Morita equivalence of dual operator algebras

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Upasana Kashyap

December 2008

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ACKNOWLEDGMENTS

I would like to thank my advisor David Blecher for his tremendous help and support. He has been very kind and generous to me. I appreciate his constant availability to clear my doubts and motivate me in every situation. I must say he has inspired me to pursue mathematics as a professional career. I am deeply grateful for all his help, support, and guidance. I would like to thank Dinesh Singh for providing me the opportunity and inspiring me to do a Ph.D. in mathematics. I would also like to thank Mark Tomforde for his generous help and support during the last year of my graduate studies. I want to express my sincere gratitude to my other committee members, Vern Paulsen and David Pitts. Last, but certainly not least, I thank my family and friends for their love, support, and encouragement.

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ABSTRACT

In this thesis, we present some new notions of Morita equivalence appropriate to weak* closed algebras of Hilbert space operators. We obtain new variants, appropriate to the dual (weak* closed) algebra setting, of the basic theory of strong Morita equivalence due to Blecher, Muhly, and Paulsen. We generalize Rieffel's theory of Morita equivalence for W^*-algebras to non-selfadjoint dual operator algebras. Our theory contains all examples considered up to this point in the literature of Morita-like equivalence in a dual (weak* topology) setting. Thus, for example, our notion of equivalence relation for dual operator algebras is coarser than the one defined recently by Eleftherakis.

In addition, we give a new dual Banach module characterization of W^*-modules, also known as selfdual Hilbert C^*-modules over a von Neumann algebra. This leads to a generalization of the theory of W*-modules to the setting of non-selfadjoint algebras of Hilbert space operators which are closed in the weak* topology. That is, we find the appropriate weak* topology variant of the theory of rigged modules due to Blecher. We prove various versions of the Morita I, II, and III theorems for dual operator algebras. In particular, we prove that two dual operator algebras are weak* Morita equivalent in our sense if and only if they have equivalent categories of dual operator modules via completely contractive functors which are also weak* continuous on appropriate morphism spaces. Moreover, in a fashion similar to the operator algebra case, we characterize such functors as the module normal Haagerup tensor product with an appropriate weak* Morita equivalence bimodule.

Contents

Chapter 1

Introduction

1.1 Morita equivalence: selfadjoint setting

One of the important and well-known perspectives of study of an algebraic object is the study of its category of representations. For example, for rings, modules are viewed as their representations, and hence rings are commonly studied in terms of their modules. Once we view an algebraic object in terms of its category of representations, it is natural to compare such categories. This leads to the notion of Morita equivalence. The notion of Morita equivalence of rings arose in pure algebra in the 1960's. Two rings are defined to be Morita equivalent if and only if they have equivalent categories of modules. A fundamental Morita theorem says that two rings A and B have equivalent categories of modules if and only if there exists a pair of bimodules X and Y such that $X \otimes_B Y \cong A$ and $Y \otimes_A X \cong B$ as bimodules. Morita equivalence is an equivalence relation and preserves many ring theoretic properties. It is a powerful tool in pure algebra and it has inspired similar notions in operator algebra theory.

In the 1970's Rieffel introduced and developed the notion of Morita equivalence for C^*-algebras and W^*-algebras [40], [41]. Rieffel defined strong Morita equivalence in terms of Hilbert C^*-modules, which may be thought of as a generalization of Hilbert space in which the positive definite inner product is C^*-algebra valued. A Hilbert C^*-module can also be viewed as the noncommutative generalization of a vector bundle. The dual (weak* topology) version of a C^*-module is called a W^*-module. These objects are fundamental tools in operator algebra theory, and they play an important role in noncommutative geometry, being intimately related to Connes' correspondences.

We recall some basic definitions of the theory. By a C^*-algebra A, we mean that A is an involutive Banach algebra satisfying the C^*-identity $\|a^*a\| = \|a\|^2$ for all $a \in A$, where $a \mapsto a^*$ denotes the involution (adjoint) on A.

A right C^*-module over a C^*-algebra B is a right B-module X endowed with B-valued sesquilinear map $\langle \cdot, \cdot \rangle_B : X \times X \to B$ such that the following conditions are satisfied:

1. $\langle \cdot, \cdot \rangle_B$ is conjugate linear in the second variable.
2. $\langle x, x \rangle_B$ is a positive element in B for all $x \in X$.
3. $\langle x, x \rangle_B = 0$ if and only if $x = 0$ for all $x \in X$.
4. $\langle x, y \rangle_B^* = \langle y, x \rangle_B$ for all $x, y \in X$.
5. $\langle x, yb \rangle_B = \langle x, y \rangle_B b$ for all $x, y \in X$, $b \in B$.
6. X is complete in the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$.

A *left C^*-module* is defined analogously. Here X is a left module over a C^*-algebra A, the A-valued inner product ${}_A\langle ., . \rangle : X \times X \to A$ is linear in the first variable, and condition (5) in the above is replaced by ${}_A\langle ax, y \rangle = a {}_A\langle x, y \rangle$, for $x, y \in X$, $a \in A$.

If X is an A-B-bimodule, then we say X is an *equivalence bimodule*, if X is a right C^*-module over B, and a left C^*-module over A, such that

1. ${}_A\langle x, y\rangle z = x\langle y, z\rangle_B$ for all $x, y, z \in X$.

2. The linear span of $\{{}_A\langle x, y\rangle \mid x, y \in X\}$, which is a two-sided ideal, in A is dense in A; likewise $\{\langle x, y\rangle_B \mid x, y \in X\}$ spans a dense two-sided ideal in B.

If there exists such an equivalence bimodule, we say that A and B are *strongly Morita equivalent.*

Let $\overline{X}$ be X with the conjugate actions of A and B, and for $x \in X$, write $\bar{x}$ when x is viewed as an element in $\overline{X}$. Thus $b\bar{x} = \overline{xb^*}$ and $\bar{x}a = \overline{a^*x}$. Then $\overline{X}$ becomes a B-A-equivalence bimodule with inner products ${}_B\langle \bar{x}, \bar{y}\rangle = \langle x, y\rangle_B^*$ and $\langle \bar{x}, \bar{y}\rangle_A = {}_A\langle x, y\rangle^*$. In fact, X and $\overline{X}$ are operator spaces.

The collection of matrices

$$\mathcal{L} = \left\{ \begin{bmatrix} a & x \\ y & b \end{bmatrix} : \ a \in A, b \in B, x \in X, y \in \overline{X} \right\},$$

may be endowed with a norm making it a C^*-algebra with multiplication

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 +_A \langle x_1, y_2\rangle & a_1x_1 + x_1b_2 \\ y_1a_2 + b_1y_2 & \langle y_1, x_2\rangle_B + b_1b_2 \end{pmatrix}$$

and involution

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix}^* = \begin{pmatrix} a^* & \bar{y} \\ \bar{x} & b^* \end{pmatrix}.$$

The C^*-algebra $\mathcal{L}$ is called the *linking algebra* of A and B determined by X.

A W^*-algebra is a C^*-algebra that has a Banach space predual; and in this case the Banach space predual is unique. We say that a right C^*-module X over a C^*-algebra A is *selfdual* if every bounded A-module map $u : X \to A$ is of the form $u(\cdot) = {}_A\langle z, \cdot\rangle$ for some $z \in X$. By a theorem of Zettl and Effros-Ozawa-Ruan [42], [24], [15, Theorem 8.5.6] this condition is equivalent to the fact that X has a predual. We say that X is a right *W^*-module* if X is a selfdual right C^*-module over a W^*-algebra.

For W^*-algebras M and N, *W^*-equivalence M-N-bimodules* are defined similarly as the equivalence bimodules for C^*-algebras, with the term C^*-module replaced with W^*-module, and the ranges of the inner products span weak*-dense ideals in M and N. If there exists such a bimodule over M and N, then we say M and N are weakly Morita equivalent. In the case of weak Morita equivalence the linking algebra turns out to be a W^*-algebra. A fundamental theorem in the theory states that weakly Morita equivalent W^*-algebras have equivalent normal (weak* continuous) Hilbert space representations. For references, see [40], [41], [15, Chapter 8].

1.2 Morita equivalence: non-selfadjoint setting

With the arrival of operator space theory in the 1990s, Blecher, Muhly, and Paulsen generalized Rieffel's C^*-algebraic notion of strong Morita equivalence to non-selfadjoint operator algebras [18]. The theory of Morita equivalence developed by Blecher, Muhly, and Paulsen focused on the category of Hilbert modules and the category of operator modules over an operator algebra. Because the appropriate morphisms in the category of operator spaces are completely bounded or completely contractive maps (defined in Section 2.1), the module operations are assumed to be completely contractive.

Let A and B be operator algebras. Let X be an operator A-B-bimodule and let Y be an operator B-A-bimodule (that is, the module actions are completely contractive). We fix a pair of completely contractive balanced (i.e., $(xa, y) = (x, ay)$ for all x, $y \in X$ and $a \in A$) bilinear bimodule maps $(\cdot, \cdot) : X \times Y \to A$, and $[\cdot, \cdot] : Y \times X \to B$. The system $(A, B, X, Y, (\cdot, \cdot), [\cdot, \cdot])$ satisfying the above hypotheses is called a *Morita context* for A and B in the case the following conditions hold:

(A) $(x_1, y)x_2 = x_1[y, x_2]$, for $x_1, x_2 \in X$, $y \in Y$.

$[y_1, x]y_2 = y_1(x, y_2)$, $y_1, y_2 \in Y$ $x \in X$.

(G) The bilinear map $(.,.) : X \times Y \to A$ induces a completely isometric isomorphism between the balanced Haagerup tensor product $X \otimes_{hB} Y$ and A.

(P) The bilinear map $[.,.] : Y \times X \to B$ induces a completely isometric isomorphism between the balanced Haagerup tensor product $Y \otimes_{hA} X$ and B.

Two C^*-algebras are Morita equivalent in the sense of Blecher, Muhly, and Paulsen if and only if they are C^*-algebraically strongly Morita equivalent in the sense of Rieffel, and moreover the equivalence bimodules are the same.

In [6], Blecher developed a theory of rigged modules over non-selfadjoint operator algebras that generalizes the theory of Hilbert C^*-modules. Let A be an operator algebra with a contractive approximate identity and let Y be a right operator A-module. That is, Y is an operator space equipped with an A-module action $Y \times A \to Y$ that is completely contractive when $Y \otimes A$ is endowed with the Haagerup tensor product. Let $C_n(A)$ denote the first column of the matrix space $M_n(A)$. Then Y is called a (right) A-rigged module if there is a net of positive integers $n(\beta)$ and completely contractive right A-module maps $\phi_\beta \colon Y \to C_{n(\beta)}(A)$ and $\psi_\beta \colon C_{n(\beta)}(A) \to Y$ such that $\psi_\beta \phi_\beta \to \mathrm{Id}_Y$ strongly on Y (i.e., for all $y \in Y$, $\psi_\beta(\phi_\beta(y)) \to y$ in Y). A basic building block example in the theory of rigged modules over an operator algebra A is $C_n(A)$. Each Hilbert C^*-module has a natural operator space structure. It turns out that the class of Hilbert C^*-modules coincides with the class of rigged modules over a C^*-algebra. Again, operator space techniques and completely bounded maps are used extensively in this theory.

In [10], Blecher proved that two operator algebras are Morita equivalent if and only if they have equivalent categories of operator modules. The functors implementing the categorical equivalences are characterized as the module Haagerup tensor product with an appropriate strong Morita equivalence bimodule.

In this thesis, we have develop a weak* version of Morita equivalence for operator algebras that are closed in the weak* topology. These operator algebras are called *dual operator algebras*, and they are the non-selfadjoint version of von Neumann algebras. In parallel with the selfadjoint setting, one can prove that an operator algebra is closed in the weak*-topology if and only if it has an operator space predual if and only if it is equal to its double commutant in a certain universal representation (e.g., see [16], [21]). In this dissertation we give a formulation of Morita equivalence for dual operator algebras, which generalizes Rieffel's Morita equivalence for von Neumann algebras. That is, two W^*-algebras are Morita equivalent in our sense if and only if they are W^*-algebraically Morita equivalent in the sense of Rieffel, and moreover the weak* equivalence bimodules are the same. This work can be viewed as a weak* version of the Morita equivalence for operator algebras of Blecher, Muhly, and Paulsen. This is analogous to Rieffel's von Neumann algebraic Morita equivalence, which is the weak* version of strong Morita equivalence for selfadjoint operator algebras.

The weak* Morita equivalence that we develop contains all examples considered up to this point in the literature of Morita-like equivalence in the dual (weak* topology) setting. Thus, our notion of equivalence relation for dual operator algebras is coarser than the one recently defined by Eleftherakis.

Also our contexts represent a natural setting for the Morita equivalence of dual algebras. It is one to which the earlier theory of Morita equivalence (from, e.g., [18] [17]) transfers in a very clean manner; indeed it may in some sense be summarized as 'just changing the tensor product' involved to one appropriate to the weak* topology. We also give a new dual Banach module characterization of W^*-modules. This leads to a generalization of the theory of W^*-modules in the setting of dual operator algebras. That is, we find the appropriate weak* topology variant of the theory of rigged modules due to Blecher, see [6].

We prove variants of Morita's celebrated fundamental theorems (known as Morita

I, Morita II, and Morita III) appropriate to dual operator algebras. For example, we prove that two dual operator algebras are weak* Morita equivalent in our sense if and only if they have equivalent categories of dual operator modules via completely contractive functors which are also weak*-continuous on appropriate morphism spaces. Moreover, in a fashion similar to the operator algebra case, such functors are characterized as a suitable tensor product (namely, the module normal Haagerup tensor product) with an appropriate weak* Morita equivalence bimodule.

Our notion of Morita equivalence focuses on the category of normal Hilbert modules (weak* continuous Hilbert space representations) and the category of dual operator modules over a dual operator algebra. The appropriate tensor product in our setting of weak* topology is the module version of the normal Haagerup tensor product, recently introduced by Eleftherakis and Paulsen in [31]. In Section 2.4 we developed some more properties of this tensor product.

We now discuss a brief outline of this thesis. In Chapter 2 we give the necessary background and preliminary results. In Chapter 3, we define our variants of Morita equivalence and present some consequences. We prove various versions of the Morita I theorems. For example, if two dual operator algebras are Morita equivalent in our sense then they have equivalent categories of dual operator modules and normal Hilbert modules. Another interesting result is that if two dual operator algebras M and N are weak* Morita equivalent then the von Neumann algebras generated by M and N are Morita equivalent in Rieffel's W^*-algebraic sense.

In Chapter 4, we present a new characterization of W^*-modules. This leads to a generalization of the theory of W^*-modules to the setting of non-selfadjoint weak* closed algebras of Hilbert space operators. This is the dual variant of the earlier theory of rigged module due to Blecher. Chapter 3 and Chapter 4 are mostly joint work with D. P. Blecher, and much of it also appears in [13] and [14] respectively.

In Chapter 5, we prove a Morita II theorem, which characterizes module category equivalences as tensoring with an invertible bimodule. We also develop the general theory of the W^*-dilation, which connects the non-selfadjoint dual operator algebra with the W^*-algebraic framework. In the case of weak* Morita equivalence, this W^*-dilation is a W^*-module over a von Neumann algebra generated by the non-selfadjoint dual operator algebra. The theory of the W^*-dilation is a key part of the proof of our main theorem. The contents of Chapter 5 appear in [33].

Chapter 2

Background and preliminary results

2.1 Operator spaces and operator algebras

By an *operator space*, we mean a norm closed subspace X of $B(H)$ for some Hilbert space H. Besides a vector space structure, an operator space has some hidden norm structure. The space of $n \times n$ matrices over X, denoted by $M_n(X)$ inherits a distinguished norm $\|.\|_n$ via the identification $M_n(X) \subseteq M_n(B(H)) \cong B(H^{(n)})$ isometrically, where $H^{(n)}$ denotes the Hilbert space direct sum of n copies of H. The appropriate morphisms in this category are the completely bounded maps, which are defined as follows: Suppose $T : X \to Y$ is a linear map between operator spaces. For $n \in \mathbb{N}$, define $T_n : M_n(X) \to M_n(Y)$ by $T_n([x_{ij}]) = [T(x_{ij})]$, for $[x_{ij}] \in M_n(X)$. We say that T is *completely bounded* if $\|T\|_{cb} \stackrel{def}{=} \sup_n \|T_n\|$ is finite. We say that T is a *complete contraction* if $\|T\|_{cb} \leq 1$, and T is a *complete isometry* if T_n is an isometry for each $n \in \mathbb{N}$. Similarly T is a *complete quotient map* if each T_n is a quotient map.

Operator space theory can be thought of as a noncommutative generalization of Banach space theory. This is regarded as the suitable category to study many problems of operator algebras and operator theory. In particular, operator theoretic and operator algebraic problems motivated by classical Banach space theory and pure algebra are often studied in the setting of operator spaces. Long before the subject of operator spaces was developed, the completely bounded maps were used to study many problems in C^*-algebras and von Neumann algebras, see [1], [36]. With the arrival of the operator space theory, it has now become clear that many features of operator algebras are best understood in this general setting. Basics on operator spaces may be found in [15], [26], [36], [39].

A fundamental theorem in the subject of operator space is *Ruan's Theorem*, which gives an abstract characterization of operator spaces.

Theorem 2.1.1. *Suppose that X is a vector space, and that for each $n \in \mathbb{N}$ we are given a norm $\|\cdot\|_n$ on $M_n(X)$. Then X is linearly completely isometrically isomorphic to a linear subspace of $B(H)$, for some Hilbert space H, if and only if condition (R_1) and (R_2) below hold:*
(R_1) $\|\alpha x \beta\|_n \leq \|\alpha\|\|x\|_n\|\beta\|$, for all $n \in \mathbb{N}$ and all $\alpha, \beta \in M_n$, and $x \in M_n(X)$.
(R_2) For all $x \in M_m(X)$ and $y \in M_n(X)$, we have

$$\left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}.$$

Turning to notation, if E and F are sets of operators in $B(H)$, then EF denotes the norm closure of the span of products xy for $x \in E$ and $y \in F$. For cardinals or sets I, J, we use the symbol $\mathbb{M}_{I,J}(X)$ for the operator space of $I \times J$ matrices over X, whose 'finite submatrices' have uniformly bounded norm. Such a matrix is normed by the supremum of the norms of its finite submatrices. We write $\mathbb{K}_{I,J}(X)$ for the norm closure of these finite submatrices. Then $C_J^w(X) = \mathbb{M}_{J,1}(X)$, $R_J^w(X) = \mathbb{M}_{1,J}(X)$, and $C_J(X) = \mathbb{K}_{J,1}(X)$ and $R_J(X) = \mathbb{K}_{1,J}(X)$. If $I = \aleph_0$ we simply denote these spaces by for e.g., $\mathbb{M}(X)$, $R^w(X)$, $C^w(X)$. We sometimes write $\mathbb{M}_I(X)$ for $\mathbb{M}_{I,I}(X)$. If X and Y are operator spaces, we denote by $CB(X, Y)$ the space of completely bounded linear maps from X to Y.

By a *concrete operator algebra* we mean a norm closed subalgebra of $B(H)$ for some Hilbert space H. Note that this subalgebra is not necessarily selfadjoint. With the arrival of operator space theory in the past few decades, there have been many new developments in the theory of general operator algebras, which are not necessarily selfadjoint. The theory in the non-selfadjoint case is not as well developed as the selfadjoint case (C^*-algebra case), but it is still necessary and worthwhile to look at non-selfadjoint operator algebras because there are many interesting non-selfadjoint examples (e.g., upper triangular matrices, nest algebras, the disc algebra $A(D)$, the bounded analytic function on the disc $H^\infty(D)$, operator algebras arising in operator function theory). When studying non-selfadjoint operator algebras, many of the techniques from the selfadjoint setting do not work, and new approaches and tools have to be developed. Recently, operator space theory has started to provide necessary tools to study non-selfadjoint operator algebras; see [15], [36].

We study operator algebras from an operator space point of view. An *abstract operator algebra* A is an operator space that is also a Banach algebra for which there exists a Hilbert space H and a completely isometric homomorphism $\pi : A \to B(H)$.

We say that an operator algebra A is *unital* if it has an identity of norm 1. We mostly consider operator algebras that are approximately unital; that is, which possess a contractive approximate identity (cai). A contractive approximate identity is a net $(e_t) \subset A$, $\|e_t\| \leq 1$ such that $e_t a \to a$ and $a e_t \to a$ for all $a \in A$. Since every C^*-algebra possesses a cai, the class of approximately unital operator algebras is very large, including all C^*-algebras. By a *representation* of an operator algebra A, we mean a completely contractive homomorphism $\pi : A \to B(H)$ for some Hilbert space

H.

The following theorem, known as the *BRS theorem*, due to Blecher, Ruan, and Sinclair, is a fundamental result that gives a criteria for a unital (or more generally an approximately unital) Banach algebra with an operator space structure, to be an operator algebra [19]. This characterizes operator algebras at least for unital or approximately unital algebras. This theorem uses Haagerup tensor product $\otimes_h$ which is defined in Section 2.4.

Theorem 2.1.2. *Let A be an operator space that is also an approximately unital Banach algebra. Let $m : A \otimes A \to A$ denote the multiplication on A. The following are equivalent:*
(i) The mapping $m : A \otimes_h A \to A$ is completely contractive.
(ii) For any $n \geq 1$, $M_n(A)$ is a Banach algebra.
(iii) A is an operator algebra; that is, there exist a Hilbert space H and a completely isometric homomorphism $\pi : A \to B(H)$.

2.2 Dual operator spaces and dual operator algebras

For any operator space X, its Banach space dual X^* is again an operator space in the following way. We assign $M_n(X^*)$ the norm pulled back via the canonical algebraic isomorphism $M_n(X^*) \cong CB(X, M_n)$ (e.g., see Section 1.4 in [15]). We call, X^* with this matrix norm structure, the *operator space dual* of X. We say that X is a *dual operator space* if X is completely isometric to the operator space dual Y^*, for an operator space Y. Dual operator spaces and weak* closed subspaces of $B(H)$, are essentially the same thing. See [15, Section 1.4], [16] for references.

Lemma 2.2.1. *Any weak* closed subspace X of $B(H)$ is a dual operator space. Conversely, any dual operator space is completely isometrically isomorphic, via a homeomorphism for the weak* topologies, to a weak* closed subspace of $B(H)$, for some Hilbert space H.*

We will often abbreviate 'weak*' to 'w^*'. For a dual space X, let X_* denote its predual.

We will be using the following variant of the Krein-Smulian theorem very often.

Theorem 2.2.2. *(Krein-Smulian)*

1. *If $T \in B(E, F)$, where E and F are dual Banach spaces, then T is w^*-continuous if and only if whenever $x_t \to x$ is a bounded net converging in the w^*-topology on E, then $T(x_t) \to T(x)$ in the w^*-topology.*

2. *Let E and F be dual Banach spaces, and $T : E \to F$ a w^*-continuous isometry. Then T has w^*-closed range, and u is a w^*-w^*-homeomorphism onto Ran(T).*

By a *concrete dual operator algebra*, we mean a unital weak* closed algebra of operators on a Hilbert space which is not necessarily selfadjoint. By Lemma 2.2.1, any concrete dual operator algebra is a dual operator space. In order to view dual operator algebras from an abstract point of view, let M be an operator algebra together with a weak* topology given by some predual for M. Then M is said to be an *abstract dual operator algebra*, if there exist a Hilbert space H and a w^*-continuous completely isometric homomorphism $\pi : M \to B(H)$. By the Krein-Smulian theorem, π is a w^*-homeomorphism onto its range which is w^*-closed. Hence $\pi(M)$ is a concrete dual operator algebra acting on H, which may be identified with M in every sense.

A *normal representation* of a dual operator algebra M is a w^*-continuous unital completely contractive representation $\pi : M \to B(H)$.

We take all dual operator algebras to be *unital*, that is we assume they each possess an identity of norm 1. We reserve the symbol M and N for dual operator algebras.

One can view a concrete dual operator algebra as a non-selfadjoint analogue of a von Neumann algebra and abstract dual operator algebra as a non-selfadjoint analogue of a W^*-algebra. A *W^*-algebra* is a C^*-algebra which is a dual Banach space. By a famous theorem of Sakai, any W^*-algebra can be represented as a von Neumann algebra on some Hilbert space via a w^*-continuous isometric $*$-isomorphism. In this case the Banach space predual of W^*-algebra is unique. The following is a non-selfadjoint version of Sakai's theorem, due to Blecher, Magajna, and Le Merdy, which gives an abstract characterization of dual operator algebras. A dual operator algebra is characterized as a unital operator algebra which is also a dual operator space (see Section 2.7 in [15]).

Theorem 2.2.3. *Let M be an operator algebra that is a dual operator space. Then M is a dual operator algebra. That is, there exists a Hilbert space H and a w^*-continuous completely isometric homomorphism $\pi : M \to B(H)$.*

The product on a dual operator algebra is separately weak* continuous since the product on $B(H)$ is separately w^*-continuous. We will use this fact very often in subtle ways.

If M is a dual operator algebra, then a *W^*-cover* of M is a pair (A, j) consisting of a W^*-algebra A and a completely isometric w^*-continuous homomorphism $j : M \to A$, such that $j(M)$ generates A as a W^*-algebra. By the Krein-Smulian theorem $j(M)$ is a w^*-closed subalgebra of A. The *maximal W^*-cover* $W^*_{\max}(M)$ is a W^*-algebra containing M as a w^*-closed subalgebra, which is generated by M as a W^*-algebra, and which has the following universal property: any normal representation $\pi : M \to B(H)$ extends uniquely to a (unital) normal $*$-representation $\tilde{\pi} : W^*_{\max}(M) \to B(H)$

(see [21]).

A normal representation $\pi : M \to B(H)$ of a dual operator algebra M, or the associated space H viewed as an M-module, will be called *normal universal*, if any other normal representation is unitarily equivalent to the restriction of a 'multiple' of π to a reducing subspace (see [21]).

Lemma 2.2.4. *A normal representation $\pi : M \to B(H)$ of a dual operator algebra M is normal universal if and only if its extension $\tilde{\pi}$ to $W^*_{\max}(M)$ is one-to-one.*

Proof. The ($\Leftarrow$) direction is stated in [21]. Thus there does exist a normal universal π whose extension $\tilde{\pi}$ to $W^*_{\max}(M)$ is one-to-one. It is observed in [21] that any other normal universal representation θ is quasiequivalent to π. It follows that the extension $\tilde{\theta}$ to $W^*_{\max}(M)$ is quasiequivalent to $\tilde{\pi}$, and it follows from this that $\tilde{\theta}$ is one-to-one. $\square$

2.3 Operator modules and dual operator modules

A *concrete left operator module* over an operator algebra A is a subspace $X \subset B(H)$ such that $\pi(A)X \subset X$ for a completely contractive representation $\pi : A \to B(H)$. An *abstract operator A-module* is an operator space X which is also an A-module, such that X is completely isometrically isomorphic, via an A-module map, to a concrete operator A-module. Similarly for right modules and bimodules. Most of the interesting modules over operator algebras are operator modules, such as Hilbert C^*-modules and *Hilbert modules*. By a Hilbert module over an operator algebra A, we mean a pair (H, π), where H is a (column) Hilbert space (see e.g. 1.2.23 in [15]), and $\pi : A \to B(H)$ is a representation of A. That is, Hilbert modules over an operator algebra are nothing but the Hilbert space representations of an operator algebra.

Let X be a left operator module over an operator algebra A. Then the module action on X is completely contractive; i.e., the spaces $M_n(X)$ are left Banach $M_n(A)$-

modules in the canonical way for every $n \in \mathbb{N}$. That is, $\|ax\|_n \leq \|a\|_n \|x\|_n$, for all $n \in \mathbb{N}$, $a \in M_n(A)$, $x \in M_n(X)$. A similar statement is true for right modules or bimodules.

The following theorem is a variation on a theorem due to Christensen, Effros, and Sinclair. We often refer to this theorem as the '*CES theorem*'.

Theorem 2.3.1. *Let A and B be approximately unital operator algebras. Let X be an operator space that is a nondegenerate A-B-bimodule such that the module actions are completely contractive. Then there exist Hilbert spaces H and K, a completely isometric linear map $\phi : X \to B(K, H)$, and completely contractive nondegenerate representations $\theta : A \to B(H)$, and $\pi : B \to B(K)$, such that*

$$\theta(a)\phi(x) = \phi(ax) \quad \textit{and} \quad \phi(x)\pi(b) = \phi(xb)$$

for all $a \in A$, $b \in B$ and $x \in X$. Thus X is completely isometric to the concrete operator A-B-bimodule $\phi(X)$ via an A-B-bimodule map.

Let M and N be dual operator algebras. A *concrete dual operator M-N-bimodule* is a w^*-closed subspace X of $B(K, H)$ such that $\theta(M)X\pi(N) \subset X$, where θ and π are normal representations of M and N on H and K respectively. An *abstract dual operator M-N-bimodule* is defined to be a nondegenerate operator M-N-bimodule X, which is also a dual operator space, such that the module actions are separately weak* continuous. Such spaces can be represented completely isometrically as concrete dual operator bimodules (see e.g., [15, 16, 25]). A similar statement is true for one sided modules (the case M or N equals $\mathbb{C}$).

We shall write ${}_M\mathcal{R}$ for the category of left dual operator modules over M. The morphisms in ${}_M\mathcal{R}$ are the w^*-continuous completely bounded M-module maps.

An important example of a left dual operator module over a dual operator algebra M, is the *normal Hilbert M-module*. By this we mean a pair (H, π), where H is a (column) Hilbert space (see 1.2.23 in [15]) and $\pi : M \to B(H)$ is a normal representation of M. The module action is expressed through the equation $m \cdot \zeta = \pi(m)\zeta$. We denote the category of normal Hilbert M-modules by ${}_M\mathcal{H}$. The morphisms are bounded linear transformation between Hilbert spaces that intertwine the representations; i.e., if (H_i, π_i), $i = 1, 2$, are objects of the category ${}_M\mathcal{H}$, then the space of morphisms is defined as: $B_M(H_1, H_2) = \{T \in B(H_1, H_2) : T\pi_1(m) = \pi_2(m)T$ for all $m \in M\}$.

If X and Y are dual operator spaces, we denote by $CB^\sigma(X, Y)$ the space of completely bounded w^*-continuous linear maps from X to Y. Similarly if X and Y are left dual operator M-modules, then $CB^\sigma_M(X, Y)$ denotes the space of completely bounded w^*-continuous left M-module maps from X to Y.

The category of operator spaces and completely bounded maps is the appropriate setting to study the Morita equivalence for operator algebras [18]. Similarly, the category of dual operator spaces and weak* continuous completely bounded maps is the appropriate setting to study the Morita equivalence for dual operator algebras.

2.4 Some tensor products

Before we begin our discussion of tensor products, we need to introduce the notions of completely bounded and completely contractive bilinear maps. Suppose that X, Y, and W are operator spaces, and that $u : X \times Y \to W$ is a bilinear map. For $n, p \in \mathbb{N}$, define a bilinear map $M_{n,p}(X) \times M_{p,n}(Y) \to M_n(W)$ by

$$(x, y) \mapsto \left[\sum_{k=1}^{p} u(x_{ik}, y_{kj}) \right]_{i,j}, \tag{2.4.1}$$

where $x = [x_{ij}] \in M_{n,p}(X)$ and $y = [y_{ij}] \in M_{p,n}(Y)$. Recall that a bilinear map $T : X \times Y \to Z$ is bounded if there exists a constant C such that $\|T(x, y)\| \leq C\|x\|\|y\|$, for all $x \in X, y \in Y$. The norm $\|T\|$ is defined as the least such C. If the norms of these bilinear maps defined by (2.4.1) are uniformly bounded over $p, n \in \mathbb{N}$, then we say that u is completely bounded, and we write the supremum of these norms as $\|u\|_{cb}$. These classes of bilinear maps were introduced by Christensen and Sinclair.

Suppose X and Y are two operator spaces. Define $\|z\|_n$ for $z \in M_n(X \otimes Y)$ as:

$$\|z\|_n = \inf \{\|a\|\|b\| \ : \ z = a \odot b, a \in M_{np}(X), b \in M_{pn}(Y), p \in \mathbb{N}\}.$$

Here $a \odot b$ stands for the $n \times n$ matrix whose i, j -entry is $\sum_{k=1}^{p} a_{ik} \otimes b_{kj}$. The algebraic tensor product $X \otimes Y$ with this sequence of matrix norms is an operator space. The completion of this operator space in the above norm is called the *Haagerup tensor product*, and is denoted by $X \otimes_h Y$. The completion of an operator space is an operator space, and hence $X \otimes_h Y$ is an operator space. The reason completely bounded bilinear maps and the Haagerup tensor product are intimately related is the well-known universal property of the Haagerup tensor product: the Haagerup tensor product linearizes completely bounded bilinear maps (see 1.5.4 in [15]).

If X and Y are respectively right and left operator A-modules, then the *module Haagerup tensor product* $X \otimes_{hA} Y$ is defined to be the quotient of $X \otimes_h Y$ by the closure of the subspace spanned by terms of the form $xa \odot y - x \odot ay$, for $x \in X$, $y \in Y$, $a \in A$. Let X be a right and Y be a left operator A-module where A is an operator algebra. We say that a bilinear map $\psi : X \times Y \to W$ is *balanced* if $\psi(xa, y) = \psi(x, ay)$ for all $x \in X$, $y \in Y$ and $a \in A$. The module Haagerup tensor product linearizes balanced bilinear maps which are completely contractive (or completely bounded). We state this important fact in the following theorem. Its proof may be found in [18].

Theorem 2.4.1. *Let X be a right operator A-module and let Y be a left operator A-module. Up to a complete isometric isomorphism, there is a unique pair $(V, \otimes_A)$, where V is an operator space and $\otimes_A : X \times Y \to V$ is a completely contractive balanced bilinear map whose range densely spans V, with the following universal property: Given any operator space W and a completely bounded bilinear balanced map ψ :*

$X \times Y \to W$, there is a unique completely bounded linear map $\tilde{\psi} : V \to W$, with $\|\tilde{\psi}\|_{cb} = \|\psi\|_{cb}$, such that $\tilde{\psi} \circ \otimes_A = \psi(x, y)$. We write $X \otimes_{hA} Y$ for V, and we write $x \otimes_A y$ for $\otimes_A(x, y)$.

