ONE-SIDED M-STRUCTURE OF OPERATOR SPACES AND OPERATOR ALGEBRAS

A Dissertation
Presented to
the Faculty of the Department of Mathematics
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In Partial Fulfillment
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Doctor of Philosophy

By
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Abstract

In this dissertation, we study the structure and properties of operator spaces and operator algebras which have a one-sided M-structure. We develop a non-commutative theory of operator spaces which are one-sided M-ideals in their bidual. We also investigate the M-ideal structure of the Haagerup tensor product of operator algebras. Further, we consider operator algebras which are, in some sense, a generalization of the algebra of the compact operators. These are called the '1-matricial algebras'. Using the Haagerup tensor product and the 1-matricial algebra, we construct a variety of examples of operator spaces and operator algebras which are one-sided M-ideals in their bidual. In the last part of the thesis, we look at operator spaces over the field of real numbers and generalize a small portion of the existing theory of complex operator spaces. In particular, we show that the injective envelope and C^*-envelope for real operator spaces exist. We also briefly consider real operator algebras and their complexification.

Contents

Chapter 1

Introduction

1.1 Ideal Structure of Operator Spaces

Ideals play an important role in the structure theory of rings and algebras. For instance, as an implication of the celebrated Wedderburn-Artin theorem, which is originally due to Cartan, a finite dimensional unital algebra over $\mathbb{C}$, is semi-simple if and only if it is a matrix algebra, $\oplus_{i=1}^{m} M_{n_i}$. Ideals occur naturally in algebras, for example, the kernel of a homomorphism is a two-sided ideal. In functional analysis, closed ideals are an important tool for the study of C^*-algebras. In 1972, Alfsen and Effros [1] generalized the notion of two-sided ideals to Banach spaces, where they introduced M-ideals. The main idea was to generalize the two-sided ideals in a C^*-algebra and obtain a variant which would serve as a tool for the study of Banach spaces. The notion of M-ideals is an appropriate generalization, since in a C^*-algebra, M-ideals coincide with the two-sided closed ideals [58]. Moreover, the definition of M-ideals is solely in terms of the norm and the linear structure of Banach spaces, and yet they encode important algebraic information. Over the years, M-ideals have been extensively studied, resulting in a vast theory. They are an important tool in functional analysis. For a comprehensive treatment and for references to the extensive literature on the subject, one may refer to the book by P. Harmand, D. Werner and W. Werner [36]. Recently, the classical theory of M-ideals has been generalized to the setting of operator spaces. In 1994, Effros and Ruan studied the "complete" M-ideals of operator spaces in [26]. The complete M-ideal theory, however, was intrinsically "two-sided". Blecher, Effros, and Zarikian developed a one-sided M-ideal theory in a series of papers (see e.g. [11, 17, 18], [15] with Smith, and also [62]). They defined two varieties of M-ideals for the non-commutative setting, the "left M-ideals" and the "right M-ideals". The intention was to create a tool for the "non-commutative functional analysis". For example, one-sided M-ideal theory has yielded several deep, general results in the theory of operator bimodules (see e.g. [11, 9]). The one-sided M-ideals also generalize some important algebraic structures in various settings. For example, the one-sided closed ideals in a C^*-algebra, one-sided submodules in a Hilbert C^*-module, and one-sided closed ideals which have a one-sided approximate identity in an approximately unital operator algebra, are one-sided M-ideals in their respective spaces.

We generalize to the non-commutative setting, the classical theory of an important and special class of M-ideals, called M-embedded spaces. The classical M-embedded spaces are Banach spaces which are M-ideals in their second dual. The study of M-embedded spaces marked a significant point in the development of M-ideal theory of Banach spaces. These spaces have a rich theory because of their stability behavior and a natural L-decomposition of their third dual. These spaces have several other nice properties such as the unique extension property, Radon Nikodým property of the dual, and many more. We study the one-sided variant of the classical theory in Chapter 3, namely the one-sided M-embedded spaces. Our main aim is to begin to import some of the rich theory of these spaces from the classical setting to the non-commutative setting. The classical theory which we generalize, consists mostly of Chapters 3 and 4 from [36].

We show that many of the interesting properties from the classical settings are retained in the non-commutative setting. For instance subspaces and quotient spaces of a right M-embedded operator space are also right M-embedded. As in the classical setting, the dual of a right M-embedded space has the unique extension property and the Radon Nikodým property. Further, if X is a right M-embedded operator space which has the completely bounded approximation property, then so does X^*. We completely characterize the C^*-algebras and TROs which are one-sided M-ideal in their bidual. The one-sided M-embedded C^*-algebras are a very nice and simple class of C^*-algebras, namely, the Kaplansky's "dual C^*-algebras". These are just the C^*-algebras of the form $\oplus_i^0 \mathbb{K}(H_i)$, where $\mathbb{K}(H_i)$ denote the space of compact operators on the Hilbert space H_i. This class has many strong properties and many interesting characterizations, which may be found in Kaplansky's works or [23, Exercise 4.7.20]. We show that the one-sided M-embedded TROs are of the form $\oplus_i^0 \mathbb{K}(H_i, H_j)$.

