=1}^N \left( \phi F^j(p) - \phi F^j(q) \right)$$

$$+ \sum_{j>N} \left( \phi F^j(p) - \phi F^j(q) \right) + \sum_{j>N} \left( \phi F^j(p) - \phi F^j(q) \right).$$

Furthermore,

$$\left| \sum_{j=1}^N (\phi F^j(p) - \phi F^j(q)) \right| \leq C (\lambda_{\text{max}}^u)^N d(p, q)$$

$$\leq C (\lambda_{\text{max}}^u)^{(1-\beta)N}$$

and by the local product structure we obtain

$$\left| \sum_{j=1}^N (\phi F^j(p) - \phi F^j(q)) \right| \leq C (\lambda_{\text{max}}^u)^N d(p, q)$$

$$\leq C (\lambda_{\text{max}}^u)^{(1-\beta)N}.$$
and

\[ \left| \sum_{j > N} (\phi F^j (q) - \phi F^j (\bar{q})) \right| \leq C (\lambda_{\text{max}}^u)^N . \]

Writing

\[ (\lambda_{\text{max}}^u)^{(1-\beta)N} = d(p, q)^{\frac{\beta-1}{\beta}} \]

and

\[ (\lambda_{\text{max}}^c)^N = (\lambda_{\text{max}}^u)^{-\gamma N} = d(p, q)^{\bar{\beta}} \]

we see that the four terms above are bounded by \( Cd(p, q)^{r_\beta} \) where

\[ r_\beta = \min \left\{ 1 - \frac{1}{\beta}, \frac{\gamma}{\bar{\beta}} \right\} . \]

Optimizing \( r_\beta \) over \( \beta > 1 \), we obtain an upper bound of \( r_\beta = \gamma / (\gamma + 1) \) for \( \beta = \gamma + 1 \). Thus, \( \tilde{\theta} \) gives us the desired sub-action in the case where \( X \) has a stable foliation.
Chapter 8

Further Work

1 Projecting a Sub-Action from a Tower to the Ambient Manifold

For dynamical systems \((T, X, \mu)\) which may be modeled by a Young Tower \((F, \Delta, \nu)\) in which the projection \(\pi : \Delta \rightarrow X\) is finite to one \(\mu\) almost everywhere, it is possible to “project” the sub-action \(\theta\) constructed on the tower to a sub-action \(\tilde{\theta}\) of the manifold by defining

\[
\tilde{\theta}(p) := \sup_{\pi(x,l)=p} \theta(x,l).
\]

Such a \(\tilde{\theta}\) will be a Hölder sub-action for the underlying system which was modeled by the tower.

However, in certain cases, such as maps of logistic type, though there is a Young Tower for the system, and the function \(\theta\) is a Hölder sub-action on the tower, the projection from the tower is infinite-to-one. In this case, we have been unable to
show that
\[ \sup_{x \in x, l \in I} \theta(x, l) \]
is bounded. The question of whether it is possible to obtain a sub-action with regularity in this case would be an interesting extension of the work done here.

2 Ergodic Optimization

Our motivation in this work was to establish the existence of regular sub-actions for a wide class of nonuniformly hyperbolic systems. Progress in this area has traditionally been on a case by case basis, and we have attempted a more axiomatic approach. Our attempts have had some success, particularly in the case of intermittent maps. In addition to our results for nonuniformly hyperbolic maps, we have presented here a unifying proof for uniformly hyperbolic systems.

In the future, we would also like to establish that for a large class of such systems maximizing measures are supported on periodic orbits. In addition to this, we would like to address the Yuan-Hunt conjecture that the maximum ergodic average may be approximated exponentially well by periodic orbits of low period.

3 A Conjecture for Future Study

Another natural problem is suggested by the results of this dissertation, and it is stated here as a conjecture.

Conjecture For a large class of nonuniformly hyperbolic systems $T : X \to X$ modeled by a Young Tower $F : \Delta \to \Delta$ there is an open and dense set of Lipschitz observables $\phi : X \to R$ such that $m(\phi, F) = m(\phi, T)$ and $m(\phi, F)$ is stably supported.
on a periodic orbit for $F : \Delta \to \Delta$ (and hence $m(\phi, T)$ is attained by a measure supported on a periodic orbit).
Bibliography


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