If X and Y are two operator spaces, then the *extended Haagerup tensor product* $X \otimes_{eh} Y$ may be defined to be the subspace of $(X^* \otimes_h Y^*)^*$ corresponding to the completely bounded bilinear maps from $X^* \times Y^* \to \mathbb{C}$ which are separately weak* continuous. If X and Y are dual operator spaces, with preduals X_* and Y_*, then this coincides with the *weak* Haagerup tensor product* defined earlier in [20], and indeed by 1.6.7 in [15], $X \otimes_{eh} Y = (X_* \otimes_h Y_*)^*$. The *normal Haagerup tensor product* $X \otimes^{\sigma h} Y$ is defined to be the operator space dual of $X_* \otimes_{eh} Y_*$. The canonical maps are complete isometries

$$X \otimes_h Y \to X \otimes_{eh} Y \to X \otimes^{\sigma h} Y.$$

The normal Haagerup tensor product was first studied by Effros and Ruan. See [27] for more details. We establish some new results about this tensor product.

Lemma 2.4.2. *For any dual operator spaces X and Y,* $\text{Ball}(X \otimes_h Y)$ *is w^*-dense in* $\text{Ball}(X \otimes^{\sigma h} Y)$.

Proof. Let $x \in \text{Ball}(X \otimes^{\sigma h} Y) \setminus \overline{\text{Ball}(X \otimes_h Y)}^{w^*}$. By the geometric Hahn-Banach theorem, there exists a $\phi \in (X \otimes^{\sigma h} Y)_*$, and $t \in \mathbb{R}$, such that $Re\ \phi(x) > t > Re\ \phi(y)$ for all $y \in \text{Ball}(X \otimes_h Y)$. Note that ϕ can be viewed as a map $\phi : X \otimes_h Y \to \mathbb{C}$ corresponding to a completely contractive bilinear map from $X \times Y \to \mathbb{C}$ which is separately w^*-continuous. It follows that $Re\ \phi(x) > t > |\phi(y)|$ for all $y \in \text{Ball}(X \otimes_h Y)$, which implies that $\|\phi\| \leq t$. Thus $|Re\ \phi(x)| \leq \|\phi\|\ \|x\| \leq t$, which is a contradiction. $\square$

Lemma 2.4.3. *The normal Haagerup tensor product is associative. That is, if X, Y, Z are dual operator spaces then $(X \otimes^{\sigma h} Y) \otimes^{\sigma h} Z = X \otimes^{\sigma h} (Y \otimes^{\sigma h} Z)$ as dual operator spaces.*

Proof. Consider $(X \otimes^{\sigma h} Y) \otimes^{\sigma h} Z \cong ((X_* \otimes^{eh} Y_*) \otimes^{eh} Z_*)^* \cong (X_* \otimes^{eh} (Y_* \otimes^{eh} Z_*))^* \cong X \otimes^{\sigma h} (Y \otimes^{\sigma h} Z)$ using associativity of the extended Haagerup tensor product (e.g., see [27]). □

We now turn to the module version of the normal Haagerup tensor product introduced in [31], and review some facts from [31]. Let X be a right dual operator M-module and Y be a left dual operator M-module. Let $(X \otimes_{hM} Y)^*_\sigma$ denote the subspace of $(X \otimes_h Y)^*$ corresponding to the completely bounded bilinear maps from $\psi : X \times Y \to \mathbb{C}$ which are separately weak* continuous and M-balanced (that is, $\psi(xm, y) = \psi(x, my)$). Define the *module normal Haagerup tensor product* $X \otimes^{\sigma h}_M Y$ to be the operator space dual of $(X \otimes_{hM} Y)^*_\sigma$. Equivalently, $X \otimes^{\sigma h}_M Y$ is the quotient of $X \otimes^{\sigma h} Y$ by the weak*-closure of the subspace spanned by terms of the form $xm \otimes y - x \otimes my$, for $x \in X$, $y \in Y$, $m \in M$. The module normal Haagerup tensor product linearizes completely contractive, separately weak* continuous, balanced bilinear maps:

Proposition 2.4.4. *[31, Proposition 2.2] If X, Y, and Z are dual operator spaces, and $\phi : X \times Y \to Z$ is a completely bounded separately w^*-continuous balanced bilinear map then there exists a w^*-continuous and completely bounded map $\tilde{\phi} : X \otimes^{\sigma h}_N Y \to Z$ such that $\tilde{\phi}(x \otimes_N y) = \phi(x, y)$ for all $x \in X$, $y \in Y$. In fact the map $\phi \mapsto \tilde{\phi}$ is a complete isometry and onto.*

We now prove some new results about the module normal Haagerup tensor product. These results are used throughout this thesis.

Lemma 2.4.5. *Let X_1, X_2, Y_1, Y_2 be dual operator spaces. If $u : X_1 \to Y_1$ and $v : X_2 \to Y_2$ are w^*-continuous, completely bounded, linear maps, then the map $u \otimes v$ extends to a well defined w^*-continuous, linear, completely bounded map from $X_1 \otimes^{\sigma h} X_2 \to Y_1 \otimes^{\sigma h} Y_2$, with $\|u \otimes v\|_{cb} \leq \|u\|_{cb}\|v\|_{cb}$.*

Proof. Since $u \otimes v = (u \otimes \mathbb{I}) \circ (\mathbb{I} \otimes v)$, we may by symmetry reduce the argument to the case that $X_2 = Y_2$ and $v = I_{X_2}$. The map $u_* : (Y_1)_* \to (X_1)_*$ is completely contractive where $(u_*)^* = u$. By the functoriality of extended Haagerup tensor product $u_* \otimes \mathbb{I} : (Y_1)_* \otimes^{eh} (X_2)_* \to (X_1)_* \otimes^{eh} (X_2)_*$ is completely contractive. Hence $(u_* \otimes \mathbb{I})^* : X_1 \otimes^{\sigma h} X_2 \to Y_1 \otimes^{\sigma h} X_2$ is a w^*-continuous, completely bounded, linear map. It is easy to check that $(u_* \otimes \mathbb{I})^* = u \otimes \mathbb{I}$. □

Corollary 2.4.6. *Let N be a dual operator algebra, let X_1 and Y_1 be dual operator spaces which are right N-modules, and let X_2, Y_2 be dual operator spaces which are left N-modules. If $u : X_1 \to X_2$ and $v : Y_1 \to Y_2$ are completely bounded, w^*-continuous, N-module maps, then the map $u \otimes v$ extends to a well defined linear, w^*-continuous, completely bounded map from $X_1 \otimes_N^{\sigma h} Y_1 \to X_2 \otimes_N^{\sigma h} Y_2$, with $\|u \otimes v\|_{cb} \leq \|u\|_{cb} \|v\|_{cb}$.*

Proof. By Lemma 2.4.5, we obtain a w^*-continuous, completely bounded, linear map $X_1 \otimes^{\sigma h} Y_1 \to X_2 \otimes^{\sigma h} Y_2$ taking $x \otimes y$ to $u(x) \otimes v(y)$. Composing this map with the w^*-continuous, quotient map $X_2 \otimes^{\sigma h} Y_2 \to X_2 \otimes_N^{\sigma h} Y_2$, we obtain a w^*-continuous, completely bounded map $X_1 \otimes^{\sigma h} Y_1 \to X_2 \otimes_N^{\sigma h} Y_2$. It is easy to see that the kernel of the last map contains all terms of form $xn \otimes_N y - x \otimes_N ny$, with $n \in N, x \in X_1, y \in Y_1$. Thus we obtain a map $X_1 \otimes_N^{\sigma h} Y_1 \to X_2 \otimes_N^{\sigma h} Y_2$ with the required properties. □

Lemma 2.4.7. *If X is a dual operator M-N-bimodule and if Y is a dual operator N-L-bimodule, then $X \otimes_N^{\sigma h} Y$ is a dual operator M-L-bimodule.*

Proof. To show $X \otimes_N^{\sigma h} Y$ is a left dual operator M-module for example, use the canonical maps

$$M \otimes_h (X \otimes^{\sigma h} Y) \to M \otimes^{\sigma h} (X \otimes^{\sigma h} Y) \to (M \otimes^{\sigma h} X) \otimes^{\sigma h} Y \to X \otimes^{\sigma h} Y.$$

Composing the map $M \otimes^{\sigma h} (X \otimes^{\sigma h} Y) \to X \otimes^{\sigma h} Y$ above with the canonical map $M \times (X \otimes^{\sigma h} Y) \to M \otimes^{\sigma h} (X \otimes^{\sigma h} Y)$, one sees the action of M on $X \otimes^{\sigma h} Y$ is separately weak* continuous (see also [31]). That $(a_1 a_2)z = a_1(a_2 z)$ for $a_i \in M$, $z \in X \otimes^{\sigma h} Y$, follows from the weak* density of $X \otimes Y$, and since this relation is true if z is finite rank. It follows from 3.3.1 in [15], that $X \otimes^{\sigma h} Y$ is a (dual) operator M-module. By 3.8.8 in [15], $X \otimes^{\sigma h}_N Y$ is a dual operator M-module. (See also Lemma 2.3 in [31].) $\square$

There is clearly a canonical map $X \otimes_{hM} Y \to X \otimes^{\sigma h}_M Y$, with respect to which:

Corollary 2.4.8. *For any dual operator M-modules X and Y, the image of* $\mathrm{Ball}(X \otimes_{hM} Y)$ *is w^*-dense in* $\mathrm{Ball}(X \otimes^{\sigma h}_M Y)$.

Proof. Consider the canonical w^*-continuous quotient map $q : X \otimes^{\sigma h} Y \to X \otimes^{\sigma h}_M Y$ as in [31, Proposition 2.1]. If $z \in X \otimes^{\sigma h}_M Y$ with $\|z\| < 1$, then there exists $z' \in X \otimes^{\sigma h} Y$ with $\|z'\| < 1$ such that $q(z') = z$. By Lemma 2.4.2, there exists a net (z_t) in $\mathrm{Ball}(X \otimes_h Y)$ such that $z_t \xrightarrow{w^*} z'$. Thus $q(z_t) \xrightarrow{w^*} q(z') = z$. $\square$

Lemma 2.4.9. *For any dual operator M-modules X and Y, and $m, n \in \mathbb{N}$, we have $M_{mn}(X \otimes^{\sigma h}_M Y) \cong C_m(X) \otimes^{\sigma h}_M R_n(Y)$ completely isometrically and weak* homeomorphically. This is also true with m, n replaced by arbitrary cardinals: $M_{IJ}(X \otimes^{\sigma h}_M Y) \cong C_I(X) \otimes^{\sigma h}_M R_J(Y)$.*

Proof. We just prove the case that $m, n \in \mathbb{N}$, the other being similar (or can be deduced easily from Proposition 2.4.11). First we claim that $M_{mn}(X \otimes^{\sigma h} Y) \cong C_m(X) \otimes^{\sigma h} R_n(Y)$. Using facts from [27] and basic operator space duality, the predual of the latter space is

$$\begin{aligned}
C_m(X)_* \otimes_{eh} R_n(Y)_* &\cong (R_m \otimes_h X_*) \otimes_{eh} (Y_* \otimes_h C_n) \\
&\cong (R_m \otimes_{eh} X_*) \otimes_{eh} (Y_* \otimes_{eh} C_n) \\
&\cong R_m \otimes_{eh} (X_* \otimes_{eh} Y_*) \otimes_{eh} C_n \\
&\cong R_m \otimes_h (X_* \otimes_{eh} Y_*) \otimes_h C_n \\
&\cong (X_* \otimes_{eh} Y_*) \widehat{\otimes} (M_{mn})_*.
\end{aligned}$$

We have used for example, 1.5.14 in [15], 5.15 in [27], and associativity of the extended Haagerup tensor product [27]. The latter space is the predual of $M_{mn}(X \otimes^{\sigma h} Y)$, by e.g., 1.6.2 in [15]. This gives the claim. If θ is the ensuing completely isometric isomorphism $C_m(X) \otimes^{\sigma h} R_n(Y) \to M_{mn}(X \otimes^{\sigma h} Y)$, it is easy to check that θ takes $[x_1\ x_2\ \ldots x_m]^T \otimes [y_1\ y_2 \ldots y_n]$ to the matrix $[x_i \otimes y_j]$. Now $C_m(X) \otimes^{\sigma h}_M R_n(Y) = C_m(X) \otimes^{\sigma h} R_n(Y)/N$ where $N = [xt \otimes y - x \otimes ty]^{-w^*}$ with $x \in C_m(X), y \in R_n(Y), t \in M$. Let $N' = [xt \otimes y - x \otimes ty]^{-w^*}$ where $x \in X, y \in Y, t \in M$, then clearly $\theta(N) = M_{mn}(N')$. Hence

$$C_m(X) \otimes^{\sigma h} R_n(Y)/N \cong M_{mn}(X \otimes^{\sigma h} Y)/\theta(N) = M_{mn}(X \otimes^{\sigma h} Y)/M_{nm}(N'),$$

which in turn equals $M_{mn}(X \otimes^{\sigma h} Y/N') = M_{mn}(X \otimes^{\sigma h}_M Y)$. □

Corollary 2.4.10. *For any dual operator M-modules X and Y, and $m, n \in \mathbb{N}$, we have that* Ball$(M_{mn}(X \otimes_{hM} Y))$ *is w^*-dense in* Ball $(M_{mn}(X \otimes^{\sigma h}_M Y))$.

Proof. If $\eta \in$ Ball $(M_{mn}(X \otimes^{\sigma h}_M Y))$, then by Lemma 2.4.9, η corresponds to an element $\eta' \in C_m(X) \otimes^{\sigma h}_M R_n(Y)$. By Corollary 2.4.8, there exists a net (u_t) in $C_m(X) \otimes_{hM} R_n(Y)$ such that $u_t \xrightarrow{w^*} \eta'$. By 3.4.11 in [15], u_t corresponds to $u'_t \in$ Ball$(M_{mn}(X \otimes_{hM} Y))$ such that $u'_t \xrightarrow{w^*} \eta$. □

Proposition 2.4.11. *The normal module Haagerup tensor product is associative. That is, if M and N are dual operator algebras, if X is a right dual operator M-module, if Y is a dual operator M-N-bimodule, and Z is a left dual operator N-module, then $(X \otimes_M^{\sigma h} Y) \otimes_N^{\sigma h} Z$ is completely isometrically isomorphic to $X \otimes_M^{\sigma h} (Y \otimes_N^{\sigma h} Z)$.*

Proof. We define $X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$ to be the quotient of $X \otimes^{\sigma h} Y \otimes^{\sigma h} Z$ by the w^*-closure of the linear span of terms of the form $xm \otimes y \otimes z - x \otimes my \otimes z$ and $x \otimes yn \otimes z - x \otimes y \otimes nz$ with $x \in X, y \in Y, z \in Z, m \in M, n \in N$. By extending the arguments of Proposition 2.2 in [31] to the threefold normal module Haagerup tensor product, one sees that $X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$ has the following universal property: If W is a dual operator space and $u : X \times Y \times Z \to W$ is a separately w^*-continuous, completely contractive, balanced, trilinear map, then there exists a w^*-continuous and completely contractive, linear map $\tilde{u} : X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z \to W$ such that $\tilde{u}(x \otimes_M y \otimes_N z) = u(x, y, z)$. We will prove that $(X \otimes_M^{\sigma h} Y) \otimes_N^{\sigma h} Z$ has the above universal property defining $X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$. Let $u : X \times Y \times Z \to W$ be a separately w^*-continuous, completely contractive, balanced, trilinear map. For each fixed $z \in Z$, define $u_z : X \times Y \to W$ by $u_z(x, y) = u(x, y, z)$. This is a separately w^*-continuous, balanced, bilinear map, which is completely bounded. Hence we obtain a w^*-continuous completely bounded linear map $u'_z : X \otimes_M^{\sigma h} Y \to W$ such that $u'_z(x \otimes_M y) = u_z(x, y)$. Define $u' : (X \otimes_M^{\sigma h} Y) \times Z \to W$ by $u'(a, z) = u'_z(a)$, for $a \in X \otimes_M^{\sigma h} Y$. Then $u'(x \otimes_M y, z) = u(x, y, z)$, and it is routine to check that u' is bilinear and balanced over N. We will show that u' is completely contractive on $(X \otimes_{hM} Y) \times Z$, and then the complete contractivity of u' follows from Corollary 2.4.10. Let $a \in M_{nm}(X \otimes_{hM} Y)$ with $\|a\| < 1$ and $z \in M_{mn}(Z)$ with $\|z\| < 1$. We want to show $\|u'_n(a, z)\| < 1$. It is well known that we can write $a = x \odot_M y$ where $x \in M_{nk}(X)$ and $y \in M_{km}(Y)$ for some $k \in \mathbb{N}$, with $\|x\| < 1$ and $\|y\| < 1$. Hence $\|u'_n(a, z)\| = \|u_n(x, y, z)\| \le \|x\|\|y\|\|z\| < 1$, proving u' is completely contractive.

By Proposition 2.2 in [31], we obtain a w^*-continuous, completely contractive, linear map $\tilde{u} : (X \otimes_M^{\sigma h} Y) \otimes_N^{\sigma h} Z \to W$ such that $\tilde{u}((x \otimes_M y) \otimes_N z) = u'(x \otimes_M y, z) = u(x, y, z)$. This shows that $(X \otimes_M^{\sigma h} Y) \otimes_N^{\sigma h} Z$ has the defining universal property of $X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$. Therefore $(X \otimes_M^{\sigma h} Y) \otimes_N^{\sigma h} Z$ is completely isometrically isomorphic and w^*-homeomorphic to $X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$. Similarly $X \otimes_M^{\sigma h} (Y \otimes_N^{\sigma h} Z) = X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$. □

Lemma 2.4.12. *If X is a left dual operator M-module then $M \otimes_M^{\sigma h} X$ is completely isometrically isomorphic to X.*

Proof. As in Lemma 3.4.6 in [15], or follows from the universal property. □

Chapter 3

Morita equivalence of dual operator algebras

3.1 Introduction

In this chapter, we introduce some notions of Morita equivalence appropriate to dual operator algebras. We obtain new variants, appropriate to the dual algebra setting, of the basic theory of strong Morita equivalence due to Blecher, Muhly, and Paulsen, and new non-selfadjoint analogues of aspects of Rieffel's W^*-algebraic Morita equivalence. That is, we generalize Rieffel's variant of W^*-algebraic Morita equivalence to dual operator algebras.

Another notion of Morita equivalence for dual operator algebras was considered in [28] and is called Δ-equivalence. In [31] it was shown that the Δ-equivalence implies weak* Morita equivalence in our sense. That is, any of the equivalences of [28] is one of our weak* Morita equivalences. Both the theories have different advantages. For example, the equivalence considered in [28] is equivalent to the very important notion of weak* stable isomorphism. On the other hand, our theory contains all examples considered up to this point in the literature of Morita-like equivalence in a dual (weak* topology) setting. Thus our notion of equivalence relation for dual operator algebras is coarser than Eleftherakis' Δ-equivalence. There are certain important examples that do not seem to be contained in the other theory but are weak* Morita equivalent in our sense. For example, in the selfadjoint setting the second dual of strongly Morita equivalent C^*-algebras are Morita equivalent in Rieffel's W^*-algebraic sense. In the non-selfadjoint case, the second dual of strongly Morita equivalent operator algebras in the sense of Blecher, Muhly and Paulsen are weak* Morita equivalent in our sense. Also, a beautiful example from [30]: two 'similar' separably acting nest algebras are Morita equivalent in our sense (using Davidson's similarity theorem). However, it is shown in [30] that these algebras are not Δ-equivalent and hence they are not weak* stably isomorphic [31]. Thus, Eleftherakis' Δ-equivalence and our notions of Morita equivalence are distinct. Eleftherakis' Morita contexts contain a W^*-algebraic Morita context (that is, his bimodules contain a bimodule implementing a W^*-algebraic Morita equivalence). Our contexts are contained in a W^*-algebraic Morita context (see Section 3.4). Also our contexts represent a natural setting for the Morita equivalence of dual algebras in the sense that the earlier theory of Morita equivalence (from e.g., [18] [17]) transfers in a very clean manner, indeed which may be in some sense be summarized as 'just changing the tensor product' involved to one appropriate to weak* topology.

In Section 3.2, we define our variant of Morita equivalence, and present some of its consequences. Section 3.3 is centered on the weak linking algebra, the key tool for dealing with most aspects of Morita equivalence. In Section 3.4 we prove that if M and N are weak* Morita equivalent dual operator algebras, then the von Neumann algebras generated by M and N are Morita equivalent in Rieffel's W^*-algebraic sense.

3.2 Morita contexts

We now define two variants of Morita equivalence for unital dual operator algebras, the first being more general than the second. There are many equivalent variants of these definitions, some of which we shall see later.

Throughout this section, we fix a pair of unital dual operator algebras, M and N, and a pair of dual operator bimodules X and Y; X will always be a M-N-bimodule and Y will always be an N-M-bimodule.

Definition 3.2.1. *We say that M is* weak* Morita equivalent *to N, if there exist a pair of dual operator bimodules X and Y as above such that $M \cong X \otimes_N^{\sigma h} Y$ as dual operator M-bimodules (that is, completely isometrically, w^*-homeomorphically, and also as M-bimodules), and similarly $N \cong Y \otimes_M^{\sigma h} X$ as dual operator N-bimodules. We call (M, N, X, Y) a* weak* Morita context *in this case.*

In this section, we will also fix separately weak* continuous completely contractive bilinear maps $(\cdot, \cdot) : X \times Y \to M$, and $[\cdot, \cdot] : Y \times X \to N$, and we will work with the 6-tuple, or *context* $(M, N, X, Y, (\cdot, \cdot), [\cdot, \cdot])$.

Definition 3.2.2. *We say that M is* weakly Morita equivalent *to N, if there exist w^*-dense approximately unital operator algebras A and B in M and N respectively, and there exists a w^*-dense operator A-B-submodule X' in X, and a w^*-dense B-A-submodule Y' in Y, such that the 'subcontext' $(A, B, X', Y', (\cdot, \cdot), [\cdot, \cdot])$ is a (strong) Morita context in the sense of [18, Definition 3.1]. In this case, we call (M, N, X, Y) (or more properly the 6-tuple above the definition), a* weak Morita context.

Remark. Some authors use the term 'weak Morita equivalence' for a quite different notion, namely to mean that the algebras have equivalent categories of Hilbert space representations.

Weak Morita equivalence, as we have just defined it, is really nothing more than the 'weak* closure of' a strong Morita equivalence in the sense of [18]. This definition includes all examples considered up to this point in the literature of Morita-like equivalence in a dual (weak* topology) setting.

Examples:

1. We shall see in Corollary 3.2.4 that every weak Morita equivalence is an example of weak* Morita equivalence.

2. We shall see in Section 3.3 that every weak Morita equivalence arises as follows: Let A, B be subalgebras of $B(H)$ and $B(K)$ respectively, for Hilbert spaces H, K, and let $X \subset B(K, H), Y \subset B(H, K)$, such that the associated subset $\mathcal{L} = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}$ of $B(H \oplus K)$ is a subalgebra of $B(H \oplus K)$, for Hilbert spaces H, K. This is the same as specifying a list of obvious algebraic conditions, such as $XY \subset A$. Assume in addition that A possesses a cai (e_t) with terms of the form xy, for $x \in \text{Ball}(R_n(X))$ and $y \in \text{Ball}(C_n(Y))$, and B possessing a cai with terms of a similar form yx (dictated by symmetry). Taking the weak* (that is, σ-weak) closure of all these spaces clearly yields a weak Morita equivalence of $\overline{A}^{w*}$ and $\overline{B}^{w*}$.

3. Every weak* Morita equivalence arises similarly to the setting in (2). The main difference is that A, B are unital, and (e_t) is not a cai, but $e_t \to 1_A$ weak*, and similarly for the net in B.

4. Von Neumann algebras which are Morita equivalent in Rieffel's W^*-algebraic sense from [40], are clearly weakly Morita equivalent. We state this in the language of TROs. We recall that a TRO is a subspace $Z \subset B(K, H)$ with $ZZ^*Z \subset Z$. Rieffel's W^*-algebraic Morita equivalence of W^*-algebras M and

N is essentially the same (see e.g. [15, Section 8.5] for more details) as having a weak* closed TRO (that is, a *WTRO*) Z, with ZZ^* weak* dense in M and Z^*Z weak* dense in N. Recall that Z^*Z denotes the norm closure of the span of products z^*w for $z, w \in Z$. Here (ZZ^*, ZZ^*, Z, Z^*) is the weak* dense subcontext.

5. More generally, the tight Morita w^*-equivalence of [16, Section 5], is easily seen to be a special case of weak Morita equivalence. In this case, the equivalence bimodules X and Y are selfdual. Indeed, this selfduality is the great advantage of the approach of [16, Section 5].

6. The second duals of strongly Morita equivalent operator algebras are weakly Morita equivalent. Recall that if A and B are approximately unital operator algebras, then A^{**} and B^{**} are unital dual operator algebras, by 2.5.6 in [15]. If X is a non-degenerate operator A-B-bimodule, then X^{**} is a dual operator A^{**}-B^{**}-bimodule in a canonical way. Let $(\cdot,\cdot)$ be a bilinear map from $X \times Y$ to A that is balanced over B and is an A-bimodule map. Then notice that by 1.6.7 in [15], there is a unique separately w^*-continuous extension from $X^{**} \times Y^{**}$ to A^{**}, which we still call $(\cdot,\cdot)$. Now the weak Morita equivalence follows easily from Goldstine Theorem.

7. Any unital dual operator algebra M is weakly Morita equivalent to $\mathbb{M}_I(M)$, for any cardinal I. The weak* dense strong Morita subcontext in this case is $(M, \mathbb{K}_I(M), R_I(M), C_I(M))$, whereas the equivalence bimodules X and Y above are $R_I^w(M)$ and $C_I^w(M)$ respectively.

8. *TRO equivalent* dual operator algebras M and N, or more generally Δ-*equivalent algebras*, in the sense of [28, 29], are weakly Morita equivalent. If $M \subset B(H)$ and $N \subset B(K)$, then TRO equivalence means that there exists a TRO $Z \subset$

$B(H,K)$ such that $M = [Z^*NZ]^{\overline{w^*}}$ and $N = [ZMZ^*]^{\overline{w^*}}$. Eleftherakis shows that one may assume that Z is a WTRO and $1_N z = z 1_M = z$ for $z \in Z$. Define X and Y to be the weak* closures of MZ^*N and NZM respectively. Define A and B to be, respectively, Z^*NZ and ZMZ^*. Define X' and Y' to be, respectively, the norm closures of Z^*YZ^* and ZXZ. Since Z is a TRO, Z^*Z is a C^*-algebra, and so it has a contractive approximate identity (e_t) where $e_t = \sum_{k=1}^{n(t)} x_k^t y_k^t$ for some $y_k^t \in Z$, and $x_k^t = (y_k^t)^*$. It is easy to check that (e_t) is a cai for A, and a similar statement holds for B. Indeed it is clear that (A, B, X', Y') is a weak* dense strong Morita subcontext of (M, N, X, Y). Hence M and N are weakly Morita equivalent.

9. Examples of weak and weak* Morita equivalence may also be easily built as at the end of [11, Section 6], from a weak* closed subalgebra A of a von Neumann algebra M, and a strictly positive $f \in M_+$ satisfying a certain 'approximation in modulus' condition. Then the weak linking algebra of such an example is Morita equivalent in the same sense to A (see Section 3.3), but again, it seems unlikely that these are always weak* stably isomorphic.

10. An example from [30], two similar separably acting nest algebras are clearly weakly Morita equivalent by the facts presented around [30, Theorem 3.5] (Davidson's similarity theorem). Indeed in this case the Morita subcontext equals the context. However, Eleftherakis shows they need not be $\Delta-$*equivalent* (that is, weak* stably isomorphic [31]).

In the theory of strong Morita equivalence, and also in our setting, it is very important that N has some kind of approximate identity (f_s) of the form

$$f_s = \sum_{i=1}^{n_s} [y_i^s, x_i^s], \qquad \|[y_1^s, \cdots, y_{n_s}^s]\| \|[x_1^s, \cdots, x_{n_s}^s]^T\| < 1, \tag{3.2.1}$$

and similarly that M has some kind of approximate identity (e_t) of form

$$e_t = \sum_{i=1}^{m_t} (x_i^t, y_i^t), \qquad \|[x_1^t, \cdots, x_{m_t}^t]\| \|[y_1^t, \cdots, y_{m_t}^t]^T\| < 1. \tag{3.2.2}$$

Here $x_i^s, x_i^t \in X, y_i^s, y_i^t \in Y$. This clearly follows from Corollary 2.4.8.

In what follows, we say that $(\cdot,\cdot)$ is a *bimodule map* if $m(x,y) = (mx,y)$ and $(x,y)m = (x,ym)$ for all $x \in X, y \in Y, m \in M$.

Theorem 3.2.3. *(M,N,X,Y) is a weak* Morita context if and only if the following conditions hold: there exists a separately weak* continuous completely contractive M-bimodule map $(\cdot,\cdot) : X \times Y \to M$ which is balanced over N, and a separately weak* continuous completely contractive N-bimodule map $[\cdot,\cdot] : Y \times X \to N$ which is balanced over M, such that $(x,y)x' = x[y,x']$ and $y'(x,y) = [y',x]y$ for $x,x' \in X, y,y' \in Y$; and also there exist nets (f_s) in N and (e_t) in M of the form in* (3.1.1) *and* (3.1.2) *above, with $f_s \to 1_N$ and $e_t \to 1_M$ weak*.*

Proof. $(\Leftarrow)$ Under these conditions, we first claim that if $\pi : X \otimes_N^{\sigma h} Y \to M$ is the canonical (w^*-continuous) M-M-bimodule map induced by $(\cdot,\cdot)$, then $\pi(u)x \otimes_N y = u(x,y)$ for all $x \in X, y \in Y$, and $u \in X \otimes_N^{\sigma h} Y$. To see this, fix $x \otimes_N y \in X \otimes_N^{\sigma h} Y$. Define $f, g : X \otimes_N^{\sigma h} Y \to X \otimes_N^{\sigma h} Y : f(u) = u(x,y)$ and $g(u) = \pi(u)x \otimes_N y$ where $u \in X \otimes_N^{\sigma h} Y$. We need to show that $f = g$. Since $X \otimes_{hN} Y$ is w^*-dense in $X \otimes_N^{\sigma h} Y$, and f, g are w^*-continuous, it is enough to check that $f = g$ on $X \otimes_{hN} Y$. For $u = x' \otimes_N y'$, we have

$$u(x,y) = x' \otimes_N y'(x,y) = x' \otimes_N [y',x]y = x'[y',x] \otimes_N y = (x',y')x \otimes_N y = \pi(u)x \otimes_N y,$$

as desired in the claim.

To see that $M \cong X \otimes_N^{\sigma h} Y$, we shall show that π above is a complete isometry. Since $M = \overline{\text{Span}(\cdot,\cdot)}^{w^*}$, it will follow from the Krein-Smulian theorem that π maps onto M. Choose an approximate identity (e_t) for M of the form in (3.2.2). Define

$\rho_t : M \to X \otimes^{\sigma h}_N Y : \rho_t(m) = \sum_{i=1}^{n_t} m x_i^t \otimes_N y_i^t$. For $[u_{jk}] \in M_n(X \otimes^{\sigma h}_N Y)$, we have by the last paragraph that

$$\rho_t \circ \pi([u_{jk}]) = [\sum_{i=1}^{n_t} \pi(u_{jk}) x_i^t \otimes_N y_i^t] = [\sum_{i=1}^{n_t} u_{jk}(x_i^t, y_i^t)] = [u_{jk} e_t] \xrightarrow{w^*} [u_{jk}],$$

the convergence by [31, Lemma 2.3]. Since ρ_t is completely contractive, we have

$$\|[u_{jk} e_t]\| = \|(\rho_t \circ \pi)([u_{jk}])\| \le \|\pi([u_{jk}])\|.$$

As $[u_{jk}]$ is the w^*-limit of the net $([u_{jk} e_t])_t$, by Alaoglu's theorem we deduce that $\|[u_{jk}]\| \le \|\pi([u_{jk}])\|$. Similarly, $N \cong Y \otimes^{\sigma h}_M X$.

($\Rightarrow$) The existence of the nets (f_s) and (e_t) follows from Corollary 2.4.8. Let f and g be the pair of completely isometric w^*-homeomorphic bimodule isomorphisms $f : X \otimes^{\sigma h}_M Y \to M$ and $g : Y \otimes^{\sigma h}_M X \to N$. We write $(x, y) = f(x \otimes_M y)$ and $[y, x] = g(y \otimes_N x)$ for $x \in X, y \in Y$. These maps have all the desired properties except the relations $(x, y)x' = x[y, x']$ and $y'(x, y) = [y', x]y$ for $x, x' \in X$ and $y, y' \in Y$. We will show that f and g may be chosen so that the following two diagram commutes which will prove the desired relations.

$$\begin{array}{ccc} X \otimes^{\sigma h}_N Y \otimes^{\sigma h}_M X & \xrightarrow{f \otimes 1_X} & M \otimes^{\sigma h}_M X \\ \downarrow{\scriptstyle 1_X \otimes g} & & \downarrow{\scriptstyle \text{canon}} \\ X \otimes^{\sigma h}_N N & \xrightarrow{\text{canon}} & X \end{array} \qquad \begin{array}{ccc} Y \otimes^{\sigma h}_M X \otimes^{\sigma h}_N Y & \xrightarrow{g \otimes 1_Y} & N \otimes^{\sigma h}_N Y \\ \downarrow{\scriptstyle 1_Y \otimes f} & & \downarrow{\scriptstyle \text{canon}} \\ Y \otimes^{\sigma h}_M M & \xrightarrow{\text{canon}} & Y \end{array}$$

Let $a : M \otimes^{\sigma h}_M X \to X$ and $b : X \otimes^{\sigma h}_N N \to X$ be the canonical maps. Each arrow in the first diagram is a completely isometrically isomorphic and w^*-homeomorphic M-N-module map. Hence there exists a w^*-continuous completely isometric M-N-bimodule map $u : X \to X$ such that $b(1 \otimes g) = ua(f \otimes 1)$. By the Corollary 2.4.6, $T = u \otimes I_Y : X \otimes^{\sigma h}_N Y \to X \otimes^{\sigma h}_N Y$ is a w^*-continuous M-M-bimodule map of $X \otimes^{\sigma h}_N Y$. Using $X \otimes^{\sigma h}_N Y \cong M$, $CB^{\sigma}_M(M) \cong M$ and following similar technique as in Proposition 1.3 in [10] , one can show that T is a w^*-continuous unitary map in the C^*-algebra sense of that term. Now replace f by Tf, which is still a completely isometric w^*-homeomorphic M-M-isomorphism from $X \otimes_N^{\sigma h} Y \to M$. Now the standard algebraic argument (e.g., see 12.12.3 in [32]) shows that both diagrams commute.

□

Corollary 3.2.4. *Every weak Morita context is a weak* Morita context.*

Proof. Let $(M, N, X, Y, (\cdot,\cdot), [\cdot,\cdot])$ be a weak Morita context with strong Morita subcontext (A, B, X', Y'). If (f_s) is a cai for B it is clear that $f_s \to 1_N$ weak*. Indeed if a subnet $f_{s_\alpha} \to f$ in the weak* topology in N, then $bf = b$ for all $b \in B$. By weak* density it follows that $bf = b$ for all $b \in N$. Similarly $fb = b$. Thus $f = 1_N$. By Lemma 2.9 in [18] we may choose (f_s) of the form (3.2.1), and similarly A has a cai (e_t) of form in (3.2.2). That $(x, y)x' = x[y, x']$ and $y'(x, y) = [y', x]y$ for $x, x' \in X, y, y' \in Y$, follows by weak* density, and from the fact that the analogous relations hold in X' and Y'. Similarly by weak* density arguments, one sees that the bilinear maps are balanced bimodule maps. □

A key point for us, is that the condition involving (3.2.1) in Theorem 3.2.3 becomes a powerful tool when expressed in terms of an asymptotic factorization of I_Y through spaces of the form $C_n(N)$ (or $C_n(B)$ in the case of a weak Morita equivalence). Indeed, define $\varphi_s(y)$ to be the column $[(x_j^s, y)]_j$ in $C_{n_s}(N)$, for y in Y, and define $\psi_s([b_j]) = \sum_j y_j^s b_j$ for $[b_j]$ in $C_{n_s}(N)$. Then $\psi_s(\varphi_s(y)) = f_s y \to y$ in the weak* topology if $y \in Y$ (or in norm if $y \in Y'$, in the case of a weak Morita equivalence, in which case we can replace $C_{n_s}(N)$ by $C_n(B)$). Similarly, the condition involving (3.2.2) may be expressed, for example, in terms of an asymptotic factorization of I_X through spaces of the form $R_n(N)$ (or $R_n(B)$ in the weak Morita case), or through $C_n(M)$ (or $C_n(A)$).

Henceforth in this section, let $(M, N, X, Y, (\cdot,\cdot), [\cdot,\cdot])$ be as in Theorem 3.2.3. We will also refer to this 6-tuple as the weak* Morita context.

In the literature of Morita equivalence of rings in pure algebra, there is popular collection of theorems known as Morita I, II, III. Morita I may be described as the consequences of a pair of bimodules being mutual inverses ($X \otimes_B Y \cong A$ and $Y \otimes_A X \cong B$). For example, one of the Morita I theorem says that such pairs of inverse bimodules give rise, by tensoring, to module category equivalences (e.g., see 12.10 in [32]). We prove this Morita I theorem below. A version of the Morita II theorem will be proved in Chapter 5.

Theorem 3.2.5. *Weak* Morita equivalent dual operator algebras have equivalent categories of dual operator modules.*

Proof. Recall that ${}_M\mathcal{R}$ denotes the category of left dual operator modules over M. The morphisms are the weak* continuous completely bounded M-module maps. If $Z \in {}_N\mathcal{R}$ and if $\mathcal{F}(Z) = X \otimes^{\sigma h}_N Z$, then $\mathcal{F}(Z)$ is a left dual operator M-module by Lemma 2.4.7. That is, $\mathcal{F}(Z) \in {}_M\mathcal{R}$. Further, if $T \in CB^{\sigma}_N(Z, W)$, for $Z, W \in {}_N\mathcal{R}$, and if $\mathcal{F}(T)$ is defined to be $I \otimes_N T : \mathcal{F}(Z) \to \mathcal{F}(W)$, then by the functoriality of the normal module Haagerup tensor product we have $\mathcal{F}(T) \in CB^{\sigma}_M(\mathcal{F}(Z), \mathcal{F}(W))$, and $\|\mathcal{F}(T)\|_{cb} \leq \|T\|_{cb}$. Thus $\mathcal{F}$ is a contractive functor from ${}_N\mathcal{R}$ to ${}_M\mathcal{R}$. Similarly, we obtain a contractive functor $\mathcal{G}$ from ${}_M\mathcal{R}$ to ${}_N\mathcal{R}$. Namely, $\mathcal{G}(W) = Y \otimes^{\sigma h}_M W$, for $W \in {}_M\mathcal{R}$, and $\mathcal{G}(T) = I \otimes_M T$ for $T \in CB^{\sigma}_M(W, Z)$ with $W, Z \in {}_M\mathcal{R}$. Similarly, it is easy to check that these functors are completely contractive; for example, $T \mapsto \mathcal{F}(T)$ is a completely contractive map on each space $CB^{\sigma}_N(Z, W)$ of morphisms. If we compose $\mathcal{F}$ and $\mathcal{G}$, we find that for $Z \in {}_N\mathcal{R}$ we have $\mathcal{G}(\mathcal{F}(Z)) \in {}_N\mathcal{R}$. By Proposition 2.4.11 and Lemma 2.4.12, we have

$$\mathcal{G}(\mathcal{F}(Z)) \cong Y \otimes^{\sigma h}_M (X \otimes^{\sigma h}_N Z) \cong (Y \otimes^{\sigma h}_M X) \otimes^{\sigma h}_N Z \cong N \otimes^{\sigma h}_N Z \cong Z.$$

where the isomorphisms are completely isometric. The rest of the proof follows as in Theorem 3.9 in [18]. □

We shall adopt the convention from algebra of writing maps on the side opposite the one on which the ring acts on the module. For example a left A-module map will be written on the right and a right A-module map will be written on the left. The pairings and actions arising in the weak Morita context give rise to eight maps:

$$\begin{aligned}
R_N &: N \to CB_M(X,X), & xR_N(b) &= x \cdot b \\
L_N &: N \to CB(Y,Y)_M, & L_N(b)y &= b \cdot y \\
R_M &: M \to CB_N(Y,Y), & yR_M(a) &= y \cdot a \\
L_M &: M \to CB(X,X)_N, & L_M(a)x &= a \cdot x \\
R^M &: Y \to CB_M(X,M), & xR^M(y) &= (x,y) \\
L^N &: Y \to CB(X,N)_N, & L^N(y)x &= [y,x] \\
R^N &: X \to CB_N(Y,N), & yR^M(x) &= [y,x] \\
L^M &: X \to CB(Y,M)_M, & L^M(x)y &= (x,y)
\end{aligned}$$

The first four maps are completely contractive since module actions are completely contractive. Also the maps L_N and L_M are homomorphisms and R_N and R_M are anti-homomorphisms. Similar proofs to the analogous results in [18] show that R^M, L^N, R^N, and L^M are completely contractive.

Theorem 3.2.6. *If $(M, N, X, Y, (\cdot,\cdot), [\cdot,\cdot])$ is a weak* Morita context, then each of the maps R^M, R^N, L^M and L^N is a weak* continuous complete isometry. The range of R^M is $CB^{\sigma}_M(X,M)$, with similar assertions holding for R^N, L^M and L^N. The map L_N (resp. R_N) is a w^*-continuous completely isometric isomorphism (resp. anti-isomorphism) onto the w^*-closed left (resp. right) ideal $CB^{\sigma}(Y)_M$ (resp. $CB^{\sigma}_M(X)$). The latter also equals the left multiplier algebra (see [15, Chapter 4]) $\mathcal{M}_\ell(Y)$ (resp. $\mathcal{M}_r(X)$). Similar results hold for L_M and R_M.*

Proof. Most of this can be proved directly, as in [18, Theorem 4.1]. For example, firstly we will show that L^N is a complete isometry. Choose a net (e_t) for M as in (3.2.2). As we said earlier, $ye_t \to y$ in the weak* topology for all $y \in Y$. Thus from Alaoglu's theorem we deduce that $\|y\| \leq \sup_t \|ye_t\|$. However

$$\begin{aligned}
\|ye_t\| &= \|\sum_{i=1}^{n_t} y(x_i^t, y_i^t)\| \\
&= \|\sum_{i=1}^{n_t} [y, x_i^t] y_i^t\| \\
&\leq \|[[y, x_1^t], [y, x_2^t], \cdots, [y, x_{n_t}^t]]\| \|[y_1^t, y_2^t, \cdots, y_{n_t}^t]^T\| \\
&\leq \|L^N(y)\|_{cb}.
\end{aligned}$$

Now take the supremum on the left hand side, to conclude that L^N is an isometry. The matricial version is similar. Now we will show that L_N is a complete isometry. Choose a net (f_s) for N as in (3.2.1). By the Theorem 3.2.3, $f_s \to 1_N$ in the w^*-topology of N. Note that for $b \in N$ we have $bf_s = \sum_{i=1}^{n_s)} [L_N(b) y_i^s, x_i^s]$. Thus

$$\begin{aligned}
\|bf_s\| &\leq \|[L_N(b)(y_1^s), L_N(b)(y_2^s), \cdots, L_N(b)(y_{n_s}^s)]\| \|[x_1^s, x_2^s, \cdots, x_{n_s}^s]^T\| \\
&\leq \|L_N(b)\|_{cb}.
\end{aligned}$$

As $bf_s \to b$ in the weak* topology in N, it follows from the above that $\|b\| \leq \|L_N(b)\|_{cb}$ which proves that L_N is isometric. A similar calculation shows that L_N is a complete isometry. To see that L_N maps onto $CB^\sigma(Y)_M$, let $T \in CB^\sigma(Y)_M$. With notation as in (3.2.1), consider the net with terms $\sum_{i=1}^{n_s} [T(y_i^s), x_i^s]$ is bounded, and so by Alaoglu's theorem has a weak* convergent subnet converging to η, say, in the w^*-topology of N. We may assume that the subnet is the entire net. Then, since L_N is w^*-continuous, for $y \in Y$ we have

$$\begin{aligned} L_N(\eta)(y) &= w^* \lim \sum_{i=1}^{n_s} [T(y_i^s), x_i^s] y \\ &= w^* \lim \sum_{i=1}^{n_s} T(y_i^s)(x_i^s, y) \\ &= w^* \lim \sum_{i=1}^{n_s} T(y_i^s(x_i^s, y)) \\ &= w^* \lim \sum_{i=1}^{n_s} T([y_i^s, x_i^s] y) \\ &= w^* \lim T(\sum_{i=1}^{n_s} [y_i^s, x_i^s] y) \\ &= T(y). \end{aligned}$$

Similar arguments work for the other seven maps.

We can also deduce the above theorem from the functoriality (Theorem 3.2.5). For example, because of the equivalence of categories via the functor $\mathcal{F} = Y \otimes_M^{\sigma h} -$, we have completely isometrically

$$M \cong CB_M^\sigma(M) \cong CB_N^\sigma(\mathcal{F}(M)) \cong CB_N^\sigma(Y),$$

and the composition of these maps is easily seen to be R_M. Thus R_M is a complete isometry. Similar proofs work for the other seven maps. To see that L_N is w^*-continuous, for example, let (b_t) be a bounded net in N converging in the w^*-topology of N to $b \in N$. Then $L_N(b_t)$ is a bounded net in $CB(Y)_M$. As the module action is separately w^*-continuous, it is easy to see that $L_N(b_t)$ converges to $L_N(b)$ in the w^*-topology. Thus L_N is a w^*-continuous isometry with w^*-closed range, by the Krein-Smulian theorem. To see that its range is a left ideal simply use the weak* density of the span of terms $[y, x]$ in N, and the equation $TL_N([y, x])(y') = L_N[Ty, x](y')$ for $T \in CB(Y, Y)_M$, $y' \in Y$. The variants for the other maps follow similarly.

To see the assertions involving multiplier algebras, note that we have obvious completely contractive maps

$$N \longrightarrow \mathcal{M}_\ell(Y) \longrightarrow CB^\sigma(Y)_M.$$

The first of these arrows arises since Y is a left operator N-module (see [15, Theorem 4.6.2]). The second arrow always exists by general properties (see e.g., [15, Chapter 4], or Theorem 4.1 in [16]) of the left multiplier algebra of a dual operator module. Both arrows are weak* continuous by e.g., Theorem 4.7.4 (ii) and 1.6.1 in [15]. Since $N \cong CB^\sigma(Y)_M$ completely isometrically and w^*-homeomorphically, we deduce that these spaces coincide with $\mathcal{M}_\ell(Y)$ too. $\square$

Remark. Note that in the case of weak Morita equivalence, $CB_A(X')$ is an operator algebra ([18], Theorem 4.9). It is not true in general that $CB_M(X)$ is an operator algebra. Nonetheless, the above shows that $CB^\sigma_M(X)$ is a dual operator algebra ($\cong N$). The following example, suggested by David Blecher, shows that in general $CB_M(X)$ is not an operator algebra.

Example. Let $M \subset M_2(B(H))$ be the algebra of matrices of the form $\begin{pmatrix} \lambda I_H & T \\ 0 & \mu I_H \end{pmatrix}$, where $T \in B(H)$ and $\lambda, \mu \in \mathbb{C}$. Let $N= \mathbb{M}(M)$, $X = R^w(M)$ and $Y = C^w(M)$. Matrix multiplication define pairings $(\cdot,\cdot)$ from $X \times Y$ into M and $[\cdot,\cdot]$ from $Y \times X$ into N. From Example (7) it follows that $(M, N, X, Y, (\cdot,\cdot), [\cdot,\cdot])$ is a weak Morita context.

From simple calculations we have that, $CB(C^w(M), M)_M$ is the space of completely bounded right M-module maps $f : C^w(M) \to M$ that takes $\begin{bmatrix} \lambda_1 & T_1 \\ 0 & \mu_1 \\ \lambda_2 & T_2 \\ 0 & \mu_2 \\ \vdots & \vdots \end{bmatrix} \mapsto$

$\begin{bmatrix} \vec{\lambda}.\vec{w} & \vec{r}.\vec{\mu} + g(\vec{T}) \\ 0 & \vec{\mu}.\vec{\nu} + \varphi(\vec{T}) \end{bmatrix}$ where $\lambda_i, \mu_i \in \mathbb{C}$, $T_i \in B(H)$, $\vec{r} \in R^w(B(H))$, $g : C^w(B(H)) \to B(H)$ is completely bounded and $g|_{C(B(H))} = \vec{w}.-$ and $\vec{w} \in l^2$, $\vec{\nu} \in l^2$, $\varphi \in C^w(B(H))^*$, $\varphi \perp C(B(H))$.

Lemma 3.2.7. *Let f and φ be as above. If f is w^*-continuous, so is φ.*

Proof. Let $\vec{c_t} \to \vec{c}$ in the w^*-topology of $C^w(B(H))$. Let $x_t = \begin{bmatrix} 0 & \vec{c_t} \\ 0 & 0 \end{bmatrix}$, $x = \begin{bmatrix} 0 & \vec{c} \\ 0 & 0 \end{bmatrix}$. Then $x_t \to x$ in the w^*-topology of $C^w(M)$ which implies $f(x_t) \to f(x)$ in the w^*-topology in M. Hence the $(2-2)$ entry of $f(x_t)$ converges to 2-2 entry of $f(x)$ in the w^*-topology, which implies $\varphi(\vec{c_t}) \to \varphi(\vec{c})$. □

Lemma 3.2.8. *There exists $f \in CB(C^w(M), M)_M$, and $T \in CB(C^w(M), C^w(M))_M$ which are not w^*-continuous.*

Proof. Choose $\varphi \in C^w(B(H))^*$ which is not w^*-continuous, and $\varphi \perp C(B(H))$. Then define $f \in CB(C^w(M), M)_M$ by

$$f\left(\begin{bmatrix} \lambda_1 & T_1 \\ 0 & \mu_1 \\ \lambda_2 & \mu_2 \\ 0 & \mu_2 \\ \vdots & \vdots \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & \varphi\left(\begin{bmatrix} T_1 \\ T_2 \\ \vdots \end{bmatrix}\right) \end{bmatrix}.$$

If f were w^*-continuous, then so is φ by above Lemma, which is false. Similarly if we define

$$T\left(\begin{bmatrix} m_1 \\ m_2 \\ \vdots \end{bmatrix}\right) = \begin{bmatrix} f\left(\begin{bmatrix} m_1 \\ m_2 \\ \vdots \end{bmatrix}\right) \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

where $m_i \in M$. Then T is not w^*-continuous as f is not. □

Corollary 3.2.9. *In the above example $C^w(M)$ is not a selfdual module over M.*

Proof. If $C^w(M)$ is a selfdual module over M, then every $f \in CB(C^w(M), M)_M$ is given by multiplication with a row vector in $R^w(M)$ which are clearly w^*-continuous. But this contradicts Lemma 3.2.8. $\square$

Corollary 3.2.10. *Let M be as above. Then $C^w(M)$ is not a rigged module over M.*

Proof. If $C^w(M)$ is a rigged module, then from Corollary 5.7 in [16], $C^w(M)$ is a selfdual module over M which is a contradiction. $\square$

Corollary 3.2.11. *Let M be as above. Then $CB(C^w(M))_M \ncong \mathbb{M}(M)$.*

Proposition 3.2.12. *Let M be as above. Then $CB(C^w(M))_M$ is not an operator algebra.*

Proof. Consider the map $\theta : C^w(M) \to CB(C^w(M))_M$ defined as $\theta(\vec{c})(\begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}) = \vec{c}b_1$ where $\vec{c}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} \in C^w(M)$. Also consider the map $\phi : CB(C^w(M), M)_M \to CB(C^w(M))_M$ defined as $\phi(f)(\vec{b}) = \begin{bmatrix} f(\vec{b}) \\ 0 \\ \vdots \end{bmatrix}$ for $f \in CB(C^w(M), M)_M$ and $\vec{b} \in C^w(M)$. It is easy to check that θ and ϕ are completely isometric. Hence $C^w(M)$ and $CB(C^w(M), M)_M$ may be identified with the subspaces of $CB(C^w(M))_M$ canonically (i.e. via the range of θ and ϕ respectively). For $f \in CB(C^w(M), M)_M$ and $\vec{b} \in C^w(M)$, the pairing $(f, \vec{b})$ is identified with $\phi(f)\theta(\vec{b})$ in $CB(C^w(M))_M$. For $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \end{bmatrix} \in C^w(M)$, $\phi(f)\theta(\vec{b})(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \end{bmatrix}) = \phi(f)(\vec{b}c_1) = \begin{bmatrix} f(\vec{b}c_1) \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} f(\vec{b})c_1 \\ 0 \\ \vdots \end{bmatrix}$. Therefore

$\phi(f)\theta(\vec{b})$ can be identified with $f(\vec{b})$ in M. Therefore, if $CB(C^w(M))_M$ is an operator algebra then the canonical map $\pi : CB(C^w(M), M)_M \otimes_h C^w(M) \to M$ that takes the pairing $(f, \vec{b})$ to $f(\vec{b})$ where $f \in CB(C^w(M), M)_M$ and $\vec{b} \in C^w(M)$ will be completely contractive. As $C^w(B(H))^* \subseteq CB(C^w(M), M)_M$ completely isometrically, therefore the restriction $\pi : C^w(B(H))^* \otimes_h C^w(B(H)) \to \mathbb{C}$ that takes the pairing $(f, \vec{b})$ to $f(\vec{b})$ where $f \in C^w(B(H))^*$ and $\vec{b} \in C^w(B(H))$, is completely contractive too. Let $N = C(B(H))$ which is a closed subspace of $C^w(B(H))$. As the map $\pi : N^\perp \otimes_h C^w(B(H)) \to \mathbb{C}$ is completely contractive, by a well known consequence of the factorization theorem for completely bounded bilinear functionals (see Corollary 9.4.2 in [10]) there exists a Hilbert space K and completely contractive mappings $\psi : N^\perp \to K_c$ and $\varphi : C^w(B(H)) \to (K_c)^*$ such that $\pi(f, b) = \psi(f)\varphi(b) = f(b)$ for $f \in N^\perp$ and $b \in C^w(B(H))$. From this it is easy to check that $\varphi^*\psi : N^\perp \to C^w(B(H))^*$ is the identity map, hence ψ is completely isometric. Thus $N^\perp$ is a column Hilbert space, hence $C^w(B(H))/C(B(H))$ is a column Hilbert space too. Let J be any set or cardinal. Without loss of generality let $H = l^2_J$. Then $B(H) \cong \mathbb{M}_J$ and $S^\infty(H) \cong \mathbb{K}_J$. Define a map from $\mathbb{M}_J \to C^w(\mathbb{M}_J)/C(\mathbb{M}_J)$ that takes an infinite matrix $(a_{ij})_{i,j\in J}$ to $A + C(\mathbb{M}_J)$, where A is an infinite column with each entry an infinite matrix supported in only one row (which is a row of $(a_{ij})_{i,j\in J}$). This is a completely isometric map with kernel $\mathbb{K}_J$, hence $\mathbb{M}_J/\mathbb{K}_J \hookrightarrow C^w(\mathbb{M}_J)/C(\mathbb{M}_J)$ completely isometrically. This implies that the Calkin algebra $B(H)/S^\infty(H)$ is a column Hilbert space, which is a contradiction since the Calkin algebra is a C^*-algebra. □

Theorem 3.2.13. *If M and N are weak* Morita equivalent dual operator algebras, then their centers are completely isometrically isomorphic via a w^*-homeomorphism.*

Proof. By Theorem 3.2.6 there is a w^*-continuous complete isometry $R_M : M \to CB^\sigma_N(Y)$. The restriction of R_M to the center $Z(M)$ of M maps into $CB^\sigma(Y)_M \cong N$, and so we have defined a w^*-continuous completely isometric homomorphism θ :

$Z(M) \to N$. One easily sees that $\theta(a)(y) = ya$, for $a \in Z(M)$. It is also easy to see that this implies that θ maps into $Z(N)$, and to argue, by symmetry, that θ must be an isomorphism. $\square$

Lemma 3.2.14. *Suppose that $T : E^* \to F^*$ is a contractive, one-one, surjection between dual Banach spaces, such that T^{-1} is weak* continuous, and such that T^{-1} restricts to an isometry on a subspace Y of F^*. Suppose also that the unit ball of Y is weak* dense in the unit ball of F^*. Then T is an isometry.*

Proof. Suppose that $\|T(x)\| \leq \delta < 1$, where $\|x\| = 1$. Then there exists a net $(y_\lambda) \in Y$ with $\|y_\lambda\| \leq \delta$ such that $y_\lambda \xrightarrow{w^*} T(x)$. Since T^{-1} is weak* continuous, $T^{-1}(y_\lambda) \xrightarrow{w^*} x$, and $\|T^{-1}(y_\lambda)\| = \|y_\lambda\| \leq \delta$. This implies that $\|x\| \leq \delta$ which is a contradiction. $\square$

Lemma 3.2.15. *In the case of weak Morita equivalence, if Z is a left dual operator N-module, then the canonical map $Y' \otimes_{hB} Z \to Y \otimes_N^{\sigma h} Z$ is completely isometric, and it maps the ball onto a w^*-dense set in* $\mathrm{Ball}(Y \otimes_N^{\sigma h} Z)$.

Proof. The canonical map here is completely contractive, let us call it θ. On the other hand, let φ_s, ψ_s be as defined just below Corollary 3.2.4, with $\psi_s(\varphi_s(y)) = f_s y \to y$. Then for $u \in M_n(Y' \otimes_B Z)$, we have

$$\|\theta_n(u)\| \geq \|(\varphi_s \otimes I)_n(\theta_n(u))\| = \|(\varphi_s \otimes I)_n(u)\| \geq \|f_s u\|.$$

Taking a limit over s, gives $\|\theta_n(u)\| \geq \|u\|$.

Let $u \in \mathrm{Ball}(Y \otimes_N^{\sigma h} Z)$. By Corollary 2.4.8, there exists a net (u_t) in the image of $\mathrm{Ball}(Y \otimes_{hN} Z)$ such that $u_t \xrightarrow{w^*} u$. We rewrite (3.1.1) and the lines below it, namely write each f_s in the form $[y, x]$ (in suggestive notation), for $y \in \mathrm{Ball}(R_m(Y'))$ and $x \in \mathrm{Ball}(C_m(X'))$.

Note, $w \odot z$ is the weak* limit of terms $f_s w \odot z$, and $e_t w \odot z = y \odot v$, where v is a column with k^{th} entry $\sum_j [x_k, w_j] z_j$. It is easy to check that $\|v\| \leq 1$. $\square$

Proposition 3.2.16. *Weak* Morita equivalence is an equivalence relation.*

Proof. This follows the usual lines, for example the transitivity follows from associativity of the normal module Haagerup tensor product and Lemma 2.4.12. □

Remark. Concerning transitivity of weak Morita equivalence, it is convenient to consider Definition 3.2.2 as defining an equivalence between pairs (M, A) and (N, B), as opposed to just between M and N. That is we also consider the weak* dense operator subalgebras, in the relation discussed in the next proposition.

Proposition 3.2.17. *Weak Morita equivalence is an equivalence relation.*

Proof. Reflexivity is a consequence of the Example (7) : In the notation there, take $X = R_1(A)$, $Y = C_1(A)$, and $B = M_1(A)$, with both $(\cdot,\cdot)$ and $[\cdot,\cdot]$ given by multiplication in A. Symmetry is evident. The only thing that requires work is transitivity. Suppose that L is weakly Morita equivalent to M and M is weakly Morita equivalent to N. Let $(L, M, X, Y, (\cdot,\cdot)_1, [\cdot,\cdot]_1)$ and $(M, N, W, Z, (\cdot,\cdot)_2, [\cdot,\cdot]_2)$ be weak Morita contexts with weak*-dense strong Morita sub-contexts $(A, B, X', Y', (\cdot,\cdot)_1, [\cdot,\cdot]_1)$ and $(B, C, W', Z', (\cdot,\cdot)_2, [\cdot,\cdot]_2)$ respectively. Set $U' = X' \otimes_{hB} W'$ and $V' = Z' \otimes_{hB} Y'$. Define $(\cdot,\cdot) : U' \times V' \to A$ by the formula $((x' \otimes_B w'), (z' \otimes_B y')) = (x', (w', z')_2 y')_1$ $= (x'(w', z')_2, y')_1$. Similarly define $[\cdot,\cdot] : V' \times U' \to$ by the formula $[z' \otimes_B y', x' \otimes_B w'] =$ $[z', [y', x']_1 w']_2 = [z'[y', x']_1, w']_2$. Then by Proposition 3.7 in [18], $(A, C, U', V', (\cdot,\cdot), [\cdot,\cdot])$ is a strong Morita context. Define $U = X \otimes^{\sigma h}_M W$ and $V = Z \otimes^{\sigma h}_M Y$, then by Corollary 2.4.8, U' and V' are w^*-dense in U and V respectively. From Proposition 2.4.3, there is a completely contractive map from $U \otimes^{\sigma h}_N V = X \otimes^{\sigma h}_M W \otimes^{\sigma h}_N Z \otimes^{\sigma h}_M Y \to X \otimes^{\sigma h}_M M \otimes^{\sigma h}_M Y$ $\cong X \otimes^{\sigma h}_M Y \to L$. By Proposition 2.2 in [31], this gives rise to a separately w^*-continuous, completely contractive, N-balanced, bilinear map $(\cdot,\cdot) : U \times V \to L$. Similarly there is a separately w^*-continuous, completely contractive, L-balanced, bilinear map $[\cdot,\cdot] : V \times U \to N$. Thus $(L, N, U, V, (\cdot,\cdot), [\cdot,\cdot])$ is a weak Morita context, which proves transitivity. □

We would like to thank G. Pisier for the following Lemma.

Lemma 3.2.18. (Pisier) *An operator space E is a Hilbert column space if and only if $E \otimes_h C_n \cong C_n(E)$ isometrically via the canonical map, for all $n \in \mathbb{N}$.*

Theorem 3.2.19. *Weak Morita equivalent dual operator algebras have equivalent categories of normal Hilbert modules. Moreover, the equivalence preserves the subcategory of modules corresponding to completely isometric normal representations.*

Proof. Suppose that H is a Hilbert space on which M is normally represented. We claim that $Y \otimes_M^{\sigma h} H^c$ is a column Hilbert space. By Lemma 3.2.18, it suffices to show that the canonical map $C_n(Y \otimes_M^{\sigma h} H^c) \to (Y \otimes_M^{\sigma h} H^c) \otimes_h C_n$ is isometric for all $n \in \mathbb{N}$. Now $C_n(Y \otimes_M^{\sigma h} H^c) \cong C_n(Y) \otimes_M^{\sigma h} H^c$, and $(Y \otimes_M^{\sigma h} H^c) \otimes_h C_n \cong Y \otimes_M^{\sigma h} C_n(H^c)$. Thus we need to show that the canonical map $C_n(Y) \otimes_M^{\sigma h} H^c \to Y \otimes_M^{\sigma h} C_n(H^c)$ is isometric. Define a map $\theta : (C_n(Y) \otimes_{hM} H^c)^*_\sigma \to (Y \otimes_{hM} C_n(H^c))^*_\sigma$ by $\theta(T)(y, (n_i)) = \sum_{i=1}^n T(y_i, n_i)$, where y_i is an n-tuple with y in the ith coordinate and otherwise zero. It is easy to check that θ is contractive. Define $\phi : (Y \otimes_{hM} C_n(H^c))^*_\sigma \to (C_n(Y) \otimes_{hM} H^c)^*_\sigma$ by $\phi(T)([y_1 \ y_2 \cdots y_n]^t, \eta) = \sum_{i=1}^n T(y_i, \eta_i)$, where η_i is an n-tuple with ζ in the ith coordinate and otherwise zero. Again it is easy to check that ϕ is bounded, and $\theta = \phi^{-1}$. Dualizing the map θ, we get a contractive, one-one, surjective map $\rho = \theta^* : Y \otimes_M^{\sigma h} C_n(H^c) \to C_n(Y) \otimes_M^{\sigma h} H^c$ with $\rho^{-1} = \phi^*$. Hence ρ and ρ^{-1} are weak* continuous. Since by Theorem 3.10 in [18], $Y' \otimes_{hA} H^c$ is a Hilbert space, Lemma 3.2.18 will yield, analogously to the above, that $Y' \otimes_{hA} C_n(H^c) = C_n(Y') \otimes_{hA} H^c$ isometrically. That is, ρ^{-1} is an isometry when restricted to $C_n(Y') \otimes_{hA} H^c$. Now the above claim follows from Lemma 3.2.15 and Lemma 3.2.14.

To see that N is normally represented on the Hilbert space $Y \otimes_M^{\sigma h} H^c$, suppose that (n_t) is a net in N converging in the w^*-topology to an element $n \in N$. Then $n_t y \to ny$ in the w^*-topology of Y, for any $y \in Y$. By Proposition 2.1 in [31], $n_t y \otimes_M \eta \to ny \otimes_M \eta$ for all $\eta \in H^c$ in the w^*-topology of K. Since for Hilbert spaces the weak topology and weak* topology coincide, and since finite sums of rank one tensors are dense in $Y \otimes^{\sigma h}_M H^c$, it is evident that $n_t\zeta \to n\zeta$ weakly in K, for any $\zeta \in Y \otimes^{\sigma h}_M H^c$. Thus $Y \otimes^{\sigma h}_M H^c$ is a normal Hilbert N-module. The last assertion is presented in the next theorem. □

Theorem 3.2.20. *Weak* Morita equivalent dual operator algebras have equivalent categories of normal Hilbert modules. Moreover, the equivalence preserves the subcategory of modules corresponding to completely isometric normal representations.*

Proof. If H is a normal Hilbert M-module, let $K = Y \otimes^{\sigma h}_M H^c$. By the discussion just below Corollary 3.2.4, combined with Corollary 2.4.6, there are nets of maps $\varphi_s : K \to C_{n_s}(M) \otimes^{\sigma h}_M H^c \cong C_{n_s}(H^c)$, and maps $\psi_s : C_{n_s}(H^c) \to K$, with $\psi_s(\varphi_s(z)) = f_s z \to z$ weak* for all $z \in K$. Here (f_s) is as in (3.1.1). Let Λ be the directed set indexing s, and let $\mathcal{U}$ be an ultrafilter on Λ with the property that $\lim_{\mathcal{U}} z_s = \lim_{\Lambda} z_s$ for scalars z_s, whenever the latter limit exists. Let $H_{\mathcal{U}}$ be the ultraproduct of the spaces $C_{n_s}(H^c)$, which is a column Hilbert space, as is well known. Define $T : K \to H_{\mathcal{U}}$ by $T(x) = (\varphi_s(x))_s$, for $x \in K$. This is a complete contraction. To see that it is an isometry, note that for any $x \in K, \rho \in \text{Ball}(K_*)$, we have

$$|\rho(x)| = \lim_{\mathcal{U}} |\rho(\psi_s(\varphi_s(x)))| \leq \lim_{\mathcal{U}} \|\varphi_s(x)\| = \|T(x)\|.$$

Similarly, T is a complete isometry. This proves that K is a column Hilbert space.

That $K = Y \otimes^{\sigma h}_M H^c$ is a normal Hilbert N-module follows from Theorem 3.2.5. Finally, suppose that M is a weak* closed subalgebra of $B(H)$, we will show that the induced representation ρ of N on K is completely isometric. Certainly this map is completely contractive. If $[b_{pq}] \in M_d(N)$, $[y_{kl}] \in \text{Ball}(M_m(Y))$, $[\zeta_{rs}] \in \text{Ball}(M_g(H^c))$, $[x_{ij}] \in \text{Ball}(M_n(X))$, then

$$\|[\rho(b_{pq})]\| \geq \|[b_{pq}y_{kl} \otimes \zeta_{rs}]\| \geq \|[(x_{ij}, b_{pq}y_{kl})\zeta_{rs}]\|.$$

Taking the supremum over all such $[\zeta_{rs}]$, gives

$$\|[\rho(b_{pq})]\| \geq \sup\{\|[(x_{ij}, b_{pq}y_{kl})]\| : [x_{ij}] \in \text{Ball}(M_n(X))\} = \|[b_{pq}y_{kl}]\|,$$

by Theorem 3.2.6. Taking the supremum over all such $[y_{kl}] \in \text{Ball}(M_m(Y))$ gives $\|[\rho(b_{pq})]\| \geq \|[b_{pq}]\|$, by Theorem 3.2.6 again. □

We summarize: the last result shows that weak* Morita equivalent operator algebras have equivalent categories of normal representations. It would be interesting to characterize when two operator algebras have equivalent categories of normal representations.

Corollary 3.2.21. *If $H \in {}_M\mathcal{H}$ then $Y \otimes^{\sigma h}_M H^c = Y \otimes_{hM} H^c = Y \widehat{\otimes}_M H^c$ completely isometrically. These are column Hilbert spaces. Here $\widehat{\otimes}_M$ is as in* 3.4.2 *of* [15]. *In the case of weak Morita equivalence, these also equal $Y' \otimes_{hA} H^c = Y' \widehat{\otimes}_A H^c$.*

Proof. By Lemma 3.2.15 and Corollary 2.4.8, $Y \otimes_{hM} H^c = Y \otimes^{\sigma h}_M H^c$. Note that since $- \otimes_h H^c = - \widehat{\otimes} H^c$ (see e.g. [26, Proposition 9.3.2]), we may replace $\otimes_{hM}$ here by $\widehat{\otimes}_M$ (where $\widehat{\otimes}_M$ is the module projective tensor product e.g., see 3.4.2 of [15]). The assertions involving Y' follow in a similar way, by Lemma 3.2.15. Note that in this case, if $\eta \in H \ominus [AH]$ and (e_t) is a cai for A, then

$$\langle \eta, \eta \rangle = \lim_t \langle e_t\eta, \eta \rangle = 0.$$

Thus A acts nondegenerately on H. □

3.3 Representations of the linking algebra

In this section again, $(M, N, X, Y, (\cdot, \cdot), [\cdot, \cdot])$ is a weak* Morita context. Suppose that M is represented as a weak*-closed nondegenerate subalgebra of $B(H)$, for a Hilbert space H. Then by Corollary 3.2.21, $K = Y \otimes^{\sigma h}_M H^c$ is a column Hilbert space. Define a right M-module map $\Phi : Y \to B(H,K)$ by $\Phi(y)(\zeta) = y \otimes_M \zeta$ where $y \in Y$ and $\zeta \in H$. It is easy to see that Φ is a completely contractive N-M-bimodule map. It is weak* continuous, since if we have a bounded net $y_t \to y$ weak* in Y, and if $\zeta \in H$, then $y_t \otimes_M \zeta \to y \otimes_M \zeta$ weakly by [31]. That is, $\Phi(y_t) \to \Phi(y)$ in the WOT, and it follows that Φ is weak* continuous. If $\|\Phi(y)\| \leq 1$, and if $\zeta \in \text{Ball}(H^{(n)})$, and $[x_{ij}] \in \text{Ball}(M_n(X))$, then

$$\|[(x_{ij}, y)]\zeta\| = \|[x_{ij} \otimes \Phi(y)]\zeta\|\| \leq \|\Phi(y)\|.$$

Taking the supremum over such ζ, and then over such $[x_{ij}]$, we obtain from Theorem 3.2.6 that $\|y\| \leq 1$. Thus Φ is an isometry, and a similar but more tedious argument shows that Φ is a complete isometry. By the Krein-Smulian theorem we deduce that the range of Φ is weak* closed. A lengthy but similar argument, shows that the map $\Psi : X \to B(K,H)$, defined by $\Psi(x)(y \otimes \zeta) = (x,y)\zeta$, is a w^*-continuous completely isometric M-N-bimodule map. As we said in Theorem 3.2.20, the induced normal representation $N \to B(K)$ is completely isometric.

We use the above to define the direct sum $M \oplus^c Y$ as follows. For specificity, we want to take H to be a universal normal representation of M, that is the restriction to M of a one-to-one normal representation of $W^*_{\max}(M)$. Define a map $\theta : M \oplus^c Y \to B(H, K \oplus H)$ by $\theta((m,y))(\zeta) = (m\zeta, y \otimes_M \zeta)$, for $y \in Y, m \in M, \zeta \in H$. One can quickly check that θ is a one-to-one, M-module map, and that θ is a weak* continuous complete isometry when restricted to each of Y and M. Also, $W = \text{Ran}(\theta)$ is easily seen to be weak* closed. We norm $M \oplus^c Y$ by pulling back the operator space structure from W via θ. Thus $M \oplus^c Y$ may be identified with the weak* closed right M-submodule W of $B(H, H \oplus K)$; and hence it is a dual operator M-module. In a similar way, we define $M \oplus^r X$ to be the canonical weak* closed left M-submodule of $B(H \oplus K, H)$.

We next define the *'weak linking algebra'* of the context, namely

$$\mathcal{L}^w = \left\{ \begin{bmatrix} a & x \\ y & b \end{bmatrix} : \; a \in M, b \in N, x \in X, y \in Y \right\},$$

with the multiplication with the given by the formula

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix} \begin{pmatrix} a' & x' \\ y' & b' \end{pmatrix} = \begin{pmatrix} aa' + (x, y') & ax' + xb' \\ ya' + by' & [y, x'] + bb' \end{pmatrix}$$

As in [18, Lemma 5.6], one easily sees that there is at most one possible sensible dual operator space structure on this linking algebra. Indeed if Λ is the set indexing t in the net in (3.2.2), and if $\beta, t \in \Lambda$, then define $\theta^{\beta,t}$ on the linear space $\mathcal{L}^w$ to be the map θ^β in [18, p. 45], but with all the y_i^β replaced by y_i^t. Then a simple modification of the argument in [18, p. 50-51], and using semicontinuity of the norm in the weak* topology, yields that any 'sensible' norm assigned to $\mathcal{L}^w$ must agree with $\sup_{\beta,t} \|\theta^{\beta,t}(\cdot)\|$.

That such a dual operator space structure does exist, one only need view $\mathcal{L}^w$ as a subalgebra $\mathcal{R}$ of $B(H \oplus K)$, using the obvious pairings $X \times K \to H$ (induced by $(\cdot, \cdot)$), $Y \times H \to K$, and $N \times K \to K$ (this is the induced representation of N on K from Theorem 3.2.20). It is easy to check that $(M, \mathcal{R}, M \oplus^r X, M \oplus^c Y)$ is also a weak* Morita context (this follows from norm equalities of the kind in e.g., the centered equations in [18, Theorem 5.12]). This all may be most easily visualized by picturing both contexts as 3×3-matrices, namely as subalgebras of $B(H \oplus H \oplus K)$. Theorem 3.2.6 gives $\mathcal{R} \cong CB^\sigma(M \oplus^c Y)_M$ completely isometrically and w^*-homeomorphically.

Note that in a weak Morita situation, the linking operator algebra of the strong Morita context (A, B, X', Y') can be identified completely isometrically as the obvious weak* dense subalgebra $\mathcal{L}$ of $\mathcal{R}$ (see e.g. [6, Proposition 6.10]). Incidentally, at this point we have already proved the assertion made at the start of Example (1) in Section 3.2, and indeed that every weak Morita equivalence arises as the weak* closure of a strong Morita equivalence, or can be viewed as the weak* closure, in some representation, of the linking operator algebra of a strong Morita equivalence. We have a strong Morita context $(A, \mathcal{L}, A \oplus^r X', A \oplus^c Y')$ (see [18, 17]), which can be viewed as a subcontext of $(M, \mathcal{R}, M \oplus^r X, M \oplus^c Y)$. Thus the latter is a weak Morita context.

Extracting from the last paragraphs, we have:

Corollary 3.3.1. *M is weak* Morita equivalent to the weak linking algebra $\mathcal{L}^w$. Indeed this is a weak Morita equivalence if (M, N, X, Y) is a weak Morita context.*

It is often useful here to know that:

Proposition 3.3.2. *With notation as in Theorem* 3.2.20*, we have $(M \oplus^c Y) \otimes^{\sigma h}_M H^c \cong (H \oplus K)^c$ as Hilbert spaces.*

Proof. We will just sketch this, since it is not used here. By Corollary 3.3.1, and Theorem 3.2.20, we have that $L = (M \oplus^c Y) \otimes^{\sigma h}_M H^c$ is a column Hilbert space. Moreover, the projections from $M \oplus^c Y$ onto M and Y respectively, induce by Corollary 2.4.6, projections P and Q from L onto $M \otimes^{\sigma h}_M H^c \cong H^c$, and K, respectively, such that $P + Q = I$. □

Mimicking the proof of [18, Theorem 5.1] we have:

Theorem 3.3.3. *Let (M, N, X, Y) be a weak* Morita context. Then there is a lattice isomorphism between the w^*-closed M-submodules of X and the lattice of w^*-closed left ideals in N. The w^*-closed M-N-submodules of X corresponds to the w^*-closed two-sided ideals in N. Similar statements for Y follows by symmetry. In particular, M and N have isomorphic lattices of w^*-closed two-sided ideals.*

Proof. Suppose the linking algebra has concrete representation on Hilbert space. All products which follows are products of operators on this Hilbert space. If U is a w^*-closed M-submodule of X, let $\mathcal{I}_U = \overline{Y.U}^{w^*}$. Since $\overline{Y.X}^{w^*} = N$ we see that $\mathcal{I}_U$ is a w^*-closed left idea of N. If $\mathcal{I}$ is a w^*-closed left ideal in N let $U_{\mathcal{I}} = \overline{X.\mathcal{I}}^{w^*}$, which is a w^*-closed M-submodule X. Clearly $U \mapsto \mathcal{I}_U$ and $\mathcal{I} \mapsto U_{\mathcal{I}}$ are inverse to each other, and are lattice isomorphisms, which establishes the first result. If U is a w^*-closed M, N-submodule, then $\mathcal{I}_U$ is a w^*-closed two-sided ideal in N. Conversely, if $\mathcal{I}$ is a w^*-closed two sided ideal in N then $U_{\mathcal{I}}$ is a w^*-closed M-N-submodule. The last statement follows by symmetry. □

We next show, analogously to [18, Section 6], that if M and N are W^*-algebras, then they are Morita equivalent in Rieffel's sense if and only if they are weakly (or equivalently, weak*) Morita equivalent in our sense. Indeed we already have remarked (Example (4) in Section 3.2) that Rieffel's Morita equivalence is an example of our weak Morita equivalence. The following gives the converse, and more:

Theorem 3.3.4. *Let (M, N, X, Y) be a weak* Morita context where N is a W^*-algebra. Then M is a W^*-algebra, and there is a completely isometric isomorphism $i : \overline{X} \to Y$ such that X becomes a W^*-equivalence M-N-bimodule (see e.g.* 8.5.12 *in [15] with inner products defined by the formulas ${}_M\langle x_1, x_2\rangle = (x_1, i(\bar{x_2}))$ and $\langle x_1, x_2\rangle_N = [i(\bar{x_1}), x_2]$.*

Proof. First we represent the linking algebra on a Hilbert space $H \oplus K$ as above. We rechoose the net (e_t) such that $e_t \to I_H$ strongly, so that $e_t^* e_t \to I_H$ thus weak*, and similarly for the net (f_s). To accomplish this, note that the WOT-closure of the convex hull of the (e_t) equals the SOT-closure, by elementary operator theory. However it is easy to see that the form in (3.2.1) is preserved if we replace e_s by convex combinations of the e_t. Now one can follow the proof of [18, Theorem 6.2] to deduce that the adjoint of any $y \in Y$ is a limit of terms in X. That is $Y \subset X^*$.

Similarly, $X \subset Y^*$. So $X = Y^*$, and so it follows that M is a W^*-algebra, and X is a WTRO (this term was defined in the list of examples in Section 3.2) setting up a W^*-algebra Morita equivalence. $\square$

The following is the non-selfadjoint analogue of a theorem of Rieffel. A special case of it is mentioned, with a proof sketch, at the end of [16].

Theorem 3.3.5. *Let (M, N, X, Y) be a weak* Morita context. Let H be a universal normal representation for M, and let K be the induced representation of N as in Theorem 3.2.20. Then $M' \cong N'$; that is there is a completely isometric w^*-continuous isomorphism $\theta : B_M(H) \cong B_N(K)$. Writing $\mathcal{R}$ for either of these commutants, we have $X \cong B_{\mathcal{R}}(K, H)$ and $Y \cong B_{\mathcal{R}}(H, K)$ completely isometrically and as dual operator bimodules.*

Proof. One uses the equivalence of categories to see that $B_M(H) \cong B_N(\mathcal{F}(H)) = B_N(K)$ completely isometrically, in the notation of Theorem 3.2.5. That is, $M' \cong N'$ as asserted, and it is easy to argue that if θ is this isomorphism then $\Phi(y)T = \theta(T)\Phi(y)$ for all $y \in Y, T \in M'$. Here Φ is as in the discussion at the start of Section 3.3. Now mimic the proof of 8.5.32 and 8.5.37 in [15]. The main point to bear in mind is that since M is weak* Morita equivalent to the weak linking algebra $\mathcal{L}^w$, the induced representation of $\mathcal{L}^w$ is also a universal normal representation, by easy category theoretic arguments. Thus by [21] it satisfies the double commutant theorem. Carefully computing the first, and then the second, commutants of $\mathcal{L}^w$ as in 8.5.32 in [15], and using the double commutant theorem, gives the result. $\square$

Example. If M and N are *finite dimensional* then weak* Morita equivalence equals strong Morita equivalence, and coincides also with the equivalence considered in [28, 29], that is, weak* stable isomorphism [31]. Indeed if (M, N, X, Y) is a weak* Morita context, then it is clearly a strong Morita context, and by [18, Lemma 2.8] we can actually factor the identity map I_Y through $C_n(M)$ for some $n \in \mathbb{N}$, so that Y is finite dimensional. Similarly, X is finite dimensional. To see that this implies that M and N are weak* stably isomorphic, note that in this situation, since $M \cong X \otimes^{\sigma h}_N Y$, there is a norm 1 element in $X \otimes_h Y$ mapping to 1_M. Similarly for 1_N, and then it is easy to argue that one has what is called a 'quasi-unit of norm 1' in [18, Section 7]. By [18, Corollary 7.9], M and N are stably isomorphic, and taking second duals and using e.g. (1.62) in [15], we see that they are weak* stably isomorphic. In the infinite dimensional case however, all these notions are distinct (e.g. see the Introduction 3.1).

3.4 Morita equivalence of generated W^*-algebras

From [17] or [6], we know that a strong Morita equivalence of operator algebras in the sense of [18] 'dilates' to, or is a subcontext of, a strong Morita equivalence in the sense of Rieffel, of containing C^*-algebras. This happens in a very tidy way. More particularly, suppose that (A, B, X, Y) is a strong Morita context of operator algebras A and B. Then any C^*-algebra C generated by A induces a C^*-algebra D generated by B, and C and D are strongly Morita equivalent in the sense of Rieffel [40], with equivalence bimodule the C^*-dilation (see [8]) $C \otimes_{hA} X$. Moreover the linking algebra for A and B is (completely isometrically) a subalgebra of the linking C^*-algebra for C and D. We see next that all of this, and the accompanying theory, will extend to our present setting. Although one may use any W^*-cover in the arguments below, for specificity, the maximal W^*-algebra $W^*_{\max}(M)$ from [21] will take the place of C above, and the maximal W^*-dilation $W^*_{\max}(M) \otimes^{\sigma h}_M X$ will play the role of the C^*-dilation. In Chapter 5, we will develop the theory for this W^*-dilation in a general setting analogously to [8, 17]. We will however state that just as in [8], any (left, say) dual operator M-module is completely isometrically embedded in its maximal W^*-dilation, via the M-module map $x \mapsto 1 \otimes x$, which is weak* continuous.

Throughout this section again, (M, N, X, Y) is a weak* Morita context. In this case, we shall show that the left and right W^*-dilations coincide, and constitutes a bimodule implementing the W^*-algebraic Morita equivalence between $W^*_{\max}(M)$ and $W^*_{\max}(N)$.

Theorem 3.4.1. *The W^*-dilation $Y \otimes^{\sigma h}_M W^*_{\max}(M)$ is a right C^*-module over $W^*_{\max}(M)$.*

Proof. With H a normal universal Hilbert M-module as usual, we may view $W^*_{\max}(M)$ as the von Neumann algebra $\mathcal{R}$ generated by M in $B(H)$. Let $K = Y \otimes^{\sigma h}_M H^c$ as usual, and let $Z = Y \otimes^{\sigma h}_M W^*_{\max}(M)$. Note that

$$Z \otimes^{\sigma h}_{W^*_{\max}(M)} H^c \cong Y \otimes^{\sigma h}_M W^*_{\max}(M) \otimes^{\sigma h}_{W^*_{\max}(M)} H^c \cong Y \otimes^{\sigma h}_M H^c = K.$$

This allows us to define a completely contractive weak* continuous $\phi : Z \to B(H, K)$ given by $\phi(y \otimes a)(\zeta) = y \otimes a\zeta$, for $y \in Y, a \in \mathcal{R}, \zeta \in H$. Note that ϕ restricted to the copy of Y is just the map Φ at the start of Section 3.3. We are following the ideas of [7, p. 286-288]. It is clear that ϕ is a $\mathcal{R}$-module map. By the discussion just below Corollary 3.2.4, combined with Corollary 2.4.6, there are nets of maps $\varphi_s \otimes I : Z \to C_{n_s}(M) \otimes^{\sigma h}_M W^*_{\max}(M) \cong C_{n_s}(W^*_{\max}(M))$, and maps $\psi_s \otimes I$, with $(\psi_s \otimes I)(\varphi_s \otimes I)(z) = f_s z \to z$ weak* for all $z \in Z$. Here (f_s) is as in (3.2.1), and the last convergence follows from e.g., [31, Lemma 2.3]. We have $\|[f_s z_{ij}]\| \leq \|[(\varphi_s \otimes I)(z_{ij})]\| \leq \|[\phi(z_{ij})]\|$. This follows, as in [7, p. 287], from the fact that there is a sequence of weak* continuous complete contractions

$$B(H, K) \to B(H, C_{n_t}(M) \otimes^{\sigma h}_M W^*_{\max}(M) \otimes^{\sigma h}_{W^*_{\max}(M)} H^c) \cong B(H, C_{n_t}(H^c))$$

that maps $\phi(y \otimes a)$ to $\varphi_s(y)a$, for $y \in Y, a \in \mathcal{R}$, and hence maps $\phi(z)$ for $z \in Z$, to $(\varphi_s \otimes I)(z)$. As in [7, p. 287], we can deduce from these facts that ϕ is a complete isometry.

Define $\langle z, w \rangle = \phi(z)^* \phi(w)$ for $z, w \in Z$. To see that this is a $\mathcal{R}$-valued inner product on Z, we will use von Neumann's double commutant theorem. Note that if

$\Delta(A) = A \cap A^*$ is the diagonal of a subalgebra of $B(H)$, then $\mathcal{R}' = \Delta(M')$, the prime denoting commutants. The proof of Theorem 3.3.5 shows that there is a completely isometric isomorphism $\theta : M' \to N'$, such that $\Phi(y)T = \theta(T)\Phi(y)$ for $y \in Y, T \in M'$, where $\Phi(y)(\zeta) = y \otimes \zeta \in K$, for $\zeta \in H$. By 2.1.2 in [15], θ restricts to a $*$-isomorphism from $\Delta(M') = \mathcal{R}'$ onto $\Delta(N')$. It follows that, in the notation of Theorem 3.9, if $y \in Y, a \in \mathcal{R}, \zeta \in H, T \in M'$ that

$$\phi(y \otimes a)(T\zeta) = y \otimes aT\zeta = y \otimes Ta\zeta = \Phi(y)T(a\zeta) = \theta(T)\Phi(y)(a\zeta) = \theta(T)\phi(y \otimes a)(\zeta).$$

Hence if $w, z \in Z$ then

$$\phi(z)^*\phi(w)T = \phi(z)^*\theta(T)\phi(w) = (\theta(T^*)\phi(z))^*\phi(w) = (\phi(z)T^*)^*\phi(w) = T\phi(z)^*\phi(w),$$

so that $\phi(z)^*\phi(w) \in \mathcal{R}'' = \mathcal{R}$.

Thus Z is a right C^*-module over $W^*_{\max}(M)$, completely isometrically isomorphic to the WTRO $\mathrm{Ran}(\phi)$. □

Theorem 3.4.2. *Suppose that (M, N, X, Y) is a weak* Morita context. Then $W^*_{\max}(M)$ and $W^*_{\max}(N)$ are Morita equivalent W^*-algebras in the sense of Rieffel, and the associated equivalence bimodule is $Y \otimes^{\sigma h}_M W^*_{\max}(M)$. Moreover, $Y \otimes^{\sigma h}_M W^*_{\max}(M) \cong W^*_{\max}(N) \otimes^{\sigma h}_N Y$ completely isometrically. Analogous assertions hold with Y replaced by X. Finally, the W^*-algebra linking algebra for this Morita equivalence contains completely isometrically as a subalgebra the linking algebra $\mathcal{L}^w$ defined earlier for the context (M, N, X, Y).*

Proof. We use the idea in [6, p. 406-407] and [17, p. 585-586]. Let H, K be as in the proof of Theorem 3.4.1. We consider the following subalgebras of $B(H \oplus K)$:

$$\begin{bmatrix} W^*_{\max}(M) & W^*_{\max}(M)X \\ YW^*_{\max}(M) & YW^*_{\max}(M)X \end{bmatrix} \quad , \quad \begin{bmatrix} XW^*_{\max}(N)Y & XW^*_{\max}(N) \\ W^*_{\max}(N)Y & W^*_{\max}(N) \end{bmatrix}.$$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the weak* closures of these two subalgebras. These are dual operator algebras which are the linking algebras for a Morita equivalence. Thus by Theorem 3.3.4, they are actually selfadjoint. Moreover, both of these can now be seen to equal the von Neumann algebra generated by $\mathcal{L}^w$, and so they are equal to each other. Now it is clear that, for example, the weak* closures of $YW^*_{\max}(M)$ and $W^*_{\max}(N)Y$ coincide, and this constitutes an equivalence bimodule (or WTRO) setting up a W^*-algebraic Morita equivalence between $W^*_{\max}(M)$ and $W^*_{\max}(N)$. The W^*-algebraic linking algebra here is just $\mathcal{L}_1 = \mathcal{L}_2$, and this clearly contains the algebra we called $\mathcal{R}$ in the discussion in the beginning of Section 3.3, that is, $\mathcal{L}^w$, as a subalgebra.

Finally, notice that the map ϕ in the proof of the last theorem is a completely isometric $W^*_{\max}(M)$-module map from $Z = Y \otimes^{\sigma h}_M W^*_{\max}(M)$ onto the weak* closure W of $YW^*_{\max}(M)$ in $B(H, K)$. Similar considerations, or symmetry, shows that $V = W^*_{\max}(N) \otimes^{\sigma h}_N Y$ agrees with the weak* closure of $W^*_{\max}(N)Y$, which by the above equals W, and thus agrees with Z. Similarly for the modules involving X. $\square$

Remark. Theorems 4 and 5 of [17] have obvious variants valid in our setting, with arbitrary W^*-dilations in place of $W^*_{\max}(M)$. We will present this in Chapter 5. Similarly, as in [17], we will show later in Chapter 5 that $W^*_{\max}(\mathcal{L}^w) = \mathcal{L}_1$.

Chapter 4

A characterization and a generalization of W^*-modules

4.1 Introduction

In this chapter, we give a new dual Banach module characterization of W^*-modules, also known as selfdual Hilbert C^*-modules, over a von Neumann algebra. This leads to a generalization of the notion, and the theory, of W^*-modules, to the setting where the operator algebras are weak* closed algebras of operators on a Hilbert space. That is, we find the appropriate weak* topology variant of the theory of rigged modules [6], which in turn generalizes the notion of Hilbert C^*-modules. The ensuing modules, the w^*-rigged modules, do not necessarily give rise to a weak* Morita equivalence in the sense of Chapter 3. Nonetheless, w^*-rigged modules have canonical 'envelopes' which are W^*-modules. Indeed, w^*-rigged modules may be defined to be a subspace of a W^*-module possessing certain properties.

A W^**-module* is a Hilbert C^*-module over a von Neumann algebra which is *selfdual*, or, equivalently, which has a predual (see e.g., [42, 24, 16]). These objects were first studied by Paschke, and then by Rieffel [37, 40] (see also [15, Section 8.7] for an account of their theory). They are by now fundamental objects in C^*-algebra theory and noncommutative geometry, being intimately related to Connes' correspondences (see, e.g., [3] for the relationship). W^*-modules have many characterizations; the one mentioned in the title of this chapter characterizes them in the setting of Banach modules in a new way. This in turn leads into a generalization of the notion of W^*-module to the setting where the operator algebra is a dual operator algebra. We also develop the basic theory of this new class of modules, which we call *w^*-rigged modules.* The theory of the space of left multipliers $\mathcal{M}_\ell(X)$ of an operator space X (see e.g., [15, Chapter 4]), plays a role in this process. Unlike the W^*-module situation, w^*-rigged modules do not necessarily give rise to a weak* Morita equivalence in the sense of Chapter 3. Thus there is limited overlap between Chapter 3 and Chapter 4. However, weak* Morita equivalence bimodules are w^*-rigged modules and there are strong connections between the two theories. Also, each w^*-rigged module has a canonical W^*-module envelope, called the W^*-dilation, and thus w^*-rigged modules give new examples of W^*-modules. This dilation is an important tool in our theory. Indeed, a w^*-rigged modules may be defined to be a subspace of a W^*-module possessing certain properties.

Much of Section 4.2 (on W^*-modules) is closely related to a paper of Blecher [4]. The main point of the this paper is that W^*-modules fall comfortably into a dual operator module setting; for example, their usual tensor product (sometimes called composition of W^*-correspondences), agrees with a certain operator space tensor product studied by Magajna. This has certain advantages, for example new results about this tensor product (see also [23]). Here we show that this tensor product also equals the normal module Haagerup tensor product recently introduced in [31], and studied further by in this thesis (see, Section 2.4). In Section 4.3 we find the variant for w^*-rigged modules of the basic theory of rigged modules from [6] (and to a lesser extent, [18]). In Section 4.4 we give several alternative equivalent definitions of w^*-rigged modules. In Section 4.5 we give examples of w^*-rigged modules.

In a couple proofs, we use multipliers of an operator space X (see, [15, Chapter 4]). We recall that the left multiplier algebra $\mathcal{M}_\ell(X)$ of X is a collection of certain operators on X, which are weak* continuous if X is a dual operator space [16].

4.2 W^*-modules

We begin this section with a useful lemma:

Lemma 4.2.1. *Let $\{H_\alpha\}$ be a collection of Hilbert spaces (resp. column Hilbert spaces) indexed by a directed set. Let Y be a dual Banach space (resp. dual operator space). Suppose there exist w^*-continuous contractive (resp. completely contractive) linear maps $\phi_\alpha : Y \to H_\alpha$, $\psi_\alpha : H_\alpha \to Y$, such that $\psi_\alpha(\phi_\alpha(y)) \xrightarrow{w^*} y$ for each $y \in Y$. Then Y is a Hilbert space (resp. column Hilbert space) with inner product given by $\langle y, z\rangle = \lim_\alpha \langle \phi_\alpha(y), \phi_\alpha(z)\rangle$, for $y, z \in Y$.*

Proof. The proof that Y is a Hilbert space (respectively column Hilbert space) follows by the ultraproduct argument in Theorem 3.2.20. For the last assertion, we will show first that $\|\phi_\alpha(y)\|^2 \to \|y\|^2$. Then by the polarization identity, it follows that $\langle y, z\rangle = \lim_\alpha \langle \phi_\alpha(y), \phi_\alpha(z)\rangle$ as desired. Suppose there exists a subnet $(\phi_{\alpha_t}(y))$ such that $\|\phi_{\alpha_t}(y)\|^2 \to \beta$. We need to prove that $\beta = \|y\|^2$. Clearly $\beta \leq \|y\|^2$. If $\beta < K < \|y\|^2$, then there exists a t_0, such that, $\|\phi_{\alpha_t}(y)\|^2 \leq K$ for all $t \geq t_0$. This implies that $\|\psi_{\alpha_t}\phi_{\alpha_t}(y)\|^2 \leq \|\phi_{\alpha_t}(y)\|^2 \leq K$ for all $t \geq t_0$. Since $\psi_{\alpha_t}\phi_{\alpha_t}(y) \xrightarrow{w^*} y$, by Alaoglu's theorem we deduce that $\|y\|^2 \leq K$, which is a contradiction. □

We now generalize the notion of W^*-modules to the setting where the operator algebras are σ-weakly closed algebras of operators on a Hilbert space. The following is the weak* variant of the notion of a rigged module studied in [6, 18, 5, 12]. The paper [12] has the most succinct definition of these objects, and [5] is a survey. In Section 4.4 there are several equivalent, but quite different looking, definitions of w^*-rigged modules.

Definition 4.2.2. *Suppose that Y is a dual operator space and a right module over a dual operator algebra M. Suppose that there exists a net of positive integers $(n(\alpha))$, and w^*-continuous completely contractive M-module maps $\phi_\alpha : Y \to C_{n(\alpha)}(M)$ and $\psi_\alpha : C_{n(\alpha)}(M) \to Y$, with $\psi_\alpha(\phi_\alpha(y)) \to y$ in the w^*-topology on Y, for all $y \in Y$. Then we say that Y is a* right w^*-rigged module *over M.*

An argument similar to that in the last few lines of the proof of Lemma 4.2.1, and using basic operator space duality principles, shows that for a w^*-rigged module Y,

$$\|[y_{ij}]\|_{M_n(Y)} = \sup_\alpha \|[\phi_\alpha(y_{ij})]\|, \qquad [y_{ij}] \in M_n(Y). \tag{4.2.1}$$

Theorem 4.2.3. *If Y is a right w^*-rigged module over a dual operator algebra M, then $CB^\sigma(Y)_M = \mathcal{M}_\ell(Y)$ completely isometrically isomorphically, and this is a weak* closed subalgebra of $CB(Y)_M$. Hence $CB^\sigma(Y)_M$ is a dual operator algebra, and Y is a left dual $CB^\sigma(Y)_M$ -module.*

Proof. By facts in the theory of multipliers of an operator space (see e.g., [15, Chapter 4] or [16]), the identity map is a weak* continuous completely contractive homomorphism $\mathcal{M}_\ell(Y) \to CB(Y)$, which maps into $CB^\sigma(Y)_M$. If $CB^\sigma(Y)_M$ is an operator algebra, and if Y is a left operator $CB^\sigma(Y)_M$-module (with the canonical action), then by the aforementioned theory there exists a completely contractive homomorphism $\pi : CB^\sigma(Y)_M \to \mathcal{M}_\ell(Y)$ with $\pi(T)(y) = T(y)$ for all $y \in Y, T \in CB^\sigma(Y)_M$. That is, $\pi(T) = T$. Thus $CB^\sigma(Y)_M = \mathcal{M}_\ell(Y)$, and it is clear from the Krein-Smulian theorem and [15, Theorem 4.7.4 (2)] that $CB(Y)_M$ is weak* closed in $CB(Y)$.

We now show that $CB^\sigma(Y)_M$ is an operator algebra by appealing to the abstract characterization of operator algebras [15, Theorem 2.3.2]. If $S = [S_{ij}], T = [T_{ij}] \in M_n(CB^\sigma(Y)_M)$, then one may use the idea in [6, Theorem 2.7] or [18, Theorem 4.9] to write the matrix $a = [\sum_k S_{ik}T_{kj}(y_{pq})]$ as an iterated weak* limit of a product of three matrices. The norm of this last product is dominated by $\|[S_{ij}]\|\|[T_{ij}]\|\|[y_{pq}]\|$. It follows by Alaoglu's theorem that $\|a\| \leq \|S\|\|T\|\|[y_{pq}]\|$, and thus $\|ST\| \leq \|S\|\|T\|$ as desired.

A similar argument shows that Y is a left operator $CB^\sigma(Y)_M$-module: If T is as above, and $y = [y_{ij}] \in M_n(Y)$, then $z = [\sum_k T_{ik}(y_{kj})]$ may be written as a weak* limit of a product of two matrices, the latter product having norm $\leq \|T\|\|y\|$. Applying Alaoglu's theorem gives $\|z\| \leq \|T\|\|y\|$, as desired.

The final assertion now follows from [15, Lemma 4.7.5]. □

Theorem 4.2.4. *Suppose that Y is a right w^*-rigged module over a dual operator algebra M. Suppose that H is a Hilbert space, and that $\theta : M \to B(H)$ is a normal representation. Then $Y \otimes_M^{\sigma h} H^c$ is a column Hilbert space. Moreover, the finite rank tensors $Y \otimes H^c$ are norm dense in the $Y \otimes_M^{\sigma h} H^c$.*

Proof. Let $e_\alpha = \phi_\alpha \psi_\alpha$ (notation as in Definition 4.2.2). By Lemma 2.4.7 and Theorem 4.2.3, $Y \otimes_M^{\sigma h} H^c$ is a left dual $\mathcal{M}_\ell(Y)$-module. By the functoriality of the module normal Haagerup tensor product, we obtain a net of complete contractions $\phi_\alpha \otimes I_H : Y \otimes_M^{\sigma h} H^c \to C_{n(\alpha)}(M) \otimes_M^{\sigma h} H^c$ and $\psi_\alpha \otimes I_H : C_{n(\alpha)}(M) \otimes_M^{\sigma h} H^c \to Y \otimes_M^{\sigma h} H^c$. Their composition $(\phi_\alpha \otimes I_H)(\psi_\alpha \otimes I_H) = e_\alpha \otimes I_H$ may be regarded as the canonical left action of $e_\alpha \in \mathcal{M}_\ell(Y)$ on $Y \otimes_M^{\sigma h} H^c$ mentioned at the start of the proof. Since the action is separately weak* continuous, the composition converges to the identity map on $Y \otimes_M^{\sigma h} H^c$ in the w^*-topology (i.e., point-weak* on $Y \otimes_M^{\sigma h} H^c$). However, for any m, we have from the facts in Section 2.4 that

$$C_m(M) \otimes_M^{\sigma h} H^c \cong (C_m \otimes^{\sigma h} M) \otimes_M^{\sigma h} H^c \cong C_m \otimes^{\sigma h} (M \otimes_M^{\sigma h} H^c) \cong C_m \otimes^{\sigma h} H^c \cong C_m(H^c),$$

and $C_m(H^c)$ is a column Hilbert space. Thus by Lemma 4.2.1, $Y \otimes^{\sigma h}_M H^c$ is a column Hilbert space. The last assertion follows from Section 2.4 in Chapter 2 and Mazur's theorem that the norm closure of a convex set equals its weak closure. But for a Hilbert space, weak closure equals its weak* closure by using the reflexivity of Hilbert spaces. □

Henceforth in this section, we stick to the case that M is a W^*-algebra.

Part (ii) (and (iii)) of the following is the Banach-module characterization of W^*-modules promised in our title. The result may be compared with e.g., [15, Corollary 8.5.25].

Theorem 4.2.5. *Let M be a W^*-algebra.*

(i) An operator M-module Y is w^*-rigged if and only if Y is a W^*-module, and the matrix norms for Y coincide with the W^*-module's canonical operator space structure.

(ii) If Y is a dual Banach space and a right M-module, then Y is a W^*-module if and only if Y satisfies the condition involving nets in Definition 4.2.2 *but with the maps ϕ_α, ψ_α contractive as opposed to completely contractive.*

(iii) If the conditions in (i) *or* (ii) *hold, and if ϕ_α is as in Definition* 4.2.2 *or* (ii), *then for $y, z \in Y$, the weak* limit $w^*\text{-}\lim_\alpha \phi_\alpha(y)^* \phi_\alpha(z)$ exists in M, and equals the W^*-module inner product.*

Proof. If Y is a W^*-module then the existence of the nets in (i) or (ii) follow easily from, e.g., Paschke's result [15, Corollary 8.5.25] or [4, Theorem 2.1].

For the other direction in (i), we follow the proof on p. 286–287 in [7]. Let ϕ_α and ψ_α be as in Definition 4.2.2. We write the k^{th} coordinate of ϕ_α as x^α_k, where x^α_k is a w^*-continuous module map from $Y \to M$, and we write the k^{th} entry of

ψ_α as $y_k^\alpha \in Y$. By hypothesis we have $\sum_{k=1}^{n(\alpha)} y_k^\alpha \, x_k^\alpha(y) \xrightarrow{w*} y$ for every $y \in Y$. Let H be a Hilbert space on which M is normally and faithfully represented. Then by Lemma 4.2.4, $K = Y \otimes_M^{\sigma h} H$ is a column Hilbert space. Define two canonical maps $\varPhi : Y \to B(H,K)$ and $\varPsi : CB_M^\sigma(Y,M) \to B(K,H)$, given respectively by $\varPhi(y)(\zeta) = y \otimes \zeta$ and $\varPsi(f)(y \otimes \zeta) = f(y)\zeta$. Then it is easily checked (or see Subsection 4.3.1 for this in a more general setting), that Φ and Ψ are weak* continuous complete isometries.

Let $e_\alpha = \sum_{k=1}^{n(\alpha)} \varPhi(y_k^\alpha)\varPsi(x_k^\alpha)$. It is easy to check that $e_\alpha \varPhi(y) = \varPhi(\psi_\alpha \phi_\alpha y)$ hence $e_\alpha \varPhi(y) \xrightarrow{w^*} \varPhi(y)$ for all $y \in Y$. Hence $e_\alpha(y \otimes \zeta) \to y \otimes \zeta$ weak* in K for all $y \in Y, \zeta \in H$. It follows by the last assertion of Theorem 4.2.4 that $e_\alpha \to I_K$ in the WOT for $B(K)$. By a argument similar to that of the proof of Theorem 3.3.4, we can rechoose the net (e_α) such that $e_\alpha \to I$ strongly on K. Continuing to follow the proof in [7], one can deduce by a small modification of the argument there, that the adjoint of any $\varPhi(y) \in \varPhi(Y)$ is a weak* limit of terms in $\varPsi(CB_M(Y,M))$. Thus for $z \in Y$, we have that $\Phi(y)^*\Phi(z)$ is a weak* limit of terms in $\varPsi(CB_M^\sigma(Y,M))\Phi(Y)$, and hence is in M, being a weak* limit of terms in M.

Define $\langle y, z \rangle = \varPhi(y)^*\varPhi(z)$ for $y, z \in Y$. As in [7], Y is a C^*-module over M and the canonical C^*-module matrix norms coincide with the operator space structure of Y, since $\varPhi$ is a complete isometry on Y. Since $\varPhi$ is w^*-continuous, it follows that the inner product on Y is separately w^*-continuous. Hence Y is a W^*-module, by Lemma 8.5.4 in [15].

The assertions (ii) and (iii) follow from (i) and (4.2.1), as in [7], replacing limits by weak* limits. Note that (4.2.1) corresponds to an isometric embedding of Y in $\oplus_\alpha^\infty C_{n_\alpha}(M)$, which is easily seen to be weak* continuous and hence a weak* homeomorphism by Krein-Smulian. Thus (4.2.1) will induce a dual operator space structure on Y. $\square$

By a *weak* approximate identity* in a unital dual Banach algebra M, we mean a net $\{e_t\}$ in M such that $e_t \xrightarrow{w^*} 1$. A *weak* iterated approximate identity* for M is a doubly indexed net $\{e_{(\alpha,\beta)}\}$ (where β and the directed set indexing β may possibly depend on α), such that for each fixed α, the weak* limit $w^*\text{-}\lim_\beta e_{(\alpha,\beta)}$ exists, and $w^*\text{-}\lim_\alpha w^*\text{-}\lim_\beta e_{(\alpha,\beta)} = 1$.

Lemma 4.2.6. *A weak* iterated approximate identity for a dual Banach algebra may be reindexed to become a weak* approximate identity.*

Proof. Suppose that $\{e_{(\alpha,\beta)}\}$ is a weak iterated approximate identity for a dual Banach algebra M. For each $a \in M$ and fixed α, define $x_\alpha(a) = w^*\text{-}\lim_\beta e_{(\alpha,\beta)}a$ and $y_\alpha(a) = w^*\text{-}\lim_\beta a e_{(\alpha,\beta)}$. We also define a new indexing set τ which consist of the set of 5-tuples $\gamma = (\alpha, \beta, V, M_*, \epsilon)$, where V is a finite subset of M, and $\epsilon > 0$ such that $|f(e_{(\alpha,\beta)}a) - f(x_\alpha(a))| < \epsilon$ and $|f(ae_{(\alpha,\beta)}) - f(y_\alpha(a))| < \epsilon$ for all $a \in V$ and $f \in M_*$. One can check that τ is a directed set with ordering $(\alpha, \beta, V, M_*, \epsilon) \leq (\alpha', \beta', V', M_*, \epsilon')$ if and only if $\alpha \leq \alpha'$, $V \subset V'$ and $\epsilon' \leq \epsilon$. We say $e_\gamma = e_{(\alpha,\beta)}$ if $\gamma = (\alpha, \beta, V, M_*, \epsilon)$.

Let $\epsilon > 0$ and $a \in M$ be given. Choose α_0 such that for $\alpha \geq \alpha_0$, $|f(x_\alpha(a)) - f(a)| < \epsilon$ and $|f(y_\alpha(a)) - f(a)| < \epsilon$. Choose β_0 in such a way that $\gamma_0 = (\alpha_0, \beta_0, a, M_*, \epsilon) \in \tau$. If $\gamma = (\alpha, \beta, V, M_*, \epsilon) \geq \gamma_0$ then $|f(e_\gamma a) - f(a)| \leq |f(e_{(\alpha,\beta)}a) - f(x_\alpha(a))| + |f(x_\alpha(a)) - f(a)| < \epsilon' + \epsilon = 2\epsilon$. Hence $\{e_\gamma\}_{\gamma\in\tau}$ is a left weak* approximate identity. Similarly it is right weak* approximate identity. ☐

Theorem 4.2.7. *Let $\{Y_i\}$ be a collection of W^*-modules over a W^*-algebra M, indexed by a directed set. Let Y be a dual Banach space (resp. dual operator space) and a right module over M. Suppose that there exist w^*-continuous contractive (resp. completely contractive) M-module maps $\phi_i : Y \to Y_i$ and $\psi_i : Y_i \to Y$, such that $\psi_i(\phi_i(y)) \xrightarrow{w^*} y$ in Y, for $y \in Y$. Then Y is a W^*-module (resp. a W^*-module with its canonical dual operator space structure). For $y, z \in Y$, the limit $w^*\text{-}\lim_i\langle\phi_i(y), \phi_i(z)\rangle$ exists in M and equals the W^*-module inner product $\langle y, z\rangle$.*

Proof. As in Theorem 4.2.5, one can focus on the operator space version. For each i choose nets $\phi^i_{\alpha_i}$, $\psi^i_{\alpha_i}$ for Y_i as in the last theorem. Let $\phi'_{i,\alpha_i} = \phi^i_{\alpha_i} \circ \phi_i$, and $\psi'_{i,\alpha_i} = \psi_i \circ \psi^i_{\alpha_i}$. By Lemma 4.2.6, reindex the net $\{\phi'_{i,\alpha_i}, \psi'_{i,\alpha_i}\}$, so that the weak* limit of $\psi'_{i,\alpha_i}\phi'_{i,\alpha_i}$ in $CB^\sigma(Y)_M$ over the new directed set coincides with the iterated weak* limit w^*-$\lim_i$ w^*-$\lim_{\alpha_i}$ $\psi'_{i,\alpha_i}\phi'_{i,\alpha_i}$, which equals I_Y. This gives a new asymptotic factorization of I_Y through spaces of form $C_n(M)$ with respect to which Y is w^*-rigged. Hence by Theorem 4.2.5, Y is a W^*-module, with the inner product

$$\langle y, z\rangle = w^*\lim \langle \phi^i_{\alpha_i}(\phi_i(y)), \phi^i_{\alpha_i}(\phi_i(z))\rangle$$

where the limit is taken over the new directed set. Carefully inspecting the directed set used in Lemma 4.2.6 (a variant of the one used in [6, Lemma 2.1]), it is easy to argue that the last inner product equals w^*- $\lim_i \langle \phi_i(y), \phi_i(z)\rangle$. □

Remark. The same proof as the above establishes the analogue of the last result, but for a dual operator module Y over a unital dual operator algebra M, taking the Y_i to be w^*-rigged modules over M, and the ϕ^i, ψ^i completely contractive (the conclusion being that Y is w^*-rigged).

Theorem 4.2.8. *If Y is a right W^*-module over M, and if Z is a left (resp. right) dual operator module over M, then $Y \otimes^{\sigma h}_M Z \cong CB^\sigma_M(\overline{Y}, Z) = CB_M(\overline{Y}, Z)$ completely isometrically and w^*-homeomorphically (resp. $Z \otimes^{\sigma h}_M \overline{Y} \cong CB^\sigma(Y, Z)_M = CB(Y, Z)_M$ completely isometrically and w^*-homeomorphically).*

Proof. We will use facts and routine techniques from [25] or [15, 1.2.26, 1.6.3, Section 3.8]. If $T \in B(Y, Z)_M$, and if $(e_\alpha)_{\alpha\in I}$ is an orthonormal basis for Y (see [37] or [15, 8.5.23]), note that by [25, Theorem 4.2 and remark after it] we have

$$T(y) = T(\sum_\alpha e_\alpha\langle e_\alpha, y\rangle) = \sum_\alpha T(e_\alpha)\langle e_\alpha, y\rangle = ag(y), \qquad y \in Y,$$

where a is the row with α^{th} entry $T(e_\alpha)$, and $g : Y \to C_I(M)$ has α^{th} entry the function $\langle e_\alpha, \cdot \rangle$. Thus T is the composition of left multiplication by $a \in R_I(Z)$ and g, both of which are weak* continuous (see e.g., the proof of [15, Corollary 8.5.25]). Thus $CB^\sigma_M(Y, Z) = CB_M(Y, Z)$.

That $Z \otimes^{\sigma h}_M \overline{Y} \cong CB^\sigma_M(Y, Z)$ is generalized later in Theorem 4.3.4. □

Corollary 4.2.9. *In the situation of the last theorem, the tensor products $\otimes^{\sigma h}_M$ coincide with Magajna's extended module Haagerup tensor product $\bar{\otimes}_{hM}$ used in* [4].

It follows that in all of the results in [4], all occurrences of the extended module Haagerup tensor product $\bar{\otimes}_{hM}$ may be replaced by the normal module Haagerup tensor product $\otimes^{\sigma h}_M$. This is very interesting because in many of these results this tensor product also coincides with the most important and commonly used tensor product for W^*-modules, the composition tensor product $Y \overline{\otimes}_\theta Z$. Thus our results gives a new way to treat this famous composition tensor product (see also [23]). Both tensor product descriptions have their own advantages: $\bar{\otimes}_{hM}$ allows one to concretely write elements as infinite sums of a nice form, whereas $\otimes^{\sigma h}_M$ has many useful general properties (see [31], Section 2.4).

We state just a couple of the many tensor product results from [4], adapted to our setting:

Corollary 4.2.10. *Let Y, Z be right W^*-modules over M and N respectively, and suppose that $\theta : M \to B(Z)$ is a normal $*$-homomorphism. Then the 'composition tensor product' $Y \overline{\otimes}_\theta Z$ equals $Y \otimes^{\sigma h}_M Z$. Also, $CB(Y \overline{\otimes}_\theta Z)_N \cong Y \otimes^{\sigma h}_M CB(Z)_N \otimes^{\sigma h}_M \overline{Y}$ completely isometrically and weak* homeomorphically.*

Proof. The first assertion is discussed above (following from Theorem 4.2.8 and [4]). For the second, just as in the proof of this result from [4], Theorem 4.2.8 gives

$$CB(Y \overline{\otimes}_\theta Z)_N \cong (Y \overline{\otimes}_\theta Z) \otimes^{\sigma h}_N (Y \overline{\otimes}_\theta Z)^- \cong (Y \otimes^{\sigma h}_M Z) \otimes^{\sigma h}_N (\overline{Z} \otimes^{\sigma h}_M \overline{Y}),$$

which equals $Y \otimes^{\sigma h}_M (Z \otimes^{\sigma h}_N \overline{Z}) \otimes^{\sigma h}_M \overline{Y} \cong Y \otimes^{\sigma h}_M B(Z)_N \otimes^{\sigma h}_M \overline{Y}$ (see [13, 31]). $\square$

A similar proof, using Theorem 4.2.8 twice and associativity of the tensor product, gives:

Corollary 4.2.11. *Let M, N be W^*-algebras, let Y be a right W^*-module over M, and let W (resp. Z) be a dual operator N-M-bimodule (resp. dual right operator N-module). Then $CB(Y, Z \otimes^{\sigma h}_N W)_M \cong Z \otimes^{\sigma h}_N CB(Y, W)_M$ completely isometrically and weak* homeomorphically.*

4.3 Some theory of w^*-rigged modules

4.3.1 Basic constructs

We begin with some notation and important constructs which will be used throughout. For a w^*-rigged module Y over a dual operator algebra M, define $\tilde{Y} = CB^\sigma(Y, M)_M$. Let ϕ_α and ψ_α be as in Definition 4.2.2. We write the k^{th} coordinate of ϕ_α as x^α_k, where x^α_k is a w^*-continuous module map from Y to M, and we write k^{th} entry of ψ_α as $y^\alpha_k \in Y$. By hypothesis we have $\sum_{k=1}^{n(\alpha)} y^\alpha_k \, x^\alpha_k(y) \overset{w^*}{\to} y$ for every $y \in Y$.

We sometimes write $\tilde{Y}$ as X, and use $(\cdot, \cdot)$ to denote the canonical pairing $\tilde{Y} \times Y \to M$. This is completely contractive, as one may see using the idea in the proof of Theorem 4.2.3 (the crux of the matter being that for $f \in \tilde{Y}, y \in Y$ we have $(f, y) = w^*\text{-}\lim_\alpha \sum_{k=1}^{n(\alpha)} f(y^\alpha_k) x^\alpha_k(y)$, a limit of a product in M).

Let H be a Hilbert space on which M is normally and faithfully (completely isometrically) represented. Then by Lemma 4.2.4, $K = Y \otimes^{\sigma h}_M H$ is a column Hilbert space. Define two canonical maps $\Phi : Y \to B(H, K)$ and $\Psi : \tilde{Y} \to B(K, H)$, given respectively by $\Phi(y)(\zeta) = y \otimes \zeta$ and $\Psi(f)(y \otimes \zeta) = f(y)\zeta$. By the argument at the beginning of Section 3.3 in Chapter 3, Φ is weak* continuous. In view of the canonical map $Y \times H^c \to K$ being completely contractive, a routine argument gives

Φ completely contractive. By the argument on p. 287 in [7], Φ is a complete isometry: one obtains, as in that calculation, that

$$\|[\phi_\alpha(y_{ij})]\| \leq \|[\Phi(y_{ij})]\|,$$

so in the limit, by (4.2.1), $\|[y_{ij}]\| \leq \|[\Phi(y_{ij})]\|$. The canonical weak* continuous complete contraction

$$\tilde{Y} \otimes^{\sigma h} K^c \cong (\tilde{Y} \otimes^{\sigma h} Y) \otimes^{\sigma h}_M H^c \to M \otimes^{\sigma h}_M H^c \to H^c,$$

corresponds to a separately weak* continuous complete contraction $\tilde{Y} \times K^c \to H^c$. The map Ψ is precisely the induced weak* continuous complete contraction $\tilde{Y} \to B(K, H)$. As before, Ψ is a complete isometry.

We define the direct sum $M \oplus^c Y$ as in Section 3.3. Namely, $\theta : M \oplus Y \to B(H, K \oplus H)$ defined by $\theta((m, y))(\zeta) = (m\zeta, y \otimes_M \zeta)$, for $y \in Y, m \in M, \zeta \in H$, is a one-to-one M-module map, which is a weak* continuous complete isometry when restricted to each of Y and M. We norm $M \oplus^c Y$ by pulling back the operator space structure via θ, then $M \oplus^c Y$ may be identified with the weak* closed right M-submodule $\mathrm{Ran}(\theta)$ of $B(H, H \oplus K)$; and hence $M \oplus^c Y$ is a dual operator M-module.

Lemma 4.3.1. *If Y is a right w^*-rigged module over M, then $M \oplus^c Y$ is a left w^*-rigged module over M. Also, $(M \oplus^c Y) \otimes^{\sigma h}_M H^c \cong (H \oplus K)^c$ as Hilbert spaces, for H, K as above.*

Proof. Define $\phi'_\alpha : M \oplus^c Y \to C_{n(\alpha)+1}(M)$ and $\psi'_\alpha : C_{n(\alpha)+1}(M) \to M \oplus^c Y$, to be $I_M \oplus \phi_\alpha$ and $I_M \oplus \psi_\alpha$ respectively. We also view $M \subset B(H)$, identify Y and $\Phi(Y)$, and write $n(\alpha) = n$. One may then view $\phi'_\alpha(m, y)$, for $m \in M, y \in Y$, as the matrix product of the $(n+1) \times 2$ matrix $I_H \oplus \Psi_{n,1}([x^\alpha_k])$ (viewed as an operator $H \oplus K \to H^{(n+1)}$), and the 2×1 matrix with entries m and $\Phi(y)$ (viewed as an operator $H \to H \oplus K$). Thus it is clear that ϕ'_α is completely contractive. Similarly, we view

$\psi'_\alpha(m, [m_k])$, for $m \in M, [m_k] \in C_n(M)$, as the matrix product of the $2 \times (n+1)$ matrix $I_H \oplus \Phi_{1,n}([y_k^\alpha])$ (viewed as an operator $H^{(n+1)} \to H \oplus K$) and the $(n+1) \times 1$ matrix with entries m and m_k (viewed as an operator $H \to H^{(n+1)}$). Thus it is clear that ψ'_α is completely contractive. It is easy to see that $\phi'_\alpha \phi'_\alpha \to I$ weak* on $M \oplus^c Y$. So $M \oplus^c Y$ is w^*-rigged.

The last assertion follows just as in Section 3.3. □

Lemma 4.3.2. *If Y is a right w^*-rigged module over M, then $\tilde{Y}$ is a weak* closed subspace of $CB(Y, M)_M$. Indeed, $\tilde{Y}$ is a left w^*-rigged module over M, which is also a dual right module over $CB^\sigma(Y)$. The canonical map $(\cdot,\cdot) : \tilde{Y} \times Y \to M$ is completely contractive and separately weak* continuous.*

Proof. Let P and Q be the canonical projections from $M \oplus^c Y$ onto Y and M respectively; and let i and j be the canonical inclusions of Y and M, respectively, into $M \oplus^c Y$. Then $\Theta(T) = jQTiP$ defines a weak* continuous completely contractive projection on $\mathcal{M}_\ell(M \oplus^c Y) = CB^\sigma(M \oplus^c Y)_M$. Thus the range of θ is weak* closed. However, this range is easily seen to be completely isometric to $CB^\sigma(Y, M)_M$. Thus the latter becomes a dual operator space, in which, from [15, Theorem 4.7.4(2)], a bounded net converges in the associated weak* topology if and only if the net converges point weak*. It follows easily that $\tilde{Y}$ is a weak* closed subspace of $CB(Y, M)_M$ (by the Krein-Smulian theorem, or by using the fact that $\mathcal{M}_\ell(M \oplus^c Y)$ is weak* closed in $CB(M \oplus^c Y)$ (see Theorem 4.2.3)).

Define nets of weak* continuous maps $f \mapsto [f(y_k^\alpha)] \in R_{n(\alpha)}(M)$, and $[m_k] \mapsto \sum_k m_k x_k^\alpha \in \tilde{Y}$, then it is easy to see that with respect to these, $\tilde{Y}$ satisfies the left module variant of Definition 4.2.2. Since $CB^\sigma(M \oplus^c Y)_M$ is a dual operator algebra, it is easy to see that its 1-2-corner $\tilde{Y}$ is a dual right module over its 2-2-corner $CB^\sigma(Y)$. We have already essentially seen the last part. □

Corollary 4.3.3. *We have $Y \cong CB^\sigma_M(\tilde{Y}, M)$ completely isometrically, and as right M-modules. That is, $\tilde{\tilde{Y}} = Y$. Also a bounded net $y_t \to y$ weak* in Y if and only if $(x, y_t) \to (x, y)$ weak* in M for all $x \in \tilde{Y}$.*

Proof. This is straightforward, using the Lemma above and the ideas in [6, 18], and routine weak* topology principles. □

We say that $T : Y \to Z$ between w^*-rigged modules over M is *adjointable* if there exists $S : \tilde{Z} \to \tilde{Y}$ such that $(w, Ty) = (S(w), y)$ for all $y \in Y, w \in \tilde{Z}$. The properties of adjointables in the first three paragraphs of p. 389 of [6] hold in our setting too, and moreover it is easy to see that T is adjointable if and only if $T \in CB^\sigma(Y, Z)_M$.

For any dual operator modules Y, Z, set $\mathbb{B}(Y, Z) = CB^\sigma(Y, Z)_M$ and set $\mathbb{B}(Y) = CB^\sigma(Y)_M$. So $\tilde{Y} = \mathbb{B}(Y, M)$. We also set $N = Y \otimes^{\sigma h}_M \tilde{Y}$. Using the canonical completely contractive and separately weak* continuous map $(\cdot, \cdot) : \tilde{Y} \times Y \to M$, one obtains by the facts in Section 2.4, a weak* continuous completely contractive map

$$N \otimes^{\sigma h} N \cong Y \otimes^{\sigma h}_M (\tilde{Y} \otimes^{\sigma h} Y) \otimes^{\sigma h}_M \tilde{Y} \to Y \otimes^{\sigma h}_M M \otimes^{\sigma h}_M \tilde{Y} \cong N.$$

This endows $N = Y \otimes^{\sigma h}_M \tilde{Y}$ with a separately weak* continuous completely contractive product, so that by [15, Theorem 2.7.9], we have that N is a dual operator algebra. We now show that N is unital. As in [18, 6], the elements $v_\alpha = \sum_{k=1}^{n(\alpha)} y_k^\alpha \otimes x_k^\alpha$ are in Ball(N), and for any $y \in Y, x \in \tilde{Y}$ we have in the product above the theorem,

$$v_\alpha(y \otimes x) = \psi_\alpha(\phi_\alpha(y)) \otimes x \;\to\; y \otimes x$$

weak* in N. If $v_{\alpha_t} \to v$ is a weak* convergent subnet, then by the above formula we have $v(y \otimes x) = y \otimes x$, and it follows that $vu = u$ for all $u \in N$. Similarly $uv = u$. We deduce from this that N has an identity of norm 1. Since such an identity is unique, we must have $v_\alpha \to 1_N$ weak*.

Theorem 4.3.4. *If Y is a right w^*-rigged module over M, and Z is a right dual operator M-module, then $\mathbb{B}(Y,Z)$ is weak* closed in $CB(Y,Z)$. Moreover, $\mathbb{B}(Y,Z) \cong Z \otimes^{\sigma h}_M \tilde{Y}$ completely isometrically and weak* homeomorphically. In particular, $\mathbb{B}(Y) \cong Y \otimes^{\sigma h}_M \tilde{Y}$ as dual operator algebras, equipping the last space with the product mentioned above.*

Proof. As in the second paragraph after Corollary 4.3.3, by the facts in Section 2.4, we have canonical weak* continuous complete contractions

$$(Z \otimes^{\sigma h}_M \tilde{Y}) \otimes^{\sigma h}_M Y \cong Z \otimes^{\sigma h}_M (\tilde{Y} \otimes^{\sigma h}_M Y) \to Z \otimes^{\sigma h}_M M \cong Z.$$

This induces a canonical completely contractive w^*-continuous linear map $\theta : Z \otimes^{\sigma h}_M \tilde{Y} \to CB(Y,Z)_M$, which satisfies $\theta(z \otimes x)(y) = z(x,y)$, and which actually maps into $\mathbb{B}(Y,Z)_M$.

In the notation we introduced prior to Theorem 4.3.4, $N = Y \otimes^{\sigma h}_M \tilde{Y}$ is a unital dual operator algebra. Set $W = Z \otimes^{\sigma h}_M \tilde{Y}$. The canonical weak* continuous maps

$$W \otimes^{\sigma h} (Y \otimes^{\sigma h}_M \tilde{Y}) \cong Z \otimes^{\sigma h}_M (\tilde{Y} \otimes^{\sigma h} Y) \otimes^{\sigma h}_M \tilde{Y} \to Z \otimes^{\sigma h}_M M \otimes^{\sigma h}_M \tilde{Y} \cong W,$$

induces a separately weak* continuous complete contraction $m : W \times N \to W$. Note that $m(z \otimes x, 1_N) = z \otimes x$ for $z \in Z, x \in \tilde{Y}$, since $m(z \otimes x, v_\alpha) = z \otimes x\psi_\alpha\phi_\alpha \to z \otimes x$ weak*. Thus $m(u, 1_N) = u$ for any $u \in W$, and so $m(u, v_\alpha) \to u$ weak*.

Now define $\mu_\alpha : CB(Y,Z)_M \to W : T \mapsto \sum_{k=1}^{n(\alpha)} T(y_k^\alpha) \otimes x_k^\alpha$. This is a weak* continuous complete contraction. We have $\mu_\alpha(\theta(z \otimes x)) = z \otimes x\psi_\alpha\phi_\alpha = m(z \otimes x, v_\alpha) \to z \otimes x$ weak* for any $z \in Z, x \in \tilde{Y}$. From the equality in the last line, and weak* density, we have for all $u \in W$ that $\mu_\alpha(\theta(u)) = m(u, v_\alpha)$. The latter, by the fact at the end of the last paragraph, converges to u. Since $\|\mu_\alpha(\theta(u))\| \leq \|\theta(u)\|$ it follows from Alaoglu's theorem that θ is an isometry. Similarly, θ is a complete isometry. Since it is weak* continuous, by Krein-Smulian θ has weak* closed range, and is a weak* homeomorphism. Since $\theta(\mu_\alpha(T)) \to T$ weak* if $T \in \mathbb{B}(Y,Z)$, we have now proved that $\mathrm{Ran}(\theta) = \mathbb{B}(Y, Z)$. Note that in the case when $Z = Y$ we have that θ is a homomorphism, because it is so on the weak* dense subalgebra $Y \otimes \tilde{Y}$. $\square$

As one immediate application of this, as on p. 391 in [6], one can argue that for any cardinals or sets I, J we have $M_{I,J}(\mathbb{B}(Y, Z)) \cong \mathbb{B}(C_J^w(Y), C_I^w(Z))$ completely isometrically and weak* homeomorphically. This uses the fact that $\widetilde{C_J^w(Y)} = R_I^w(Y) \cong Y \otimes^{\sigma h} R_I$.

4.3.2 The weak linking algebra, and its representations

If Y is a w^*-rigged module over M, with $\tilde{Y}$, set

$$\mathcal{L}^w = \left\{ \begin{bmatrix} a & x \\ y & b \end{bmatrix} : \ a \in M, b \in \mathbb{B}(Y), x \in \tilde{Y}, y \in Y \right\},$$

with the obvious multiplication. As in Section 3.3, one may easily adapt the proof of the analogous fact in [18], that there is at most one possible sensible dual operator space structure on this linking algebra, and so the linking algebra with this structure must coincide with $\mathbb{B}(M \oplus^c Y)$. Another description proceeds as follows. Let H be any Hilbert space on which M is normally completely isometrically represented, and set $K = Y \otimes_M^{\sigma h} H^c$. We saw at the start of Section 4.3 the canonical maps $\Phi : Y \to B(H, K)$ and $\Psi : \tilde{Y} \to B(K, H)$.

Lemma 4.3.5. *The weak* closure in $B(K)$ of $\Phi(Y)\Psi(\tilde{Y})$ is completely isometrically isomorphic, via a weak* homeomorphism, to $\mathbb{B}(Y)$.*

Proof. Let N be this weak* closure, which is a weak* closed operator algebra. Clearly $N\Phi(Y) \subset \Phi(Y)$, so that by 4.6.6 in [15] we have a completely contractive homomorphism $\mu : N \to \mathcal{M}_\ell(Y)$. Conversely, since Y is a dual left operator $\mathcal{M}_\ell(Y)$-module by Theorem 4.2.3, so is K by Lemma 2.4.7. Thus by the proof of [15, Theorem 4.7.6], there is a normal representation $\theta : \mathcal{M}_\ell(Y) \to B(K)$. If $y \otimes f$ denotes the obvious operator in $CB^\sigma(Y)$, for $y \in Y$, and $f \in \tilde{Y}$, then $\theta(y \otimes f)(y' \otimes \zeta) = yf(y') \otimes \zeta = \Phi(y)\Psi(f)(y' \otimes \zeta)$ for all $y' \in Y, \zeta \in H$. Thus $\theta(y \otimes f) = \Phi(y)\Psi(f) \in N$. However, it is easy to see from the fact that $T\psi_\alpha\phi_\alpha \to I_Y$ weak*, that the span of such $y \otimes f$ is weak* dense in $CB^\sigma(Y)_M$, and it follows that θ maps into a weak* dense subset of N. Clearly $\mu(\theta(y \otimes f)) = y \otimes f$, and so $\mu \circ \theta = I$. Thus θ is a complete isometry, and the proof is completed by an application of the Krein-Smulian theorem. □

Thus the linking algebra $\mathcal{L}^w$ of the w^*-rigged module may also be taken to be the subalgebra of $B(H \oplus K)$ with 'four corners' $\Phi(Y), \Psi(\tilde{Y}), M$, and the weak* closure in $B(K)$ of $\Phi(Y)\Psi(\tilde{Y})$.

4.3.3 Tensor products of w^*-rigged modules

If Y is a right w^*-rigged module over M, if Z is a right w^*-rigged module over a dual operator algebra $\mathcal{R}$, and if Z is a also left dual operator M-module, then $Y \otimes^{\sigma h}_M Z$ is a right dual operator $\mathcal{R}$-module (see Section 2.4). As in the proof of Theorem 4.2.4, we obtain a net of completely contractive right $\mathcal{R}$-module maps $\phi_\alpha \otimes I_Z : Y \otimes^{\sigma h}_M Z \to C_{n(\alpha)}(M) \otimes^{\sigma h}_M Z \cong C_{n(\alpha)}(Z)$, and $\psi_\alpha \otimes I_Z : C_{n(\alpha)}(Z) \to Y \otimes^{\sigma h}_M Z$, such that the composition $(\phi_\alpha \otimes I_Z)(\psi_\alpha \otimes I_Z) = e_\alpha \otimes I_Z$ converges weak* to the identity map on $Y \otimes^{\sigma h}_M Z$. Thus by the remark after Theorem 4.2.7, $Y \otimes^{\sigma h}_M Z$ is a w^*-rigged module over $\mathcal{R}$. In particular, if $\mathcal{R}$ is a W^*-algebra then $Y \otimes^{\sigma h}_M Z$ is a W^*-module over $\mathcal{R}$ by Theorem 4.2.5 (i).

In the setting of the last paragraph (and $\mathcal{R}$ possibly non-selfadjoint again),

$$(Y \otimes^{\sigma h}_M Z)\tilde{} = CB^\sigma(Y \otimes^{\sigma h}_M Z, \mathcal{R})_{\mathcal{R}} \cong \tilde{Z} \otimes^{\sigma h}_M \tilde{Y}, \tag{4.3.1}$$

completely isometrically and weak* homeomorphically. We give one proof of this (another route is to use the method on p. 402–403 in [6]). Note that the canonical weak* continuous complete contractions

$$(\tilde{Z} \otimes^{\sigma h}_M \tilde{Y}) \otimes^{\sigma h} (Y \otimes^{\sigma h}_M Z) \to \tilde{Z} \otimes^{\sigma h}_M M \otimes^{\sigma h}_M Z \to \tilde{Z} \otimes^{\sigma h}_M Z \to \mathcal{R},$$

induce a weak* continuous complete contraction $\sigma : \tilde{Z} \otimes^{\sigma h}_M \tilde{Y} \to CB^\sigma(Y \otimes^{\sigma h}_M Z, \mathcal{R})_{\mathcal{R}}$. On the other hand, the complete contraction from the operator space projective tensor product to $Y \otimes^{\sigma h}_M Z$, induces a complete contraction $CB^\sigma(Y \otimes^{\sigma h}_M Z, \mathcal{R})_{\mathcal{R}} \to CB(Y, CB(Z, \mathcal{R}))$ that is easily seen to map into $CB(Y, \tilde{Z})$, and in fact into $CB^\sigma(Y, \tilde{Z})_M$. Now it is easy to check that this map $CB^\sigma(Y \otimes^{\sigma h}_M Z, \mathcal{R})_{\mathcal{R}} \to CB^\sigma(Y, \tilde{Z})_M$ is also weak*-continuous. By Theorem 4.3.4, we have constructed a weak*-continuous complete contraction $\rho : CB^\sigma(Y \otimes^{\sigma h}_M Z, \mathcal{R})_{\mathcal{R}} \to \tilde{Z} \otimes^{\sigma h}_M \tilde{Y}$. It is easy to check that $\rho\sigma = Id$, thus σ is completely isometric, and by Krein-Smulian σ has weak* closed range. Any $f \in CB^\sigma(Y \otimes^{\sigma h}_M Z, \mathcal{R})_{\mathcal{R}}$ is a weak* limit of $f \circ (\psi_\alpha \phi_\alpha \otimes I_Z)$. The latter function is easily checked to lie in $\text{Ran}(\sigma)$, using the fact that for any $y \in Y$ the map $f(y \otimes \cdot)$ on Z is in $\tilde{Z}$. Hence σ has weak* dense range, and hence is surjective, proving (4.3.1).

Just as in the proof of Corollary 4.2.10, one may deduce from (4.3.1) the relation $\mathbb{B}(Y \otimes^{\sigma h}_M Z) \cong Y \otimes^{\sigma h}_M \mathbb{B}(Z)_{\mathcal{R}} \otimes^{\sigma h}_M \tilde{Y}$.

In fact the weak* variants of all the theorems in Section 6 of [6] are valid. In next subsection, we merely focus on Section 6.8 from that paper, which we shall need at the end of the next section.

4.3.4 The W^*-dilation

This important tool is a canonical W^*-module envelope of a w^*-rigged module Y over M. If $\mathcal{R}$ is a W^*-algebra containing M as a weak* closed subalgebra with $1_{\mathcal{R}} = 1_M$, then $E = Y \otimes^{\sigma h}_M \mathcal{R}$ is a W^*-module over $\mathcal{R}$, and it is called a W^*-dilation of Y. We may identify Y with $Y \otimes 1$. This is a completely isometric weak* homeomorphic identification, since by (4.2.1) we have for $[y_{ij}] \in M_n(Y)$ that

$$\|[y_{ij} \otimes 1]\|_{M_n(E)} = \sup_\alpha \|[(\psi_\alpha \otimes I_Z)(y_{ij} \otimes 1)]\| = \sup_\alpha \|[\psi_\alpha(y_{ij})]\| = \|[y_{ij}]\|_{M_n(Y)}.$$

Thus every w^*-rigged module over M is a weak* closed M-submodule of a W^*-module over $\mathcal{R}$. Usually we assume $\mathcal{R}$ is generated as a W^*-algebra by M.

Similarly, it is easy to see that $\tilde{Y}$ is a weak* closed left M-submodule of $\mathcal{R} \otimes_M^{\sigma h} \tilde{Y}$. By (4.3.1) above, $\mathcal{R} \otimes_M^{\sigma h} \tilde{Y} = \tilde{E}$, which in turn is just the conjugate C^*-module (see e.g. [15, 8.1.1 and 8.2.3(2)]) $\bar{E}$ of E. We claim that $\mathbb{B}(Y)$ may be regarded as a weak* closed subalgebra of $\mathbb{B}(E)$ having a common identity element. By a principle we have met several times now (e.g., in the proof of Lemma 4.3.5), there is a canonical weak* continuous completely contractive unital homomorphism $\mathbb{B}(Y) \to \mathbb{B}(E)$. However, since $Y \cong Y \otimes 1 \subset E$ as above, it is easy to see that the last homomorphism is a completely isometric weak* homeomorphism. Thus we have established the variant in our setting of [6, Theorem 6.8].

The W^*-dilation is discussed in a more general setting in Chapter 5.

4.3.5 Direct sums

If Y is a w^*-rigged module over M, and if $P \in \mathbb{B}(Y)$ is a contractive idempotent, then it is easy to see from the remark after Theorem 4.2.7, that $P(Y)$ is a w^*-rigged module over M, called an *orthogonally complemented submodule* of Y.

As in the discussion at the top of p. 409 of [6], if $\{Y_k\}_{k \in I}$ is a collection of w^*-rigged modules over M, and if $E_k = Y_k \otimes_M^{\sigma h} \mathcal{R}$ is the W^*-dilation of Y_k, for a W^*-algebra $\mathcal{R}$ containing M, then we can define the column direct sum $\oplus_{k \in I}^c Y_k$, to be $\oplus_{k \in I}^c Y_k = \{(y_k) \in \oplus_{k \in I}^c E_k : y_k \in Y_k \text{ for all } k \in I\}$, where $\oplus_{k \in I}^c E_k$ is the W^*-module direct sum (see [37] or [15, 8.5.26]). A key principle, which is used all the time when working with this direct sum, is the following. Consider the directed set of finite subsets Δ of I, and for $z \in \oplus_{k \in I}^c E_k$, write z_Δ for the tuple z, but with entries z_k

switched to zero if $k \notin \Delta$. Then $(z_\Delta)_\Delta$ is a net indexed by Δ, which converges weak* to z. For example, it follows from this principle that $\oplus^c_{k\in I} Y_k$ is the weak* closure inside $\oplus^c_{k\in I} E_k$ of the finitely supported tuples (y_k) with $y_k \in Y_k$ for all k.

Theorem 4.3.6. *If $\{Y_k : k \in I\}$ is a collection of w^*-rigged modules over M, then $\oplus^c_{k\in I} Y_k$ is again a w^*-rigged module over M.*

Proof. We first observe that this holds if I is finite. For simplicity, we just want to consider the case of two modules, which is similar to the proof of Lemma 4.3.1. In fact one may use Definition 4.4.5 below in Section 4.4 to see quickly that $Y_1 \oplus^c Y_2$ is w^*-rigged if Y_1 and Y_2 are: note that if $E = E_1 \oplus^c E_2$ in the notation of the last paragraph, and if $(z_1, z_2) \in E$ is such that $\langle (z_1, z_2)|(y_1, y_2)\rangle = \langle z_1|y_1\rangle + \langle z_2|y_2\rangle \in M$ for all $y_1, y_2 \in Y$, then $z_k^* \in X_k = \tilde{Y}_k$ by Definition 4.4.5. One can then see that the conditions of Definition 4.4.5 are satisfied, so that $Y_1 \oplus^c Y_2$ is w^*-rigged.

If I is infinite we proceed as follows. In the notation of the paragraph above the theorem, we have that $\oplus^c_{k\in\Delta} Y_k$ is w^*-rigged by the last paragraph. There are canonical maps ϕ_Δ and ψ_Δ between $Y = \oplus^c_{k\in I} Y_k$ and $\oplus^c_{k\in\Delta} Y_k$. Namely, ϕ_Δ is essentially the map $z \mapsto z_\Delta$, and ψ_Δ is the inclusion, indeed $\phi_\Delta \circ \psi_\Delta = Id$. It is easy to see that these maps are completely contractive and weak* continuous, since when one tensors them with $I_{\mathcal{R}}$ they have these properties. Also, $\psi_\Delta \circ \phi_\Delta \to I_Y$ weak*, using the principle above the theorem that $z_\Delta \to z$. It follows from the remark after Theorem 4.2.7, that Y is w^*-rigged. □

The following universal property shows that the direct sum $\oplus^c_{k\in I} Y_k$ does not dependent on the specific construction of it above:

Theorem 4.3.7. *Suppose that $\{Y_k\}_{k\in I}$ is a collection of dual operator modules over M, that Y is a fixed w^*-rigged module over M, and that there exist weak* continuous contractive M-module maps $i_k : Y_k \to Y$, $\pi_k : Y \to Y_k$ with $\pi_k \circ i_m = \delta_{km} Id_{Y_m}$*

for all k, m. Here δ_{km} is the Kronecker delta. Then each Y_k is w^*-rigged, and Y is completely isometrically weak homeomorphically M-isomorphic to the column direct sum $Z \oplus^c (\oplus_k^c Y_k)$ defined above, where Z is a submodule of Y which is also w^*-rigged. If $\sum_{k \in I} i_k \pi_k = I_Y$ in the weak*-topology of $\mathbb{B}(Y)$, then $Z = (0)$.*

Proof. The ranges $i_k(Y_k)$ are orthogonally complemented submodules of Y, and hence they are w^*-rigged, and so is Y_k. The sum $R = \sum_k i_k \pi_k$ is a increasing net of contractive projections in the dual operator algebra $\mathbb{B}(Y)$, indexed by the finite subsets of I directed upwards by inclusion. Hence it converges in the weak* topology in $\mathbb{B}(Y)$ to a contractive projection $R \in \mathbb{B}(Y)$. Let $Z = \text{Ran}(I - R)$, which again is w^*-rigged. Define $Z \oplus^c (\oplus_k^c Y_k)$ as above the theorem, a weak* closed M-submodule of the W^*-module direct sum $(Z \otimes_M^{\sigma h} \mathcal{R}) \oplus^c (\oplus_k^c (Y_k \otimes_M^{\sigma h} \mathcal{R}))$. Tensoring all maps with $I_{\mathcal{R}}$ we obtain maps back and forth between $Y_k \otimes_M^{\sigma h} \mathcal{R}$ and $Y \otimes_M^{\sigma h} \mathcal{R}$, and between $Z \otimes_M^{\sigma h} \mathcal{R}$ and $Y \otimes_M^{\sigma h} \mathcal{R}$, satisfying the hypotheses of [4, Theorem 2.2]. Note that $i_k \pi_k \in \mathbb{B}(Y)_M$, and $Y \otimes_M^{\sigma h} \mathcal{R}$ is a left dual operator $\mathbb{B}(Y)_M$-module (since Y is). It follows that $\sum_k (i_k \pi_k \otimes I_{\mathcal{R}}) = R \otimes I_{\mathcal{R}}$, so that $\sum_k (i_k \pi_k \otimes I_{\mathcal{R}}) + (I - R) \otimes I_{\mathcal{R}} = I$. From [4, Theorem 2.2], it follows that the canonical map is a completely isometric weak* homeomorphic, $\mathcal{R}$-isomorphism between $(Z \otimes_M^{\sigma h} \mathcal{R}) \oplus^c (\oplus_k^c (Y_k \otimes_M^{\sigma h} \mathcal{R}))$ and $Y \otimes_M^{\sigma h} \mathcal{R}$. The restriction of this isomorphism to the copy of $Z \oplus (\oplus_n^w Y_n)$ is the desired map. □

As in [6, Section 7] it follows that the column direct sum is associative and commutative. We also have the obvious variant of [6, Theorem 7.4] valid in our setting, concerning the direct sum $\oplus_k T_k$ of maps $T_k \in \mathbb{B}(Y_k, Z_k)$. Again, the proof of this is now familiar: apply the W^*-module case of this result to the maps $T_k \otimes I_{\mathcal{R}}$ between the W^*-dilations, and then restrict to the appropriate subspace. Also, we obtain from Theorem 4.3.7 and functoriality of the tensor product $\otimes_M^{\sigma h}$, as in [6, p. 411], both left and right distributivity of this tensor product $\otimes_M^{\sigma h}$ over column direct sums of w^*-rigged modules:

$$(\oplus_k^c Y_k) \otimes_M^{\sigma h} Z \cong \oplus_k^c (Y_k \otimes_M^{\sigma h} Z),$$

and

$$Y \otimes_M^{\sigma h} (\oplus_k^c Z_k) \cong \oplus_k^c (Y \otimes_M^{\sigma h} Z_k).$$

All spaces in these formulae are right w^*-rigged modules, and the Z and Z_k are also left dual operator M-modules. For the last formula, for infinitely many Z_k, one needs to use the fact that if $T_t \to T$ weak* in $\mathbb{B}(Z)_M$, then $I_Y \otimes T_t \to I \otimes T$ weak* in $\mathbb{B}(Y \otimes_M^{\sigma h} Z)$. Indeed, if we have a weak* convergent subnet $I_Y \otimes T_{t_\mu} \to S \in \mathbb{B}(Y \otimes_M^{\sigma h} Z)$, then $S(y \otimes z) = y \otimes T(z)$ for $y \in Y, z \in Z$. Since finite rank tensors are weak* dense, we have $S = I \otimes T$, and it follows that $I_Y \otimes T_t \to I \otimes T$ weak*.

Remark. Theorem 4.3.7 also shows that our definition at the start of Section 4.3 of $M \oplus^c Y$, agrees with the column direct sum in the present subsection. Thus the last relation in Lemma 4.3.1 is a very simple special case of the second last centered formula.

4.4 Equivalent definitions of w^*-rigged modules

4.4.1 One may prefer some of the following four descriptions of w^*-rigged modules, each of which involves a *pair* X, Y of modules. In each case, the first paragraph of the subsection constitutes the alternative definition. One must show that every w^*-rigged module Y satisfies (or is completely isometrically, weak* homeomorphically, M-isomorphic to a module which satisfies) the given alternative description; and that conversely any Y satisfying the description is w^*-rigged, and that moreover $X \cong \tilde{Y}$.

We will be a little informal in this section, as the objectives here are quite clear — we are just adapting four theorems from [6, Section 5] to the present setting of weak* topology.

4.4.2 Second definition of a w^*-rigged module

Fix two unital dual operator algebras M and N, and two dual operator bimodules X and Y, with X an M-N-bimodule and Y an N-M-bimodule. We also assume there exists a separately weak*-continuous completely contractive M-bimodule map $(\cdot,\cdot) : X \times Y \to M$ which is balanced over N, and a separately weak* continuous completely contractive N-bimodule map $[\cdot,\cdot] : Y \times X \to N$ which is balanced over M, such that $(x,y)x' = x[y,x']$ and $y'(x,y) = [y',x]y$ for $x,x' \in X, y,y' \in Y$; and such that $[\cdot,\cdot]$ induces a weak* continuous quotient map $Y \otimes^{\sigma h} X \to N$.

As in [6, Section 5], any w^*-rigged module in the earlier sense of Definition 4.2.2, satisfies the conditions in the last paragraph, with $N = \mathbb{B}(Y)$ (or $N = Y \otimes^{\sigma h}_M \tilde{Y}$), and $X = \tilde{Y}$, by our earlier results. Conversely, given the conditions in the last paragraph, suppose that $u \in \text{Ball}(Y \otimes^{\sigma h} X)$ maps to 1_N, and that (f_s) is a net of finite rank tensors in $\text{Ball}(Y \otimes_{\text{h}} X)$ which converges weak* to u (using Corollary 2.4.8). The image of f_s in N converges weak* to 1_N. From this it is easy to see that Y satisfies Definition 4.2.2, following similar assertions in [6] (see the bottom of p. 384 of [6]). Moreover, a by-now-routine modification of the last two paragraphs of the proof of [18, Theorem 4.1], one sees that the canonical map $X \to CB^\sigma(Y,M)_M$ is a weak* continuous surjective complete isometry. That is $X \cong \tilde{Y}$ as dual operator M-modules. We have a canonical weak* continuous complete quotient map $\theta : Y \otimes^{\sigma h}_M \tilde{Y} \to N$. A simple modification of the last paragraph of the proof of Theorem 4.3.4, which is essentially the proof of ($\Leftarrow$) in Theorem 3.2.3, shows that θ is a complete isometry, so that $[\cdot,\cdot]$ induces a weak* homeomorphic complete isometry $Y \otimes^{\sigma h}_M \tilde{Y} \cong N$.

4.4.3 Third definition of a w^*-rigged module

A pair consisting of a dual left M-module X, and a dual right M-module Y, with a separately weak* continuous completely contractive pairing $(\cdot,\cdot) : X \times Y \to M$, such that if we equip $N = Y \otimes_M^{\sigma h} X$ with the canonical separately weak* continuous completely contractive product induced by $(\cdot, \cdot)$, as in the discussion above Theorem 4.3.4, then this (dual operator) algebra has an identity of norm 1. We also assume that the canonical actions of N on Y and on X are non-degenerate (that is, $1_N y = y, x 1_N = x$ for $y \in Y, x \in X$).

Again, clearly any w^*-rigged module in the earlier sense, satisfies the conditions in the last paragraph, by Theorem 4.3.4 and the remarks above it. Conversely, suppose that X, Y, $(\cdot, \cdot)$ are as in the last paragraph. We shall verify the conditions of Definition 4.4.2 above. It is by now routine to see that X, Y are dual operator modules over N. To see that $(\cdot, \cdot)$ is N-balanced, follows by showing that for $x \in X, y \in Y$, the two weak* continuous functions $(x, \cdot\, y)$ and $(x\, \cdot\, , y)$ on N, are equal on the weak* dense subset $Y \otimes X$ of N. The rest is straightforward.

4.4.4 Fourth description of w^*-rigged modules

Let M, N be weak* closed unital subalgebras of $B(H)$ and $B(K)$ respectively, for Hilbert spaces H, K, and let $X \subset B(K, H), Y \subset B(H, K)$ be weak* closed subspaces, such that the associated subset $\mathcal{L}$ of $B(H \oplus K)$ is a subalgebra of $B(H \oplus K)$, for Hilbert spaces H, K. This is the same as specifying a list of obvious algebraic conditions, such as $XY \subset M$. Assume in addition that the weak* closure N in $B(K)$ of YX, possesses a net (e_t) with terms of the form yx, for $x \in \text{Ball}(C_n(X))$ and $y \in \text{Ball}(R_n(Y))$, such that $e_t \to 1_N$ weak*.

That every w^*-rigged module Y is essentially of this form, follows by replacing Y by $\Phi(Y)$, $\tilde{Y}$ by $X = \Psi(\tilde{Y})$, and looking at the weak linking algebra $\mathcal{L}^w$ at the end of Section 4.2. As in the proof of Theorem 4.2.5, one sees that the net (e_t) defined there converges to I_K weak*, which gives the condition in the last paragraph. Conversely, given the setup in the last paragraph, we will verify the conditions of Definition 4.4.2

above. The canonical map $\theta : Y \otimes^{\sigma h} X \to N$ is completely contractive and weak* continuous, we need to show it is a quotient map. If $T \in \text{Ball}(N)$, and if we write the x and y in the last paragraph as $x = [x_k], y = [y_k]$, then $u_t = \sum_k Ty_k \otimes x_k \in \text{Ball}(Y \otimes^{\sigma h} X)$. Consider a weak* convergent subnet $u_{t_\beta} \to u \in \text{Ball}(Y \otimes^{\sigma h} X)$. Then $\theta(u_{t_\beta}) \to \theta(u)$. On the other hand, $\theta(u_{t_\beta}) = Te_{t_\beta} \to T$ weak*. So $T = \theta(u)$, so that θ is a quotient map.

4.4.5 Fifth definition of a w^*-rigged module

Let $\mathcal{R}$ be a W^*-algebra containing M as a weak*-closed subalgebra with $1_{\mathcal{R}} = 1_M$, and suppose that Z is a right W^*-module over $\mathcal{R}$, and that Y is a weak* closed M-submodule of Z. Define $W = \{z \in Z : \langle z|y \rangle \in M\}$, and set N to be the weak* closure in $\mathbb{B}(Z)_{\mathcal{R}}$ of the span of terms of the form $|y\rangle\langle w|$ for $y \in Y, w \in W$. Here we are using bra-ket notation, $|y\rangle\langle w|$ is the rank one operator on Z (see e.g., [15, 8.1.7]). Suppose that there is a net (e_t) converging to I_Z weak* in $\mathbb{B}(Z)$, with terms of the form $e_t = \sum_{k=1}^n |y_k\rangle\langle w_k|$, where $y_k \in Y, w_k \in W$ with $\sum_{k=1}^n |y_k\rangle\langle y_k| \leq 1$ and $\sum_{k=1}^n |w_k\rangle\langle w_k| \leq 1$.

We claim that under the hypotheses in the last paragraph, Y is w^*-rigged, $\tilde{Y} \cong W$, and $\mathbb{B}(Y)_M \cong N$. To see this, we follow the proof of [6, Theorem 5.10], working inside the linking W^*-algebra $\mathcal{L}^w(Z)$ for Z, where all inner products and module actions become concrete operator multiplication. Note first that W is a weak* closed right M-submodule of Z, and hence $X = W^*$ is a weak* closed left M-submodule of Z^*. The subspace of $\mathcal{L}^w(Z)$ with four corners M, X, Y, N, is a weak* closed subalgebra, and one can see that the criteria of the Definition 4.4.4 above are met for these subspaces of $\mathcal{L}^w(Z)$. Hence the criteria of Definition 4.4.4 above are met, and we are done by facts from there.

Conversely, to see that every w^*-rigged module Y is essentially of this form, set

$Z = Y \otimes^{\sigma h}_M \mathcal{R}$, which we saw in 4.3.4 is a W^*-module over $\mathcal{R}$, containing Y as a weak* closed M-submodule. Also we saw there that $\tilde{Y} \subset \bar{Z}$ (resp. $\mathbb{B}(Y)_M \subset \mathbb{B}(Z)_{\mathcal{R}}$) as a weak* closed M-submodule (resp. weak* closed subalgebra with a common identity). Now apply a simple variant of the argument in the last paragraph of [6, p. 405].

4.5 Examples

(1) As we saw in Section 4.2, W^*-modules are w^*-rigged. Thus so are *WTROs*, where a WTRO is a weak* closed space Z of Hilbert space operators with $ZZ^*Z \subset Z$ (see [15, 8.5.11 and 8.5.18]).

(2) For finite dimensional modules over a finite dimensional operator algebra M, the notions of rigged and w^*-rigged coincide, as is easily seen from Definition 4.2.2.

(3) By Definition 4.4.2 and Theorem 3.2.3, every weak* Morita equivalence bimodule in the sense of Chapter 3 is w^*-rigged. In Section 3.2, a long list of examples of these bimodules is given. Indeed, a weak* Morita equivalence bimodule is essentially the same thing as a left-right symmetric variant of second definition (that is, we also assume there that $(\cdot,\cdot)$ induces a weak* continuous quotient map $X \otimes^{\sigma h} Y \to M$).

There are simple examples of w^*-rigged modules which give rise to no kind of weak* Morita equivalence (in contrast to the W^*-module case). For example, consider $Y = R_2$, a right w^*-rigged module over the upper triangular 2×2 matrices. A partial result in the positive direction here: if Y is a w^*-rigged M-module which is w^*-*full*, that is the span of the range of $(\cdot,\cdot)$ is weak* dense in M, and if $\mathcal{R}$ is a W^*-algebra generated by M, then the W^*-dilation $E = Y \otimes^{\sigma h}_M \mathcal{R}$ gives a von Neumann algebraic Morita equivalence (see [40] or [15, Section 8.7]) between $\mathcal{R}$ and $\mathbb{B}(E)$. This will follow if E is w^*-full over $\mathcal{R}$ (see [15, 8.5.12]).

To this end, note $\bar{E} = \tilde{E} = \mathcal{R} \otimes^{\sigma h}_M \tilde{Y}$ by (4.3.1). Thus $\tilde{Y}Y$, and therefore also M, is contained in the weak* closure of $\tilde{E}E$. So E is w^*-full, since the latter is an ideal of $\mathcal{R}$, and because M generates $\mathcal{R}$.

(4) The second dual of a rigged module over an operator algebra A is w^*-rigged over A^{**}. This is evident by taking the second dual of all objects in the definition of a rigged module from [12] (note that $C_n(A)^{**} = C_n(A^{**})$ by basic operator space duality).

(5) If P is a weak*-continuous completely contractive idempotent M-module map on $C^w_I(M)$, for a cardinal or set I, then $\text{Ran}(P)$ is a w^*-rigged module (see Section 4.3.5).

(6) Examples of w^*-rigged may be built analogously to the rigged modules in [11] (see the end of Section 6 of [11]).

(7) There is a stronger variant of 'w^*-rigged' which we call 'weakly rigged'. The distinction between this notion and w^*-rigged, is exactly like the difference between the notions we called *weak* and *weak** Morita equivalence in Chapter 3. Indeed one definition of 'weakly rigged' is just as in Definition 3.2.2 of Chapter 3, but replacing the phrase '(strong) Morita context' by '(P)-context' (see [18, p. 20]). An adaption of the proof of Corollary 3.2.4 shows that weakly rigged modules are w^*-rigged. One may then show that any weakly rigged module pair (Y, X) arises as a weak* closure of a rigged module situation, just as in Example (2) after Definition 3.2.2 of Chapter 3, but dropping the requirement on the cai for A there. This proceeds by showing that the linking algebra for the 'subcontext' is a weak* dense subalgebra of the weak linking algebra for (Y, X) (this uses 6.10 in [6], see the argument above Corollary 3.3.1 in Chapter 3).

(8) Let Z be any WTRO (see Example (1)), and suppose that Z^*Z is contained in

a dual operator algebra M. Then $Y = \overline{ZM}^{w^*}$ is a w^*-rigged M-module. We call this example a *WTRO-rigged module*. We also have $\tilde{Y} \cong \overline{MZ^*}^{w^*}$. To see this, denote the last space by X, and set N to be the weak* closure of ZMZ^*, a dual operator algebra containing ZZ^*. If (e_t) is the usual approximate identity for ZZ^* with terms of the form $\sum_{k=1}^n z_k z_k^*$, then it is routine to see that (e_t) converges weak* to an identity 1_N for N. One can check that M, Y, X, N satisfy Definition 4.4.4, and we are done.

We remark that the above is a generalization of Eleftherakis' recent notion of *TRO-equivalence* (see e.g., [29, 31]). Indeed, a WTRO-rigged module gives a TRO-equivalence of M and N iff the identity e of the weak* closure of Z^*Z is 1_M. For the most difficult part of this, note that if the latter holds, and if $f_s \to e$ weak* with $f_s \in Z^*Z$, then any $m \in M$ is an iterated weak* limit of the $f_s m f_{s'}$, and it follows that M equals the weak* closure of Z^*NZ.

(9) The *selfdual* rigged modules over a dual operator algebra M, considered at the start of the last section in [16], together with their unique dual space structure making $(\cdot, \cdot)$ separately weak* continuous (see [16, Lemma 5.1]), are w^*-rigged. Indeed one can see from the last mentioned continuity that Definition 4.2.2 is satisfied.

Chapter 5

A Morita theorem for dual operator algebras

5.1 Introduction

In this chapter, we prove that two dual operator algebras are weak* Morita equivalent in our sense if and only if they have equivalent categories of dual operator modules via completely contractive functors which are also weak* continuous on appropriate morphism spaces. Moreover, in a fashion similar to the operator algebra case, we characterize such functors as the module normal Haagerup tensor product with an appropriate weak* Morita equivalence bimodule. We also develop the theory of the W^*-dilation, which connects the non-selfadjoint dual operator algebra with the W^*-algebraic framework. In the case of weak* Morita equivalence, this W^*-dilation is a W^*-module over a von Neumann algebra generated by the non-selfadjoint dual operator algebra.

In the literature on Morita equivalence in pure algebra, there is a popular collection of theorems known as Morita I, II and III. Morita I deals with the consequences of a pair of bimodules being mutual inverses ($X \otimes_N Y \cong M$ and $Y \otimes_M X \cong N$). For dual operator algebras, most of the appropriate version of Morita I is proved in Chapter 3 (Section 3.2). Morita II characterizes module category equivalences as tensoring with an invertible bimodule, and our main theorem here is a Morita II theorem for dual operator algebras. The Morita III theorem states that there is a bijection between the set of isomorphism classes of invertible bimodules and the set of equivalence classes of category equivalences; its appropriate version for dual operator algebras follows as in pure algebra.

In Chapter 3, we proved that two dual operator algebras which are weak* Morita equivalent in our sense have equivalent categories of dual operator modules. In this chapter, we prove the converse, a Morita II theorem: if two dual operator algebras have equivalent categories of dual operator modules then they are weak* Morita equivalent in our sense. The functors implementing the categorical equivalences are characterized as the module normal Haagerup tensor product with an appropriate weak* Morita equivalence bimodule. In Section 5.2, we develop the theory of the W^*-dilation, which connects the non-selfadjoint dual operator algebra with the W^*-algebraic framework. In particular, we use the maximal W^*-algebra $\mathcal{C}$ generated by a dual operator algebra M. Every dual operator M-module dilates to a dual operator module over $\mathcal{C}$, which is called the maximal dilation. We show that every dual operator module is a weak* closed submodule of its maximal dilation. Indeed, in the case of weak* Morita equivalence this maximal dilation turns out to be a W^*-module over $\mathcal{C}$ as we saw in Chapter 3. The theory of the W^*-dilation is a key part of the proof of our main theorem. In Section 5.3, we discuss some weak* Morita equivalence and W^*-dilation results. In Section 5.4 and 5.5, we prove our main theorem.

Many of the techniques and ideas in this chapter are taken from [8], [10], [9], [17]. In some places we just need to modify the arguments in the present setting of weak*-topology, or merely change the tensor product. However, we need to develop new techniques to deal with a number of subtleties that arise in the weak* topology setting.

In Chapter 3, we showed that weak* Morita equivalent dual operator algebra have equivalent categories of normal Hilbert space representations (also known as normal Hilbert modules). However, the converse of this is still an open problem. The characterization theorem in [28] is in terms of equivalence of categories of normal Hilbert modules which intertwines not only the representations of the dual operator algebras, but also their restrictions to the diagonals.

In this chapter, we refer to Rieffel's W^*-algebraic Morita equivalence [40] as 'weak Morita equivalence' for W^*-algebras, and the associated equivalence bimodules as 'W^*-equivalence-bimodules' (see Section 8.5 in [15]).

5.2 Dual operator modules over a generated W^*-algebra and W^*-dilations

We begin this section with a weak* topology version of Theorem 3.1 in [8].

Theorem 5.2.1. *Let D be a W^*-algebra, B a Banach algebra which is also a dual Banach space, and $\theta : D \to B$ a unital w^*-continuous contractive homomorphism. Then the range of θ is w^*-closed, and possesses an involution with respect to which θ is a $*$-homomorphism and the range of θ is a W^*-algebra.*

Proof. It is known that (see Theorem A.5.9 in [15]) the range of a contractive homomorphism between a C^*-algebra and a Banach algebra is a C^*-algebra and moreover such homomorphisms are $*$-homomorphisms. To see that the range of θ is w^*-closed, consider the quotient map $D/ker(\theta) \to B$ which is an isometry, and apply the Krein-Smulian theorem. □

Thus if X is a left dual operator module over a W^*-algebra D, and if we let $\theta : D \to CB(X)$ be the associated unital w^*-continuous contractive (equivalently completely contractive by Proposition 1.2.4 in [15]) homomorphism, then the range of θ is a W^*-algebra.

Theorem 5.2.2. *Suppose that X is a left dual operator module over a dual operator algebra M. Let $\theta : M \to CB(X)$ be the associated completely contractive homomorphism. Suppose that D is any W^*-algebra generated by M. Then the M-action on X can be extended to a D-action with respect to which X is a dual operator D-module if and only if θ is the restriction to M of a w^*-continuous contractive (equivalently completely contractive) homomorphism $\phi : D \to CB(X)$. This extended D-action, or equivalently the homomorphism ϕ, is unique if it exists.*

Proof. If θ is the restriction to M of a w^*-continuous completely contractive homomorphism $\phi : D \to CB(X)$ then the M-action on X can be extended to a D-action via $d \cdot x = \phi(d) \cdot x$. Note that the D-module action $x \mapsto dx$ on X, for $x \in X$ and $d \in D$, is a multiplier (see e.g., Section 4.5 in [15]), hence it is weak* continuous by Theorem 4.1 in [16]. The D-module action on X is separately w^*-continuous and completely contractive. Hence X is a dual operator D-module. The converse is obvious. To see the uniqueness assertion, suppose that ϕ_1 and ϕ_2 are two w^*-continuous contractive homomorphisms $D \to CB(X)$, extending θ. By Theorem 5.2.1, the ranges $\mathcal{E}_1$ and $\mathcal{E}_2$, of ϕ_1 and ϕ_2 respectively, are each W^*-algebras, but with possibly different involutions and weak* topologies. We will write these involutions as $\star$ and $\#$ respectively. With respect to these involutions ϕ_1 and ϕ_2 are $*$-homomorphisms. Note that $CB(X)$ is a unital Banach algebra and $\mathcal{E}_1$ and $\mathcal{E}_2$ may be viewed as unital subalgebras of $CB(X)$ with the same unit. Let $a \in M$ and f be a state on $CB(X)$. Then $f|\mathcal{E}_i$ is a state on $\mathcal{E}_i$ for $i = 1, 2$. Thus $f(\phi_1(a)^\star) = \overline{f(\phi_1(a))} = \overline{f(\phi_2(a))} = f(\phi_2(a)^\#)$. Thus $u = \phi_1(a)^\star - \phi_2(a)^\#$ is a Hermitian element in $CB(X)$ with numerical radius 0, and hence $u = 0$. This implies that $\phi_1(a^*) = \phi_2(a^*)$, since ϕ_1 and ϕ_2 are $*$-homomorphisms. Hence ϕ_1 equals ϕ_2 on the $*$-subalgebra generated by M in D. By weak*-density, it follows that $\phi_1 = \phi_2$ on D. □

This immediately gives the following:

Corollary 5.2.3. *Let D be a W^*-algebra generated by a dual operator algebra M. If X_1 and X_2 are two dual operator D-modules, and if $T : X_1 \to X_2$ is a w^*-continuous completely isometric and surjective M-module map, then T is a D-module map.*

Corollary 5.2.4. *Let D be a W^*-algebra generated by a dual operator algebra M. Then the category ${}_D\mathcal{R}$ of dual operator modules over D is a subcategory of the category ${}_M\mathcal{R}$ of dual operator modules over M. Similarly, ${}_D\mathcal{H}$ is a subcategory of ${}_M\mathcal{H}$.*

Next we discuss the W^*-dilation which we call the *D-dilation* of a dual operator M-module X, where D is a W^*-algebra generated by M. Strictly speaking, it should be called the W^*-D-dilation, but for brevity we will use the shorter term.

Definition 5.2.5. *A pair (E, i) is said to be a D-*dilation *of a left dual operator M-module X if the following hold:*

1. *E is a left dual operator D-module and $i : X \to E$ is a w^*-continuous completely contractive M-module map.*
2. *For any left dual operator D-module X', and any w^*-continuous completely bounded M-module map $T : X \to X'$, there exists a unique w^*-continuous completely bounded D-module map $\tilde{T} : E \to X'$ such that $\tilde{T} \circ i = T$, and also $\|T\|_{cb} = \|\tilde{T}\|_{cb}$.*

*Some authors also use the terminology D-*adjunct *for D-dilation (see [8]).*

The assertion in (2) above implies that $i(X)$ generates E as a dual operator D-module. To see this, let $E' = \overline{Di(X)}^{w*}$, and consider the quotient map $q : E \to E/E'$. Then E/E' is a left dual operator D-module such that $q \circ i = 0$. Hence the assertion in (2) in the above definition implies that the map $q = 0$. Thus $E = E'$.

Up to a complete isometric module isomorphism there is a unique pair (E, i) satisfying (1) and (2) in the above definition. To see this, let (E', i') be any other pair satisfying (1) and (2), then there exists a unique w^*-continuous completely contractive D-module linear maps $\rho : E \to E'$ and $\phi : E' \to E$ such that $\rho \circ i = i'$ and $\phi \circ i'$ $= i$. One concludes that $\rho \circ \phi$ is the identity map on $i'(X)$ and $\phi \circ \rho$ is the identity map on $i(X)$. Since $i(X)$ and $i'(X)$ generate E as a dual operator D-module, and since ϕ and ρ are w^*-continuous complete contractions, this implies that ϕ and ρ are complete isometries.

Remark 5.2.6. From the above it is clear that the D-dilation (E, i) is the unique pair satisfying (1), and such that for all dual operator D-modules X', the canonical map $i^* : CB^{\sigma}_D(E, X') \to CB^{\sigma}_M(X, X')$, given by composition with i, is an isometric isomorphism. Note that by using (1.7) and Corollary 1.6.3 in [15], it is easy to see that $M_n(CB^{\sigma}(X, Y)) \cong CB^{\sigma}(X, M_n(Y))$ completely isometrically for dual operator spaces X and Y. If X is a left dual operator M-module, then $M_n(X)$ is also a left dual operator M-module via $m \cdot [x_{ij}] = [m \cdot x_{ij}] = I_n \otimes m \cdot [x_{ij}]$, where $I_n \otimes m$ denotes the diagonal matrix in $M_n(M)$ with diagonal entries m. Indeed, if X is a dual operator M-module, the above module action is completely contractive and by Corollary 1.6.3 in [15], this action is separately w^*-continuous. This proves that $M_n(X)$ is a dual operator M-module if X is a dual operator M-module. Since i^* is an isometry for all dual operator D-modules X', it follows that $CB^{\sigma}_D(E, M_n(X')) \cong CB^{\sigma}_M(X, M_n(X'))$ for all dual operator D-modules X', which implies that i^* is a complete isometry.

Thus the D-dilation E of X satisfies:

$$CB_D^\sigma(E, X') \;\cong\; CB_M^\sigma(X, X') \tag{5.2.1}$$

completely isometrically.

By the dual operator module version of Christensen-Effros-Sinclair theorem (see Theorem 3.3.1 in [15]), X' in Definition 5.2.5 can be taken to be $B(H, K)$, where K is a normal Hilbert D-module and H is a Hilbert space. In fact, by a modification of Theorem 3.8 in [8], we may take $X' = K$. We are going to prove this important fact in the next theorem but before that we need to recall some tensor product facts.

For operator spaces X, Y and Z, we let $CB(X \times Y, Z)$ denotes the space of completely bounded bilinear maps from $X \times Y \to Z$ (in the sense of Christensen and Sinclair). It is well known that $CB(X \times Y, Z) \cong CB(X \otimes_h Y, Z)$ completely isometrically (see 1.5.4 in [15]).

If X and Y are two dual operator spaces, we use $(X \otimes_h Y)^*_\sigma$ to denote the subspace of $(X \otimes_h Y)^*$ corresponding to the completely bounded bilinear maps from $X \times Y \to \mathbb{C}$ which are separately w^*-continuous. Recall that the *normal Haagerup tensor product* $X \otimes^{\sigma h} Y$ is then defined to be the operator space dual of $(X \otimes_h Y)^*_\sigma$. If Z is another dual operator space, we denote by $CB^\sigma(X \times Y, Z)$ the space of completely bounded bilinear maps from $X \times Y \to Z$ which are separately w^*-continuous. By the matrical version of (5.22) in [25], $CB^\sigma(X \times Y, Z) \cong CB^\sigma(X \otimes^{\sigma h} Y, Z)$ completely isometrically.

Suppose X is a right dual operator M-module and Y is a left dual operator M-module. We let $(X \otimes_{hM} Y)^*_\sigma$ denote the subspace of $(X \otimes_{hM} Y)^*$ corresponding to the completely bounded balanced bilinear maps from $X \times Y \to \mathbb{C}$ that are separately w^*-continuous, where $\otimes_{hM}$ denotes the module Haagerup tensor product. Recall, by Proposition 2.1 in [31], the *module normal Haagerup tensor product* $X \otimes^{\sigma h}_M Y$ may be defined to be the operator space dual of $(X \otimes_{hM} Y)^*_\sigma$. If Z is another dual operator space, we denote by $CB^{M\sigma}(X \times Y, Z)$ the space of completely bounded balanced separately w^*-continuous bilinear maps. By Proposition 2.2 in [31], $CB^{M\sigma}(X \times Y, Z) \cong CB^{\sigma}(X \otimes_M^{\sigma h} Y, Z)$ completely isometrically.

In order to prove the next lemma, we will introduce some notation. Let $CB^{S\sigma}(X \widehat{\otimes} Y, Z)$ denote the subspace of $CB(X \widehat{\otimes} Y, Z)$ consisting of completely bounded maps from $X \widehat{\otimes} Y$ to Z that are induced by the jointly completely bounded bilinear maps from $X \times Y \to Z$ which are separately w^*-continuous, where $\widehat{\otimes}$ denotes the operator space projective tensor product (see e.g. 1.5.11 in [15]). In the case, when $Z = \mathbb{C}$, we denote $CB^{S\sigma}(X \widehat{\otimes} Y, \mathbb{C})$ by $(X \widehat{\otimes} Y)^*_\sigma$.

Lemma 5.2.7. *For any Hilbert spaces H and K and dual operator space X, $CB^{\sigma}(X, B(H, K)) \cong CB^{\sigma}(X \otimes^{\sigma h} H^c, K^c) \cong (\overline{K}^r \otimes^{\sigma h} X \otimes^{\sigma h} H^c)_*$ completely isometrically.*

Proof. For any dual operator space X, we have the following isometries:

$$\begin{aligned} CB^{\sigma}(X \otimes^{\sigma h} H^c, K^c) &\cong CB^{\sigma}(X \times H^c, K^c) \\ &\cong CB^{S\sigma}(X \widehat{\otimes} H^c, K^c) \\ &\cong CB^{\sigma}(X, CB(H^c, K^c) \\ &\cong CB^{\sigma}(X, B(H, K)) \end{aligned}$$

using Proposition 1.5.14 (1) and (1.50) from [15]. Consider

$$\begin{aligned} CB^{\sigma}(X \otimes^{\sigma h} H^c, K^c) &\cong (\overline{K}^r \widehat{\otimes} (X \otimes^{\sigma h} H^c))^*_\sigma \\ &\cong (\overline{K}^r \otimes_h (X \otimes^{\sigma h} H^c))^*_\sigma \\ &\cong (\overline{K}^r \otimes^{\sigma h} (X \otimes^{\sigma h} H^c))_* \\ &\cong (\overline{K}^r \otimes^{\sigma h} X \otimes^{\sigma h} H^c)_*, \end{aligned}$$

using (1.51) and Proposition 1.5.14 (1) in [15], and associativity of the normal Haagerup tensor product. □

Similarly we have the module version of the above lemma:

Lemma 5.2.8. *Let X be a left dual operator M-module and K be a normal Hilbert M-module. Then for any Hilbert space H, $CB^{\sigma}_M(X, B(H,K)) \cong CB^{\sigma}_M(X \otimes^{\sigma h} H^c, K^c)$ $\cong (\overline{K}^r \otimes^{\sigma h}_M X \otimes^{\sigma h} H^c)_*$ completely isometrically.*

Proof. The first isomorphism follows as above with completely bounded maps replaced with module completely bounded maps. Consider

$$\begin{aligned} CB^{\sigma}_M(X \otimes^{\sigma h} H^c, K^c) &\cong (\overline{K}^r \widehat{\otimes}_M (X \otimes^{\sigma h} H^c))^*_{\sigma} \\ &\cong (\overline{K}^r \otimes_{hM} (X \otimes^{\sigma h} H^c))^*_{\sigma} \\ &\cong (\overline{K}^r \otimes^{\sigma h}_M (X \otimes^{\sigma h} H^c))_* \\ &\cong (\overline{K}^r \otimes^{\sigma h}_M X \otimes^{\sigma h} H^c)_*, \end{aligned}$$

using Corollary 3.5.10 in [15], $K^r \otimes_{hM} - = K^r \widehat{\otimes}_M -$ and a variant of Proposition 2.4.11. □

We would like to thank David Blecher for the proof of the following lemma.

Lemma 5.2.9. *Let $S : X \to Y$ be a w^*-continuous linear map between dual operator spaces. The following are equivalent:*
(i) S is a complete isometry and surjective.
(ii) For some Hilbert space H, $S \otimes I_H : X \otimes^{\sigma h} H^c \to Y \otimes^{\sigma h} H^c$ is a complete isometry and surjective.

Proof. Firstly, suppose S is a completely isometric and w^*-homeomorphic map. Then, by the functoriality of the normal Haagerup tensor product $S \otimes I_H$ and $S^{-1} \otimes I_H$ are completely contractive w^*-continuous maps, where I_H denotes the identity map on H. Also $(S^{-1} \otimes I_H) \circ (S \otimes I_H) = Id$ on a weak* dense subset $X \otimes H$. By w^*-density, $(S^{-1} \otimes I_H) \circ (S \otimes I_H) = Id$ on $X \otimes^{\sigma h} H^c$. Similarly, $(S \otimes I_H) \circ (S^{-1} \otimes I_H) = Id$. Thus $S \otimes I_H$ is a completely isometric and w^*-homeomorphic map.

Conversely, suppose (ii) holds. Fix a $\eta \in H$ with $\|\eta\| = 1$. Let $v : X \to X \otimes^{\sigma h} \eta : x \mapsto x \otimes \eta$. Since $X \subseteq X \otimes_h H^c$ completely isometrically via v, and $X \otimes_h H^c \subseteq X \otimes^{\sigma h} H^c$ completely isometrically, this implies that v is a complete isometry. If $S \otimes I_H$ is a complete isometry, then $S \otimes I_H$ restricted to $X \otimes^{\sigma h} \eta$ is a complete isometry. Similarly, let $u : Y \to Y \otimes^{\sigma h} \eta : y \mapsto y \otimes \eta$. Thus, $S = u^{-1} \circ (S \otimes I_H) \circ v$ is a complete isometry. To see S is onto, suppose for the sake of contradiction that it is not. Then by the Krein-Smulian theorem $G = \text{Ran}(S)$ is a weak* closed proper subspace of Y. Let $\varphi \in G_\perp$ and $\varphi \neq 0$. Consider a map $r : Y \otimes^{\sigma h} H^c \to \mathbb{C} \otimes^{\sigma h} H^c : \; y \otimes \zeta \mapsto \varphi(y) \otimes \zeta$. Then $r \circ (S \otimes I_H) = 0$, since this vanishes on a w^*-dense subset $Y \otimes H^c$. So $r = 0$. Hence $\varphi(y) \otimes \zeta = 0$ for all $\zeta \in H$ and $y \in Y$. This implies $\varphi = 0$, which is a contradiction. □

Theorem 5.2.10. *Suppose E is a left dual operator D-module and $i : X \to E$ is a w^*-continuous completely contractive M-module map. Then (E, i) is the D-dilation of X if and only if the canonical map $i^* : CB^\sigma_D(E, K) \to CB^\sigma_M(X, K)$ as defined above is a complete isometric isomorphism, for all normal Hilbert D-modules K. It is sufficient to take K to be the normal universal representation of D or any normal generator for ${}_D\mathcal{H}$ in the sense of [21], [40].*

Proof. Consider the following sequence of complete contractions:

$$\overline{K}^r \otimes^{\sigma h}_M X \xrightarrow{id \otimes i} \overline{K}^r \otimes^{\sigma h}_M E \cong \overline{K}^r \otimes^{\sigma h}_D D \otimes^{\sigma h}_M E \to \overline{K}^r \otimes^{\sigma h}_D E$$

where the last map in the sequence comes from the multiplication $D \times E \to E$. Taking the composition of the above maps, we get a complete contraction $S : \overline{K}^r \otimes^{\sigma h}_M X \to \overline{K}^r \otimes^{\sigma h}_D E$. Tensoring S with the identity map on H, we get a w^*-continuous, completely contractive linear map $S_1 = S \otimes id_H : \overline{K}^r \otimes^{\sigma h}_M X \otimes^{\sigma h} H^c \to \overline{K}^r \otimes^{\sigma h}_D E \otimes^{\sigma h} H^c$ by Lemma 2.4.5. From a well known weak* topology fact, $S_1 = T^*$ for some $T : (\overline{K}^r \otimes^{\sigma h}_D E \otimes^{\sigma h} H^c)_* \to (\overline{K}^r \otimes^{\sigma h}_M X \otimes^{\sigma h} H^c)_*$. From Lemma 5.2.8, and standard weak*

density arguments, it follows that T equals i^*, as defined earlier. Indeed, we use the duality pairing, namely, $\langle \psi \otimes x \otimes \eta, T\rangle = \langle T(x)(\eta), \psi\rangle$, for $T \in CB^\sigma_M(X, B(H,K))$, $x \in X$, $\eta \in H$, $\psi \in K^*$, to check that $(i^*)^* = S_1$ on the weak* dense subset $\overline{K}^r \otimes X \otimes H^c$. Then by weak* density, it follows that $(i^*)^* = S_1 = T^*$, so $i^* = T$. Hence, i^* is an isometric isomorphism if and only if S_1 is an isometric isomorphism if and only if S is an isometric isomorphism by Lemma 5.2.9. Note that with $H = \mathbb{C}$ in Lemma 5.2.8, $CB^\sigma_M(X, K^c) = (\overline{K}^r \otimes^{\sigma h}_M X)_*$. From Lemma 5.2.8, it is clear that $CB^\sigma_D(E, K^c) \cong CB^\sigma_M(X, K^c)$ if and only if $CB^\sigma_D(E \otimes^{\sigma h} H^c, K^c) \cong CB^\sigma_M(X \otimes^{\sigma h} H^c, K^c)$ for all normal Hilbert D-modules K. For the last assertion, note that every nondegenerate normal Hilbert D-module K is a complemented submodule of a direct sum of I copies of the normal universal representation or normal generator, for some cardinal I (see [21]). Therefore we need to show that if $CB^\sigma_D(E, K) \cong CB^\sigma_M(X, K)$ completely isometrically then $CB^\sigma_D(E, K^I) \cong CB^\sigma_M(X, K^I)$ completely isometrically as well, where K^I denotes the Hilbert space direct sum of I-copies of K. This follows from the operator space fact that $CB^\sigma_M(X, Y^I) \cong M_{I,1}(CB^\sigma_M(X, Y))$ completely isometrically for any dual operator spaces X and Y which are also M-modules (see p. 156 in [27]). Here $M_{I,1}(X)$ denotes the operator space of columns of length I with entries in X, whose finite subcolumns have uniformly bounded norm. □

The following lemma shows the existence of the D-dilation. The normal module Haagerup tensor product $D \otimes^{\sigma h}_M X$ (which is a dual operator D-module by Lemma 2.4.7) acts as the D-dilation of X. We note that, since by Lemma 2.4.12 $M \otimes^{\sigma h}_M X \cong X$, there is a canonical w^*-continuous completely contractive M-module map $i : X \to D \otimes^{\sigma h}_M X$ taking $x \mapsto 1 \otimes_M x$.

Lemma 5.2.11. *For any left dual operator module X over M, the dual operator D-module $E = D \otimes^{\sigma h}_M X$ is the D-dilation of X.*

Proof. If $T : X \to X'$ is as in Definition 5.2.5, then by the functoriality of the normal module Haagerup tensor product, $\mathbb{I}_D \otimes T : D \otimes^{\sigma h}_M X \to D \otimes^{\sigma h}_M X'$ is w^*-continuous completely bounded. Composing this with the w^*-continuous module action $D \otimes^{\sigma h}_M X' \to X'$ gives the required map $\tilde{T}$. It is routine to check that $\tilde{T}$ has the required properties. □

Lemma 5.2.12. *If X is a left dual operator M-module, and if D is a W^*-algebra generated by M, then the following are equivalent:*

1. *There exists a dual operator D-module X' and a completely isometric w^*-continuous M-module map $j : X \to X'$.*
2. *The canonical w^*-continuous M-module map $i : X \to D \otimes^{\sigma h}_M X$, is a complete isometry.*

Proof. The direction (2) implies (1) is obvious. For the other direction, suppose that m is the module action on X'. Then we have the following sequence of canonical w^*-continuous completely contractive M-module maps:

$$X \xrightarrow{i} D \otimes^{\sigma h}_M X \xrightarrow{\mathbb{I} \otimes j} D \otimes^{\sigma h}_M X' \xrightarrow{m} X'.$$

The composition of these maps equals j, which is a complete isometry. This forces i to be a complete isometry, which proves the assertion. □

In the case that $D = \mathcal{C} = W^*_{max}(M)$, we call $\mathcal{C} \otimes^{\sigma h}_M X$ the maximal W^*-dilation or maximal dilation. This is the key point in proving our main theorem (Section 5.4). The reason we work mostly with maximal dilation instead of any arbitrary dilation is the following result.

Corollary 5.2.13. *For any left dual operator M-module X, the canonical M-module map $i : X \to \mathcal{C} \otimes^{\sigma h}_M X$ is a w^*-continuous complete isometry.*

Proof. This follows from the previous result, the Christensen-Effros-Sinclair representation theorem for dual operator modules, and the fact that every normal Hilbert

M-module is a normal Hilbert $\mathcal{C}$-module for the maximal W^*-algebra generated by M (i.e., the universal property of $\mathcal{C}$). □

Hence, we may regard X as a w^*-closed M-submodule of $\mathcal{C} \otimes_M^{\sigma h} X$. There is a similar notion of W^*-dilation for right dual operator modules or dual operator bimodules. The results in this section carry through analogously to these cases.

5.3 Morita equivalence and W^*-dilation

In this section, M and N are again dual operator algebras. We reserve the symbols $\mathcal{C}$ and $\mathcal{D}$ for the maximal W^*-algebras $W^*_{max}(M)$ and $W^*_{max}(N)$ generated by M and N respectively.

We begin with the following normal Hilbert module characterization of W^*-algebras which is proved in Proposition 7.2.12 in [15].

Proposition 5.3.1. *Let M be a dual operator algebra. Then M is a W^*-algebra if and only if for every completely contractive normal representation $\pi : M \to B(H)$, the commutant $\pi(M)'$ is selfadjoint.*

Corollary 5.3.2. *Suppose M and N are dual operator algebras such that the categories ${}_M\mathcal{H}$ and ${}_N\mathcal{H}$ are completely isometrically equivalent; i.e., there exist completely contractive functors $F : {}_M\mathcal{H} \to {}_N\mathcal{H}$ and $G : {}_N\mathcal{H} \to {}_M\mathcal{H}$, such that $FG \cong Id$ and $GF \cong Id$ completely isometrically, then*

1. *If M is a W^*-algebra then so is N.*

2. *Also ${}_\mathcal{C}\mathcal{H}$ and ${}_\mathcal{D}\mathcal{H}$ are completely isometrically equivalent.*

Proof. Suppose $F : {}_M\mathcal{H} \to {}_N\mathcal{H}$ and $G : {}_N\mathcal{H} \to {}_M\mathcal{H}$, are functors as in the statement of the corollary. If M is a W^*-algebra, then for $H \in {}_M\mathcal{H}$, $B_M(H)$ is a W^*-algebra by Proposition 5.3.1. The map $T \mapsto F(T)$ from $B_M(H)$ to $B_N(F(H))$ is a surjective isometric homomorphism (see Lemma 2.2 in [10] or Lemma 5.4.4 below). Hence by Theorem A.5.9 in [15], this is a $*$–homomorphism if M is a W^*-algebra, and consequently its range $B_N(F(H))$ is a W^*-algebra. Thus, if M is a W^*-algebra, then $B_N(H)$ is a W^*-algebra for all normal Hilbert N-modules H. From Proposition 5.3.1, it follows that N is a W^*-algebra. For $H \in {}_M\mathcal{H}$, we have $B_{\mathcal{C}}(H)$ is a subalgebra of $B_M(H)$. The proof that F restricts to a functor from ${}_{\mathcal{C}}\mathcal{H}$ to ${}_{\mathcal{D}}\mathcal{H}$ and similar assertion for G, follows identically to the C^*-algebra case (see Proposition 5.1 in [8]). □

Definition 5.3.3. *1. Suppose that $\mathcal{E}$ and $\mathcal{F}$ are weakly Morita equivalent W^*-algebras in the sense of Rieffel [40], and that Z is a W^*-equivalence $\mathcal{F}$-$\mathcal{E}$-bimodule (see 8.5.12 in [15]), and that $W = \overline{Z}$ is the conjugate $\mathcal{E}$-$\mathcal{F}$- bimodule of Z. Then we say that $(\mathcal{E}, \mathcal{F}, W, Z)$ is a* W^*-Morita context *(or* W^*-context *for short).*

2. *Suppose that M and N are dual operator algebras, and suppose that $\mathcal{E}$ and $\mathcal{F}$ are W^*-algebras generated by M and N respectively. Suppose that $(\mathcal{E}, \mathcal{F}, W, Z)$ is a W^*-Morita context, X is a w^*-closed M-N-submodule of W, and Y is a w^*-closed N-M-submodule of Z. Suppose that the natural pairings $Z \times W \to \mathcal{F}$ and $W \times Z \to \mathcal{E}$ restrict to maps $Y \times X \to N$, and $X \times Y \to M$ respectively, both with w^*-dense range. Then we say (M, N, X, Y) is a* subcontext *of $(\mathcal{E}, \mathcal{F}, W, Z)$. If further, $\mathcal{E}$ and $\mathcal{F}$ are maximal W^*-covers of M and N respectively, then we say that (M, N, X, Y) is a* maximal subcontext .

3. *A subcontext (M, N, X, Y) of a W^*-Morita context $(\mathcal{E}, \mathcal{F}, W, Z)$ is* left dilatable *if W is the left $\mathcal{E}$-dilation of X, and Z is the left $\mathcal{F}$-dilation of Y. In this case we say that M and N are* left weakly subequivalent *and (M, N, X, Y) is a* left subequivalence context.

There is a similar definition and symmetric theory where we replace the word 'left' by 'right' or 'two-sided'.

Remark 5.3.4. Note that (2) in the above definition implies that X and Y are nondegenerate dual operator modules over M and N.

Write $\mathcal{L}^w$ for the set of 2×2 matrices

$$\mathcal{L}^w = \left\{ \begin{bmatrix} a & x \\ y & b \end{bmatrix} : \ a \in M, b \in N, x \in X, y \in Y \right\}.$$

Write $\mathcal{L}'$ for the same set, but with entries from the W^*-context $(\mathcal{E}, \mathcal{F}, W, Z)$. It is well known that $\mathcal{L}'$ is canonically a W^*- algebra, called the linking W^*-algebra of the W^*-context $(\mathcal{E}, \mathcal{F}, W, Z)$ (see e.g., 8.5.10 in [15]). Saying that (M, N, X, Y) is a subcontext of $(\mathcal{E}, \mathcal{F}, W, Z)$ implies that $\mathcal{L}^w$ is a w^*-closed subalgebra of $\mathcal{L}'$. Thus a subcontext gives a linking dual operator algebra $\mathcal{L}^w$. Clearly $\mathcal{L}^w$ has a unit. We shall see that $\mathcal{L}^w$ generates $\mathcal{L}'$ as a W^*-algebra.

The proof of the following theorem is similar to the proof of Theorem 3.4.2 with an arbitrary W^*-dilation in place of $W^*_{max}(M)$ and hence we omit it.

Theorem 5.3.5. *Suppose that dual operator algebras M and N are linked by a weak*Morita context (M, N, X, Y) in our sense. Suppose that M is represented normally and completely isometrically as a subalgebra of $B(H)$ nondegenerately, for some Hilbert space H, and let $\mathcal{E}$ be the W^*-algebra generated by M in $B(H)$. Then $Y \otimes^{\sigma h}_M \mathcal{E}$ is a right W^*-module over $\mathcal{E}$. Also (as in the proof of Theorem 3.4.2) $Y \otimes^{\sigma h}_M \mathcal{E} \cong \overline{Y\mathcal{E}}^{w*}$ completely isometrically and w^*-homeomorphically and hence $Y \otimes^{\sigma h}_M \mathcal{E}$ contains Y as a w^*-closed M-submodule completely isometrically. Also, via this module, $\mathcal{E}$ is weakly Morita equivalent (in the sense of Rieffel) to the W^*-algebra $\mathcal{F}$ generated by the completely isometric induced normal representation of N on $Y \otimes^{\sigma h}_M H$.*

If C is a W^*-algebra generated by M, then we shall write $\mathcal{F}(C)$ for $Y \otimes^{\sigma h}_M C \otimes^{\sigma h}_M X$. By an obvious modification of Theorem 3.4.2, we have that $\mathcal{F}(C)$ is a W^*-algebra containing a copy of N, which is $*$-isomorphic and w^*-homeomorphic to $(YCX)^{-w*}$. The copy of N may be identified with $(YMX)^{-w*}$. Thus, Theorem 5.3.5 tells us that C is weakly Morita equivalent to $\mathcal{F}(C)$ as W^*-algebras.

Similarly, if D is a W^*-algebra generated by N, then we write $\mathcal{G}(D)$ for $X \otimes^{\sigma h}_N D \otimes^{\sigma h}_N Y$. Again $\mathcal{G}(D) \cong (XDY)^{-w*}$ $*$-isomorphically and w^*-homeomorphically. By associativity of the module normal Haagerup tensor product and Lemma 2.4.12, $\mathcal{G}(\mathcal{F}(C)) \cong C$, and $\mathcal{F}(\mathcal{G}(D)) \cong D$ $*$-isomorphically. One can think of $\mathcal{F}$ as a mapping between W^*-covers of M and N. There is a natural ordering of W^*-covers of a dual operator algebra. If (A, j) and (A', j') are W^*-covers of M, we then define $(A, j) \leq (A', j')$ if and only if there is a w^*-continuous $*$-homomorphism $\pi : A' \to A$ such that $\pi \circ j' = j$. It is an easy exercise (using that the range of π is w^*-closed) to check that π is surjective.

Theorem 5.3.6. *The correspondence $C \mapsto \mathcal{F}(C)$ is bijective and order preserving.*

Proof. From the above discussion, the bijectivity is clear. Suppose $\phi : C_1 \to C_2$ is a w^*-continuous quotient $*$-homomorphism between two W^*-algebras generated by M, such that $\phi|_M = Id_M$. Then by Corollary 2.4.6, $\tilde{\phi} = Id_Y \otimes \phi \otimes Id_X : Y \otimes^{\sigma h}_M C_1 \otimes^{\sigma h}_M X \to Y \otimes^{\sigma h}_M C_2 \otimes^{\sigma h}_M X$ is a w^*-continuous completely contractive map with w^*-dense range, which equals the identity when restricted to the copy of N. It is easy to check that $\tilde{\phi}$ is a homomorphism on the w^*-dense subset $Y \otimes C_1 \otimes X$. Therefore by w^*-density, $\tilde{\phi}$ is a homomorphism. Hence by Proposition A.5.8 in [15], $\tilde{\phi}$ is a $*$-homomorphism and is onto. Hence, ϕ is order preserving. □

Corollary 5.3.7. *If $\mathcal{L}^w$ is the linking dual operator algebra for a weak* Morita equivalence of dual operator algebras M and N, and if $\mathcal{L}'$ is the corresponding linking W^*-algebra of the weak Morita equivalence of W^*-algebras $W^*_{max}(M)$ and $W^*_{max}(N)$, then*

$W^*_{max}(\mathcal{L}^w) = \mathcal{L}'$.

Proof. Suppose $W^*_{max}(M)$ is normally and faithfully represented on $B(H)$ for some Hilbert space H. Then, by Lemma 2.2.4, H is a normal universal Hilbert M-module. Also M is weak* Morita equivalent to $\mathcal{L}^w$, via the dual bimodule $M \oplus^c Y$ (see Corollary 3.3.1). By Theorem 3.2.20, this induces a normal representation of $\mathcal{L}^w$ on the Hilbert space $(M \oplus_c Y) \otimes^{\sigma h}_M H^c$. By Proposition 3.3.2 we have that

$$(M \oplus^c Y) \otimes^{\sigma h}_M H^c \cong (H \oplus K)^c$$

unitarily, where $K = Y \otimes^{\sigma h}_M H^c$ and K is also a normal universal Hilbert N-module (see the remark on p. 6 in [21]). As in the proof of Theorem 3.4.2, $W^*_{max}(\mathcal{L}^w)$ may be taken to be the W^*-algebra generated by $\mathcal{L}^w$ in $B(H \oplus K)$, which is $\mathcal{L}'$. $\square$

The above corollary has a variant valid for arbitrary W^*-covers. That is, if $\mathcal{L}'$ is the corresponding linking W^*-algebra of the weak Morita equivalence of arbitrary W^*-covers then $\mathcal{L}'$ is a W^*-cover of $\mathcal{L}^w$.

Proposition 5.3.8. *If (M, N, X, Y) is a subcontext of a W^*-Morita context $(\mathcal{E}, \mathcal{F}, W, Z)$, then*

1. *X and Y generate W and Z respectively as left dual operator modules; i.e., W is the smallest w^*-closed left $\mathcal{E}$-submodule of W containing X. Similar assertions hold as right dual operator modules, by symmetry.*

2. *The linking algebra $\mathcal{L}$ of (M, N, X, Y) generates the linking W^*-algebra $\mathcal{L}'$ of $(\mathcal{E}, \mathcal{F}, W, Z)$.*

3. *If M or N is a W^*-algebra, then $(M, N, X, Y) = (\mathcal{E}, \mathcal{F}, W, Z)$.*

Proof. Since the pairing $[\cdot, \cdot] : Y \times X \to N$ has w^*-dense range, we can pick a net e_t in N which is a sum of terms of the form $[y, x]$, for $y \in Y$, $x \in X$, such that

$e_t \xrightarrow{w^*} 1_N$. Hence $we_t \xrightarrow{w^*} w$ for all $w \in W$. Thus, sums of terms of the form $w[y,x]$, for $w \in W, x \in X, y \in Y$ are w^*-dense in W. However, $w[y,x] = (w,y)x \in \mathcal{E}X$ which shows that $\mathcal{E}X$ is w^*-dense in W. Thus, X generates W as a left dual operator $\mathcal{E}$-module. Assertions (2) and (3) follow from (1). For example, if M is a W^*-algebra, then clearly $X = W$. Since Y generates Z as a right dual operator module, we have $Z = \overline{Y\mathcal{E}}^{w*} = \overline{YM}^{w*} = Y$. Since the ranges of the natural pairings $Z \times W \to \mathcal{F}$ and $Y \times X \to N$ are weak* dense, this implies that $\mathcal{F} = N$. □

Theorem 5.3.9. *If (M, N, X, Y) is a weak* Morita context which is a subcontext of a W*-Morita context $(\mathcal{E}, \mathcal{F}, W, Z)$, then it is a dilatable subcontext.*

Proof. By Proposition 5.3.8, X and Y generate W and Z, respectively, as left dual operator modules. Hence we have a w^*-continuous complete contraction $\mathcal{E} \otimes_M^{\sigma h} X \to W$ with w^*-dense range. On the other hand,

$$W \cong W \otimes_N^{\sigma h} N \cong W \otimes_N^{\sigma h} Y \otimes_M^{\sigma h} X \cong (W \otimes_N^{\sigma h} Y) \otimes_M^{\sigma h} X$$

completely isometrically and w^*-homeomorphically. However, the pairing $(\cdot,\cdot)$: $W \times Y \to \mathcal{E}$ determines a w^*-continuous complete contraction $W \otimes_M^{\sigma h} Y \to \mathcal{E}$, and so we obtain a w^*-continuous complete contraction $W \to \mathcal{E} \otimes_M^{\sigma h} X$. Recall from Chapter 3 that N has an 'approximate identity' of the form $\sum_{i=1}^{n_t} [y_i^t, x_i^t]$. Under the above identifications,

$$w \mapsto w \otimes_N 1_N \mapsto w \otimes_N w^*\text{-}\lim_t \sum_{i=1}^{n_t} y_i^t \otimes_M x_i^t \mapsto w^*\text{-}\lim_t \sum_{i=1}^{n_t} (w \otimes_N y_i^t) \otimes_M x_i^t$$

$$\mapsto w^*\text{-}\lim_t \sum_{i=1}^{n_t} (w, y_i^t) x_i^t \mapsto w^*\text{-}\lim_t \sum_{i=1}^{n_t} w[y_i^t, x_i^t] = w.$$

Hence, the composition of these maps

$$\mathcal{E} \otimes_M^{\sigma h} X \to W \to \mathcal{E} \otimes_M^{\sigma h} X$$

is the identity map, from which it follows that $W \cong \mathcal{E} \otimes^{\sigma h}_{M} X$. Similarly Z is the dilation of Y. ☐

Theorem 5.3.10. *If (M, N, X, Y) is a left dilatable maximal subcontext of a W^*-context, then M and N are weak* Morita equivalent dual operator algebras. Indeed, it also follows that (M, N, X, Y) is a weak* Morita context. Conversely, every weak* Morita equivalence of dual operator algebras occurs in this way. That is, every weak* Morita context is a left dilatable maximal subcontext of a W^*-Morita context.*

Proof. Every weak* Morita context is a left dilatable maximal subcontext of a W*-Morita context is proved in Theorem 3.4.2 in Chapter 3. For the converse, let $\mathcal{C}$ and $\mathcal{D}$ be the usual maximal W^*-algebras of M and N respectively, and let (M, N, X, Y) be a left dilatable subcontext of $(\mathcal{C}, \mathcal{D}, W, Z)$. Using Corollary 5.2.13 and Lemma 5.2.11, we have

$$Y \otimes^{\sigma h}_{M} X \subset (\mathcal{D} \otimes^{\sigma h}_{N} Y) \otimes^{\sigma h}_{M} X \cong Z \otimes^{\sigma h}_{M} X \cong (Z \otimes^{\sigma h}_{\mathcal{C}} \mathcal{C}) \otimes^{\sigma h}_{M} X \cong Z \otimes^{\sigma h}_{\mathcal{C}} W \cong \mathcal{D},$$

complete isometrically and w^*-homeomorphically. On the other hand, we have the canonical w^*-continuous complete contraction

$$Y \otimes^{\sigma h}_{M} X \to N \subset \mathcal{D}$$

coming from the restricted pairing in Definition 5.3.3 (2). It is easy to check that the composition of maps in these two sequences agree. Thus the canonical map $Y \otimes^{\sigma h}_{M} X \to N$ is a w^*-continuous completely isometric isomorphism. Similarly, $X \otimes^{\sigma h}_{N} Y \to M$ is a w^*-continuous completely isometric isomorphism. Hence by the Krein-Smulian theorem, $X \otimes^{\sigma h}_{N} Y \cong M$ and $Y \otimes^{\sigma h}_{M} X \cong N$ completely isometrically and w^*-homeomorphically. Thus M and N are weak* Morita equivalent dual operator algebras. ☐

5.4 The main theorem

Definition 5.4.1. *Two dual operator algebras M and N are* (left) dual operator Morita equivalent *if there exist completely contractive functors $F : {}_M\mathcal{R} \to {}_N\mathcal{R}$ and $G : {}_N\mathcal{R} \to {}_M\mathcal{R}$ which are weak* continuous on morphism spaces (see below), such that $FG \cong Id$ and $GF \cong Id$ completely isometrically. Such F and G will be called* dual operator equivalence functors*.*

Note that by Corollary 3.5.10 in [15], $CB_M(V, W)$ for $V, W \in {}_M\mathcal{R}$ is a dual operator space, but $CB^\sigma_M(V, W)$ is not a w^*-closed subspace of $CB_M(V, W)$. In the above definition, by the functor F being w^*-continuous on morphism spaces, we mean that if $(f_t) \subseteq CB^\sigma_M(V, W)$, $f_t \overset{w^*}{\to} f$ in $CB_M(V, W)$, and if f also lies in $CB^\sigma_M(V, W)$, then $F(f_t) \overset{w^*}{\to} F(f)$ in $CB_N(F(V), F(W))$. Similarly for the functor G. We also assume that the natural transformations coming from $GF \cong Id$ and $FG \cong Id$ are weak* continuous in the sense that for all $V \in {}_M\mathcal{R}$, the natural transformation $w_V : GF(V) \to V$ is a weak* continuous map. A similar statement is true for $FG \cong Id$.

There is an obvious analogue to right dual operator Morita equivalence, where we are concerned with right dual operator modules. Throughout, we write $\mathcal{C}$ and $\mathcal{D}$ for $W^*_{max}(M)$ and $W^*_{max}(N)$ respectively.

We now state our main theorem:

Theorem 5.4.2. *Two dual operator algebras are weak* Morita equivalent if and only if they are left dual operator Morita equivalent if and only if they are right dual operator Morita equivalent. Suppose that F and G are the left dual operator equivalence functors, and set $Y = F(M)$ and $X = G(N)$. Then X is a weak* Morita equivalence M-N-bimodule. Similarly Y is a weak* Morita equivalence N-M-bimodule; that is, (M, N, X, Y) is a weak* Morita context. Moreover, $F(V) \cong Y \otimes^{\sigma h}_M V$ completely isometrically and weak* homeomorphically (as dual operator N-modules) for all $V \in$*

${}_M\mathcal{R}$. *Thus,* $F \cong Y \otimes^{\sigma h}_M -$ *and* $G \cong X \otimes^{\sigma h}_N -$ *completely isometrically. Also* F *and* G *restrict to equivalences of the subcategory* ${}_M\mathcal{H}$ *with* ${}_N\mathcal{H}$, *the subcategory* ${}_\mathcal{C}\mathcal{H}$ *with* ${}_\mathcal{D}\mathcal{H}$, *and the subcategory* ${}_\mathcal{C}\mathcal{R}$ *with* ${}_\mathcal{D}\mathcal{R}$.

One direction of the main theorem (i.e., the weak* Morita equivalent dual operator algebras are left dual operator Morita equivalent) is proved in Chapter 3 (Theorem 3.2.5) with the exception that the functors implementing the categorical equivalences are weak* continuous in the sense described above. See [14] for the proof of this part.

We will use techniques similar to those of [9] and [10] to prove our main theorem. Mostly this involves the change of tensor product and modification of arguments in the present setting of weak* topology.

The following lemmas will be very useful to us. Their proofs are almost identical to analogous results in [9] and therefore are omitted.

Lemma 5.4.3. *Let* $V \in {}_M\mathcal{R}$. *Then* $v \mapsto r_v$ *where* $r_v(m) = mv$, *is a* w^**-continuous complete isometry of* V *onto* $CB_M(M, V)$. *In this case,* $CB_M(M, V) = CB^\sigma_M(M, V)$; *i.e.,* $V \cong CB^\sigma_M(M, V)$ *completely isometrically and* w^**-homeomorphically.*

Lemma 5.4.4. *If* V, $V' \in {}_M\mathcal{R}$ *then the map* $T \mapsto F(T)$ *gives a completely isometric surjective linear isomorphism* $CB^\sigma_M(V, V') \cong CB^\sigma_N(F(V), F(V'))$. *If* $V = V'$, *then this map is a completely isometric surjective homomorphism.*

Lemma 5.4.5. *For any* $V \in {}_M\mathcal{R}$, *we have* $F(R_m(V)) \cong R_m(F(V))$ *and* $F(C_m(V)) \cong C_m(F(V))$ *completely isometrically.*

Lemma 5.4.6. *The functors* F *and* G *restrict to a completely isometric functorial equivalence of the subcategories* ${}_M\mathcal{H}$ *and* ${}_N\mathcal{H}$.

Proof. Let $H \in {}_M\mathcal{H}$. Recall that H with its column Hilbert space structure H^c is a left dual operator M-module. We need to show that $K = F(H^c) \in {}_N\mathcal{H}$ or equivalently

$F(H^c)$ is a column Hilbert space. For any dual operator space X and $m \in \mathbb{N}$, we have $X \otimes_h C_m = X \otimes^{\sigma h} C_m$. Hence by Proposition 2.4 in [9], it suffices to show that the identity map $K \otimes_{min} C_m \to K \otimes^{\sigma h} C_m$ is a complete contraction for all $m \in \mathbb{N}$. Since all operator space tensor products coincide for Hilbert column spaces, we have $C_m(H^c) \cong H^c \otimes_{min} C_m \cong H^c \otimes_h C_m \cong H^c \otimes^{\sigma h} C_m$. Thus

$$\begin{aligned} K \otimes_{min} C_m &\cong C_m(F(H^c)) \\ &\cong F(C_m(H^c)) \\ &\cong F(H^c \otimes^{\sigma h} C_m) \\ &\cong F(G(K) \otimes^{\sigma h} C_m) \end{aligned}$$

using Lemma 5.4.5 and $G(K) \cong H^c$. Also, using Lemma 5.4.3 and Lemma 5.4.4 we have

$$\begin{aligned} G(K) &\cong CB_M(M, G(K)) \\ &\cong CB_N^\sigma(Y, FG(K)) \\ &\cong CB_N^\sigma(Y, K). \end{aligned}$$

By Lemma 2.4.5, we get a complete contraction $G(K) \otimes^{\sigma h} C_m \to CB_N^\sigma(Y, K) \otimes^{\sigma h} C_m$. Now $CB_N^\sigma(Y, K) \otimes^{\sigma h} C_m \to CB_N^\sigma(Y, K \otimes^{\sigma h} C_m) : T \otimes z \mapsto y \mapsto T(y) \otimes z$ for $T \in CB_N^\sigma(Y, K)$ and $z \in C_m$, is a complete contraction. Again using Lemma 5.4.3 and Lemma 5.4.4, we have $CB_N^\sigma(Y, K \otimes^{\sigma h} C_m) \cong CB_M^\sigma(M, G(K \otimes^{\sigma h} C_m)) \cong G(K \otimes^{\sigma h} C_m)$. Taking the composition of above maps gives a complete contraction $G(K) \otimes^{\sigma h} C_m \to G(K \otimes^{\sigma h} C_m)$. Applying F to this map, we get a complete contraction $F(G(K) \otimes^{\sigma h} C_m) \to K \otimes^{\sigma h} C_m$. This together with $K \otimes_{min} C_m \cong F(G(K) \otimes^{\sigma h} C_m)$ gives the required complete contraction $K \otimes_{min} C_m \to K \otimes^{\sigma h} C_m$. $\square$

Corollary 5.4.7. *The functors F and G restrict to a completely isometric equivalence of ${}_{\mathcal{C}}\mathcal{H}$ and ${}_{\mathcal{D}}\mathcal{H}$.*

The above is Corollary 5.3.2 proved earlier. Also, this restricted equivalence is a normal $*$-equivalence in the sense of Rieffel [40], and so $\mathcal{C}$ and $\mathcal{D}$ are weak Morita equivalent in the sense of Definition 7.4 in [40].

Lemma 5.4.8. *For a dual operator M-module V, the canonical map $\tau_V : Y \otimes V \to F(V)$ given by $y \otimes v \mapsto F(r_v)(y)$ is separately w^*-continuous and extends uniquely to a completely contractive map on $Y \otimes^{\sigma h}_M V$. Moreover, this map has w^*-dense range.*

Proof. Since the functor F is w^*-continuous on morphism spaces, it is easy to check that $\tau_V : Y \times V \to F(V)$ is a separately w^*-continuous bilinear map. To see that τ_V has w^*-dense range, suppose the contrary. Let $Z = F(V)/N$ where $N = \overline{Range(\tau_V)}^{w*}$ and let $Q : F(V) \to Z$ be the nonzero w^*-continuous quotient map. Then $G(Q) : G(F(V)) \to G(Z)$ is nonzero. Thus there exists $v \in V$ such that $G(Q)w_V^{-1}r_v \neq 0$ as a map on M, where w_V is the w^*-continuous completely isometric natural transformation $GF(V) \to V$ coming from $GF \cong Id$. Hence $FG(Q)F(w_V^{-1})F(r_v) \neq 0$, and thus $QTF(r_v) \neq 0$ for some w^*-continuous module map $T : F(V) \to F(V)$ since w_V^{-1} is w^*-continuous by the Krein-Smulian theorem. By Lemma 5.4.4, $T = F(S)$ for some w^*-continuous module map $S : V \to V$, so that $QF(r_{v'}) \neq 0$ for $v' = S(v) \in V$. Hence $Q \circ \tau_V \neq 0$, which is a contradiction. Again as in the proof of Lemma 2.6 in [10], τ_V is a complete contraction. Thus, τ_V is a separately w^*-continuous completely contractive bilinear map. The result follows from the universal property of $Y \otimes^{\sigma h}_M V$. $\square$

Let $(M, N, \mathcal{C}, \mathcal{D}, F, G, X, Y)$ be as above. We let $H \in {}_M\mathcal{H}$ be the Hilbert space of the normal universal representation of $\mathcal{C}$ and let $K = F(H)$. By Lemma 5.4.6 and Corollary 5.4.7, F and G restrict to equivalences of ${}_M\mathcal{H}$ with ${}_N\mathcal{H}$, and restrict further to normal $*$-equivalences of ${}_\mathcal{C}\mathcal{H}$ with ${}_\mathcal{D}\mathcal{H}$. By Proposition 1.3 in [40] , $\mathcal{D}$ acts faithfully on K. Hence, we can regard $\mathcal{D}$ as a subalgebra of $B(K)$. Define $Z = F(\mathcal{C})$ and $W = G(\mathcal{D})$.

From Lemma 5.4.8, with $V = M$, it follows that Y is a right dual operator M-module with module action $y \cdot m = F(r_m)(y)$, for $y \in Y$, $m \in M$ and $r_m : M \to M$: $c \mapsto cm$ is simply right multiplication by m. Similarly, X is a right dual operator N-module, and Z and W are dual operator N-$\mathcal{C}$- and M-$\mathcal{D}$-bimodules respectively. The inclusion i of M in $\mathcal{C}$ induces a completely contractive w^*-continuous inclusion $F(i)$ of Y in Z. One can check that $F(i)$ is a N-M-module map. By Lemma 5.4.9 below and its proof, it is easy to see that $F(i)$ is a complete isometry. Hence we may regard Y as a w^*-closed N-M-submodule of Z and similarly X may be regarded as a w^*-closed M-N-submodule of W.

With $V = X$ in Lemma 5.4.8, there is a left N-module map $Y \otimes X \to F(X)$ defined by $y \otimes x \mapsto F(r_x)(y)$. Since $F(X) = FG(N) \cong N$, we get a left N-module map $[.] : Y \otimes X \to N$. In a similar way we get a module map $(.) : X \otimes Y \to M$. In what follows we use the same notation for the unlinearized bilinear maps, so for example we use the symbol $[y, x]$ for $[y \otimes x]$. These maps $(.)$ and $[.]$ have natural extensions to $Y \otimes W \to \mathcal{D}$ and $X \otimes Z \to \mathcal{C}$ respectively, which we denote by the same symbols. Namely, $[y, w]$ is defined via τ_W for $y \in Y$ and $w \in W$. By Lemma 5.4.8, these maps have weak* dense ranges.

Lemma 5.4.9. *The canonical maps $X \to CB^\sigma_N(Y, N)$ and $Y \to CB^\sigma_M(X, M)$, induced by $[.]$ and $(.)$ respectively, are completely isometrically isomorphic. Similarly, the extended maps $W \to CB^\sigma_N(Y, \mathcal{D})$ and $Z \to CB^\sigma_M(X, \mathcal{C})$ are complete isometries.*

Proof. By Lemma 5.4.3 and Lemma 5.4.4, we have $X \cong CB^\sigma_M(M, X) \cong CB^\sigma_N(Y, F(X))$ $\cong CB^\sigma_N(Y, N)$ completely isometrically. Taking the composition of these maps shows that $x \in X$ corresponds to the map $y \mapsto [y, x]$ in $CB^\sigma_N(Y, N)$. Similarly for the other maps. □

Next consider maps $\phi : Z \to B(H, K)$, and $\rho : W \to B(K, H)$ defined by $\phi(z)(\zeta)$ $= F(r_\zeta)(z)$, and $\rho(w)(\eta) = \omega_H G(r_\eta)(w)$, for $\zeta \in H$ and $\eta \in K$ where $\omega_H : GF(H) \to$

H is the w^*-continuous M-module map coming from the natural transformation $GF \cong Id$. Again $r_\zeta : \mathcal{C} \to H$ and $r_\eta : \mathcal{D} \to K$ are the obvious right multiplications. As ω_H is an isometric onto map between Hilbert spaces, ω_H is unitary and hence also a $\mathcal{C}$-module map by Corollary 5.2.3. One can check that:

$$\rho(x)\phi(z) = (x, z) \quad and \quad \phi(y)\rho(w) = [y, w]V \tag{5.4.1}$$

for all $x \in X$, $y \in Y$, $z \in Z$, $w \in W$ and where $V \in B(K)$ is a unitary operator in $\mathcal{D}'$ composed of two natural transformations. A calculation similar to that in Lemma 4.3 in [10], shows that the unitary V is in the center of $\mathcal{D}$, hence $\phi(y)\rho(w) \in \mathcal{D}$ for all $y \in Y$ and $w \in W$.

Lemma 5.4.10. *The map ϕ (respectively ρ) is a completely isometric w^*-continuous N-$\mathcal{C}$-module map (respectively, M-$\mathcal{D}$-module map). Moreover, $\phi(z_1)^*\phi(z_2) \in \mathcal{C}$ for all $z_1, z_2 \in Z$, and $\rho(w_1)^*\rho(w_2) \in \mathcal{D}$, for all $w_1, w_2 \in W$.*

Proof. We will prove that the maps ϕ and ρ are w^*-continuous. The rest of the assertions follow as in Lemma 4.2 in [10] and by von Neumann's double commutant theorem. To see that ϕ is w^*-continuous, let (z_t) be a bounded net in Z such that $z_t \xrightarrow{w^*} z$ in Z. For $\zeta \in H$, we have $F(r_\zeta) \in CB^\sigma_N(Z, K)$. Hence $F(r_\zeta)(z_t) \to F(r_\zeta)(z)$ weakly. That is, $\phi(z_t) \to \phi(z)$ in the WOT and it follows that ϕ is weak* continuous. A similar argument works for ρ. □

We will follow the approach of [9] to prove the selfadjoint analogue of our main theorem, which involves a change of the tensor product. Nonetheless, for completeness we will give the proof.

Theorem 5.4.11. *Two W^*-algebras A and B are weakly Morita equivalent in the sense of Rieffel if and only if they are dual operator Morita equivalent in the sense of Definition 5.4.1. Suppose that F and G are the dual operator equivalence functors,* *and set* $Z = F(A)$ *and* $W = G(B)$*. Then,* W *is a* W^**-equivalence* A*-*B*-bimodule,* Z *is a* W^**-equivalence* B*-*A*-bimodule, and* Z *is unitarily and* w^**-homeomorphically isomorphic to the conjugate* W^**-bimodule* $\overline{W}$ *of* W*. Moreover,* $F(V) \cong Z \otimes_A^{\sigma h} V$ *completely isometrically and weak* homeomorphically (as dual operator* B*-modules) for all* $V \in {}_A\mathcal{R}$*. Thus* $F \cong Z \otimes_A^{\sigma h} -$ *and* $G \cong W \otimes_B^{\sigma h} -$ *completely isometrically. Also* F *and* G *restrict to equivalences of the subcategory* ${}_A\mathcal{H}$ *with* ${}_B\mathcal{H}$*.*

Proof. In Chapter 3 we saw that the weakly Morita equivalent W^*-algebras (in the sense of Rieffel) are weak* Morita equivalent. Hence by Theorem 3.2.5, they have equivalent categories of dual operator modules and the assertion about the form of the functors also holds.

For the other direction, observe that by Corollary 5.4.6, the functors F and G restrict to a completely isometric equivalence of ${}_A\mathcal{H}$ and ${}_B\mathcal{H}$. Hence, by Definition 7.4 in [40], A and B are weakly Morita equivalent in the sense of Rieffel. We will follow [9] to prove the rest of the assertions.

By the polarization identity and Lemma 5.4.10, W is a right C^*-module over B with inner product $\langle w_1, w_2\rangle_B = \rho(w_1)^*\rho(w_2)$, for $w_1, w_2 \in W$. Similarly, W is a left C^*-module over A by setting ${}_A\langle w_1, w_2\rangle = \rho(w_1)\rho(w_2)^*$. To see that this inner product lies in A, note that, since the range of (.) is w^*-dense in A, we can choose a net in A of the form $e_\alpha = \sum_{k=1}^{n(\alpha)}(w_k, z_k) = \sum_{k=1}^{n(\alpha)} \rho(w_k)\phi(z_k)$ where $z_k \in Z$ and $w_k \in W$, such that $e_\alpha \xrightarrow{w^*} 1_A$. Then, $e_\alpha^* \xrightarrow{w^*} 1_A$. Since ρ is a weak* continuous A-module map, $\rho(w)^* = w^*\text{-}\lim_\alpha \rho(e_\alpha^* w)^* = w^*\text{-}\lim_\alpha \rho(w)^* e_\alpha$, it follows that $\rho(w)\rho(w)^*$ is a weak* limit of finite sums of terms of the form $\rho(w)(\rho(w)^*\rho(w_k))\phi(z_k) = \rho(w)\phi(bz_k) = (w, bz_k) \in A$, where $b = \rho(w)^*\rho(w_k) \in B$. Thus $\rho(w)\rho(w)^* \in A$. By the polarization identity $\rho(w_1)\rho(w_2)^* \in A$. Similarly, Z is both a left and a right C^*-module. To see that Z is a w^*-full right C^*-module over A, rechoose a net in A of the form $e_\alpha = \sum_{k=1}^{n(\alpha)} \rho(w_k)\phi(z_k)$ such that $e_\alpha \to I_H$ strongly, so that $e_\alpha^* e_\alpha \to I_H$ weak* as done in

Theorem 3.3.4. However $e_\alpha^* e_\alpha = \sum_{k,l} \phi(z_k)^* b_{kl} \phi(z_l)$ where $b_{kl} = \rho(w_k)^* \rho(w_l) \in B$. Since $P = [b_{kl}]$ is a positive matrix, it has a square root $R = [r_{ij}]$, with $r_{ij} \in B$. Thus $e_\alpha^* e_\alpha = \sum_k \phi(z_k^\alpha)^* \phi(z_k^\alpha)$ where $z_k^\alpha = \sum_j r_{kj} z_j$. From this one can easily deduce that the A-valued inner product on Z has w^*-dense range. Similarly Z is a weak* full left C^*-module over B. Similarly for W. Since ρ and ϕ are w^*-continuous, the inner products are separately w^*-continuous. Hence, by Lemma 8.5.4 in [15], W and Z are W^*-equivalence bimodules, implementing the weak Morita equivalence of A and B. Note that by Corollary 8.5.8 in [15], $CB_A(W, A) = CB_A^\sigma(W, A)$. Thus by (8.18) in [15] and Lemma 5.4.9, $Z \cong \overline{W}$ completely isometrically.

Let $V \in {}_{\mathcal{A}}\mathcal{R}$. By Lemma 5.4.3, Lemma 5.4.4, Theorem 4.2.8, and the fact that $Z \cong \overline{W}$, we have the following sequence of isomorphisms:

$$F(V) \cong CB_B^\sigma(B, F(V)) \cong CB_A^\sigma(W, V) \cong Z \otimes_A^{\sigma h} V$$

as left dual operator B-modules. Thus the conclusions of the theorem hold. □

Now we will come back to the setting where M and N are dual operator algebras and $\mathcal{C}$ and $\mathcal{D}$ are maximal W^*-algebras generated by M and N respectively.

Theorem 5.4.12. *The W^*-algebras $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent. In fact Z, which is a dual operator N-$\mathcal{C}$-bimodule, is a W^*-equivalence $\mathcal{D}$-$\mathcal{C}$-bimodule. Similarly, W is a W^*-equivalence $\mathcal{C}$-$\mathcal{D}$-bimodule, and W is unitarily and w^*-homeomorphically isomorphic to the conjugate W^*-bimodule $\overline{Z}$ of Z (and as dual operator bimodules).*

Proof. By Lemma 5.4.10, it follows that $\rho(W)$ is a w^*-closed TRO (a closed subspace $Z \subset B(K, H)$ with $ZZ^*Z \subset Z$). Hence, by 8.5.11 in [15] and Lemma 5.4.10, W (or equivalently $\rho(W)$) is a right W^*-module over $\mathcal{D}$ with inner product $\langle w_1, w_2 \rangle_{\mathcal{D}} = \rho(w_1)^* \rho(w_2)$. Since ρ is a complete isometry, the induced norm on W coming from the inner product coincides with the usual norm. Similarly Z is a right W^*-module over $\mathcal{C}$. Also, W (or equivalently $\rho(W)$) is a w^*-full left W^*-module over $\mathcal{E}$ = weak* closure of $\rho(W)\rho(W)^*$, with the obvious inner product ${}_{\mathcal{E}}\langle w_1, w_2\rangle = \rho(w_1)\rho(w_2)^*$. We will show that $\mathcal{E} = \mathcal{C}$. Analogous statements hold for $\mathcal{D}$ and ϕ. It will be understood that whatever a property is proved for W, by symmetry, the matching assertions for Z hold.

Let $\mathcal{L}^w$ be the linking W^*-algebra for the right W^*-module W, viewed as a weak* closed subalgebra of $B(H \oplus K)$. We let $\mathcal{A}$ equal the weak* closure of $\rho(W)\phi(Y)$. It is easy to check, using the fact that $\phi(Y)\rho(W) \in \mathcal{D}$ (see above Lemma 5.4.10) and Lemma 5.4.10, that $\mathcal{A}$ is a dual operator algebra. By the last assertion of Lemma 5.4.8 and (5.4.1), $M = \overline{\rho(X)\phi(Y)}^{w*} \subseteq \mathcal{A}$ and the identity of M is an identity of $\mathcal{A}$. We let $\mathcal{U}$ be the weak* closure of $\mathcal{D}\phi(Y)$, and we define $\mathcal{L}$ to be the following subset of $B(H \oplus K)$

$$\begin{bmatrix} \mathcal{A} & \rho(W) \\ \mathcal{U} & \mathcal{D} \end{bmatrix}.$$

Using (5.4.1) and Lemma 5.4.10, it is easy to check that $\mathcal{L}$ is a subalgebra of $B(H\oplus K)$. By explicit computation and Cohen's factorization theorem, $\mathcal{L}^w\mathcal{L} = \mathcal{L}$ and $\mathcal{L}\mathcal{L}^w = \mathcal{L}^w$. Indeed, by Lemma 5.4.10 and the fact that $\rho(W)$ is a TRO, it follows that $\mathcal{L}^w\mathcal{L} \subseteq \mathcal{L}$. Again by using (5.4.1), Lemma 5.4.10 and the fact that $\rho(W)^*$ is a left W^*-module over $\mathcal{D}$, it follows that $\mathcal{L}\mathcal{L}^w \subseteq \mathcal{L}^w$. As $\rho(W)$ is a right W^*-module over $\mathcal{D}$ so $\rho(W)$ is a nondegenerate D-module (see 8.1.3 in [15]), hence $\rho(W) = \rho(W)\mathcal{D}$ by Cohen's factorization theorem (A.6.2 in [15]). For the same reason, $\rho(W) = \rho(W)\rho(W)^*\rho(W)$. Now one can easily check that $\mathcal{L} \subseteq \mathcal{L}^w\mathcal{L}$ and similarly $\mathcal{L}^w \subseteq \mathcal{L}\mathcal{L}^w$. Hence $\mathcal{L}^w\mathcal{L} = \mathcal{L}$ and $\mathcal{L}\mathcal{L}^w = \mathcal{L}^w$. Therefore, we conclude that $\mathcal{L}^w = \mathcal{L}$. Comparing corners of these algebras gives $\mathcal{E} = \mathcal{A}$ and $\mathcal{U} = \rho(W)^*$. Thus, $M \subseteq \mathcal{E}$, from which it follows that $\mathcal{C} \subseteq \mathcal{E}$, since $\mathcal{C}$ is the W^*-algebra generated by M in $B(H)$. Thus $\rho(W)$ is a left $\mathcal{C}$-module, so W can be made into a left $\mathcal{C}$-module in a unique way (by Theorem 5.2.2). Also by Corollary 5.2.3, ρ is a left $\mathcal{C}$-module map. By symmetry, Z is a left $\mathcal{D}$-module and ϕ is a $\mathcal{D}$-module map, so that $\rho(W)^* = \mathcal{U} = \overline{\mathcal{D}\phi(Y)}^{w^*} \subset \phi(Z)$. By symmetry,

$\phi(Z)^* \subset \rho(W)$, so that $\rho(W)^* = \phi(Z)$. Since, $\phi(Z) = \overline{\mathcal{D}\phi(Y)}^{w*}$, by symmetry, $\rho(W) = \overline{\mathcal{C}\rho(X)}^{w*}$. Also, $\rho(W)\phi(Y) \subset \overline{\mathcal{C}\rho(X)\phi(Y)}^{w^*} \subset \mathcal{C}$ and thus $\mathcal{E} = \mathcal{A} \subset \mathcal{C}$. Thus $\mathcal{E} = \mathcal{A} = \mathcal{C}$, and that $\mathcal{D} = \overline{\phi(Z)\phi(Z)^*}^{w^*} = \overline{\rho(W)^*\rho(W)}^{w^*}$. This proves the theorem. □

5.5 W^*-restrictable equivalences

Definition 5.5.1. *We say that a dual operator equivalence functor F is* W^*-restrictable*, if F restricts to a functor from ${}_{\mathcal{C}}\mathcal{R}$ into ${}_{\mathcal{D}}\mathcal{R}$.*

We prove our main theorem under the assumption that the functors F and G are W^*-restrictable. Later we will prove that this condition is automatic; i.e., the functors F and G are automatically W^*-restrictable.

Remark 5.5.2. The canonical equivalence functors coming from a given weak* Morita equivalence are W^*-restrictable. Suppose that M and N are weak* Morita equivalent and let (M, N, X, Y) be a weak* Morita context. Then from Theorem 3.4.2 we know that $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent W^*-algebras, with W^*-equivalence $\mathcal{D}$-$\mathcal{C}$-bimodule $Z = Y \otimes^{\sigma h}_M \mathcal{C}$. From Theorem 3.2.5, $F(V) = Y \otimes^{\sigma h}_M V$, for V a dual operator M-module. However, if V is a dual operator $\mathcal{C}$-module, $Y \otimes^{\sigma h}_M V \cong Y \otimes^{\sigma h}_M \mathcal{C} \otimes^{\sigma h}_{\mathcal{C}} V \cong Z \otimes^{\sigma h}_{\mathcal{C}} V$. Hence, F restricted to ${}_{\mathcal{C}}\mathcal{R}$ is equivalent to $Z \otimes^{\sigma h}_{\mathcal{C}} -$, and thus F is W^*-restrictable.

Theorem 5.5.3. *Suppose that the dual operator equivalence functors F and G are W^*-restrictable. Then the conclusions of the Theorem 5.4.2 all hold.*

Proof. Clearly, F and G give a dual operator Morita equivalence of ${}_{\mathcal{C}}\mathcal{R}$ and ${}_{\mathcal{D}}\mathcal{R}$ when restricted to these subcategories. Set $Y = F(M)$, $Z = F(\mathcal{C})$, $X = G(N)$, and W

$= G(\mathcal{D})$ as before. By Theorem 5.4.11, $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent von Neumann algebras with Z and W as W^*-equivalence bimodules. From the discussion above Lemma 5.4.9, Y is a right dual operator M-module and X is a right dual operator N-module. Also Y is a w^*-closed N-M-submodule of Z and X is a w^*-closed M-N-submodule of W.

For any left dual operator $\mathcal{C}$-module X', we have the following sequence of canonical complete isometries by Lemma 5.4.3 and Lemma 5.4.4:

$$\begin{aligned} CB^{\sigma}_{M}(X, X') &\cong CB^{\sigma}_{N}(N, F(X')) \\ &\cong F(X') \\ &\cong CB^{\sigma}_{\mathcal{D}}(\mathcal{D}, F(X')) \\ &\cong CB^{\sigma}_{\mathcal{C}}(W, X'). \end{aligned}$$

Hence, by the discussion following Definition 5.2.5, and by Lemma 5.2.11, we have $W \cong \mathcal{C} \otimes^{\sigma h}_{M} X$ completely isometrically and as $\mathcal{C}$-modules. It can be checked that this isometry is a right N-module map. Similarly, $Z \cong \mathcal{D} \otimes^{\sigma h}_{N} Y$.

For any dual operator M-module V, we have, $Y \otimes^{\sigma h}_{M} V \subset (\mathcal{D} \otimes^{\sigma h}_{N} Y) \otimes^{\sigma h}_{M} V \cong Z \otimes^{\sigma h}_{M} V$ completely isometrically, since any dual operator module is contained in its maximal dilation. On the other hand, using Lemma 5.4.8, Lemma 5.4.4, and Theorem 5.4.11, we have the following sequence of canonical completely contractive N-module maps:

$$Y \otimes^{\sigma h}_{M} V \to F(V) \to F(\mathcal{C} \otimes^{\sigma h}_{M} V) \cong Z \otimes^{\sigma h}_{\mathcal{C}} (\mathcal{C} \otimes^{\sigma h}_{M} V) \cong Z \otimes^{\sigma h}_{\mathcal{C}} V.$$

The composition of the maps in this sequence coincides with the the composition of complete isometries in the last sequence. Hence, the canonical map $Y \otimes^{\sigma h}_{M} V \to F(V)$ is a w^*-continuous complete isometry. Since this map has w^*-dense range, by the Krein-Smulian theorem it is a complete isometric isomorphism. Thus $F(V) \cong Y \otimes^{\sigma h}_{M} V$, and similarly $G(U) \cong X \otimes^{\sigma h}_{N} U$. Finally, $M \cong GF(M) \cong X \otimes^{\sigma h}_{N} Y$, using Lemma 2.4.12 and similarly $N \cong Y \otimes_M^{\sigma h} X$ completely isometrically and w^*-homeomorphically. □

Corollary 5.5.4. *Dual operator equivalence functors are W^*-restrictable.*

Proof. Firstly, we will show that W is the maximal dilation of X, and Z is the maximal dilation of Y. In Theorem 5.4.12, we saw that the set $\mathcal{U}$ equals Z. This implies that Y generates Z as a left dual operator $\mathcal{D}$-module. Similarly, X generates W as a left dual operator $\mathcal{C}$-module.

By Lemma 5.4.3 and Lemma 5.4.4, we have the following sequence of maps

$$CB_M^\sigma(X, H) \cong CB_N^\sigma(N, K) \cong K \cong CB_{\mathcal{D}}^\sigma(\mathcal{D}, K) \to CB_M^\sigma(W, H).$$

One can check that $\eta \in K$ corresponds under the last two maps in the sequence to the map $w \mapsto \rho(w)(\eta)$, which lies in $CB_{\mathcal{C}}^\sigma(W, H)$, since ρ is a left $\mathcal{C}$-module map. Thus, the composition R of the maps in the above sequence has range contained in $CB_{\mathcal{C}}^\sigma(W, H)$. Also, R is an inverse to the restriction map $CB_{\mathcal{C}}^\sigma(W, H) \to CB_M^\sigma(X, H)$. Thus $CB_{\mathcal{C}}^\sigma(W, H) \cong CB_M^\sigma(X, H)$. Since H is a normal universal representation of $\mathcal{C}$ (see the paragraph below Lemma 5.4.8), it follows from Theorem 5.2.10, that W is the maximal dilation of X. Similarly Z is the maximal dilation of Y.

Let $V \in {}_{\mathcal{C}}\mathcal{R}$. By Lemma 5.4.3, Lemma 5.4.4, Definition 5.2.5, Theorem 4.2.8, and Theorem 5.4.12, we have the following sequence of isomorphisms

$$F(V) \cong CB_N^\sigma(N, F(V)) \cong CB_M^\sigma(X, V) \cong CB_{\mathcal{C}}^\sigma(W, V) \cong Z \otimes_{\mathcal{C}}^{\sigma h} V,$$

as left dual operator N-modules. Since $Z \otimes_{\mathcal{C}}^{\sigma h} V$ is a left dual operator $\mathcal{D}$-module, we see that $F(V)$ is a left dual operator $\mathcal{D}$-module and by Theorem 5.2.2, this $\mathcal{D}$-module action is unique. Also by Corollary 5.2.3 the map $Z \otimes_{\mathcal{C}}^{\sigma h} V \to F(V)$ coming from the composition of the above isomorphisms is a $\mathcal{D}$-module map. This map $Z \otimes_{\mathcal{C}}^{\sigma h} V \to$

$F(V)$ is defined analogously to the map τ_V defined in Lemma 5.4.8. One can check that if $T : V_1 \to V_2$ is a morphism in ${}_{\mathcal{C}}\mathcal{R}$, then the following diagram commutes:

$$\begin{array}{ccc} Z \otimes^{\sigma h}_{\mathcal{C}} V_1 & \longrightarrow & F(V_1) \\ \big\downarrow{\scriptstyle I_Z \otimes T} & & \big\downarrow{\scriptstyle F(T)} \\ Z \otimes^{\sigma h}_{\mathcal{C}} V_2 & \longrightarrow & F(V_2) \end{array}$$

By Corollary 2.4.6, $I_Z \otimes T$ is a w^*-continuous $\mathcal{D}$-module map and both the horizontal arrows above are w^*-continuous $\mathcal{D}$-module maps. Hence $F(T)$ is a w^*-continuous $\mathcal{D}$-module map; that is, $F(T)$ is a morphism in ${}_{\mathcal{D}}\mathcal{R}$. Thus F is W^*-restrictable. By Theorem 5.5.3, our main theorem is proved. $\square$

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