We end Chapter 3 with a discussion of one-sided L-embedded spaces. The dual of a right M-embedded operator space is left L-embedded. However, we show that, not all one-sided L-embedded spaces arise in this manner (Proposition 3.3.8).

We thank our Ph.D. adviser, Dr. David Blecher, for proposing this project in Chapter 3 and continually supporting the work. We are grateful for his insightful comments and very many suggestions and corrections.

In Chapter 4, we extend several known results about the Haagerup tensor products of C^*-algebras (mainly from [7, 18]) to operator algebras. We also investigate the one-sided M-ideal structure of the Haagerup tensor product of non-selfadjoint operator algebras.

Among other things, we show that if A and B are approximately unital operator algebras, and both have dimensions greater than 1, then $A \otimes_h B$ has no non-trivial complete M-ideals. Further, under some suitable hypothesis, we show that the right M-ideals in $A \otimes_{\mathrm{h}} B$ are precisely of the form $J \otimes_{\mathrm{h}} B$, for some closed right ideal J in A such that J has a left contractive approximate identity. Thus we completely characterize the one-sided M-ideals of $A \otimes_{\mathrm{h}} B$. Our results provide examples of operator spaces which are right but not left ideals (or M-ideals) in their second dual. We also generate examples of algebras which are ideals in their bidual, using an interesting class of operator algebras called 1-matricial algebras, defined in [3, Section 4]. The 1-matricial algebras are, in some sense, a generalization of the compact operators. This class can be constructed in a very simple manner using invertible operators on a Hilbert space. We see that 1-matricial algebras contain many algebras which are one-sided M-ideals in their bidual. Thus using various properties and results about 1-matricial algebras, we explicitly construct M-embedded operator algebras which are different from Kaplansky's "dual C^*-algebras", and have interesting features. We end the chapter with a discussion of some results related to the "Wedderburn-Artin type" structure theorems for operator algebras. These types of theorems have been studied in [3, 37, 41]. Most of Chapter 4 is joint work with M. Almus and D. P. Blecher and appears in [3]. The parts which do not appear in [3], are joint work with my advisor, D. P. Blecher.

1.2 Real Operator Spaces

In functional analysis, the underlying objects of study are vector spaces over a field, where the field is usually either the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$. The field of complex numbers has been preferred more by mathematicians, since the field of reals is a little more restrictive. For instance, every polynomial over the field of reals has a roots in $\mathbb{C}$, but need not have any root in $\mathbb{R}$, or a $n \times n$ matrix need not have real eigenvalues. Thus, usually most of the theory is developed with the assumption that the underlying field is $\mathbb{C}$. The theory of real spaces, however, occurs naturally in all areas of mathematics and physics. They come up naturally in the theory of C^*-algebras, for instance, the self-adjoint part of every C^*-algebra is a real space, also in graded C^*-algebras and in the theory of real TROs in graded C^*-algebras. See [33, 43, 55, 60] for the theory of real C^*-algebras and real W^*-algebras. They also come up in JB^*-triples [38] and KK theory [6, 20]. Thus it becomes important to study the analogues theory for the case when the field is the real scalars and know which results hold true and which result fail.

The theory of real operator spaces is the study of subspaces of bounded operators on real Hilbert spaces. In the general theory of (complex) operator spaces, the underlying Hilbert space is assumed to be a complex Hilbert space.

In two recent papers [53, 54], Ruan studies the basic theory of real operator spaces. He shows that with appropriate modifications, many complex results hold for real operator spaces. It is shown among other things, that Ruan's characterization, Stinespring's theorem, Arveson's extension theorem, injectivity of $B(H)$ for real Hilbert space H, hold true for real operator spaces. In [54], Ruan defines the notion of complexification of a real operator space and studies the relationship between the properties of real operator spaces and the properties of their complexification. We want to continue this program, and develop more theory of real operator spaces and real operator algebras. This is a work in progress, and hopefully will include a satisfactory theory of real M-ideals and real M-embedded spaces.

We show here among other things that the real injective envelope of a real operator space exists. We also study the relation between the real injective envelope and the injective envelope of its complexification. We begin to develop the one-sided real M-ideal theory. We briefly consider real operator algebras and their complexification. We show that the BRS characterization theorem of operator algebras holds for real operator algebras.

Chapter 2

Preliminaries

2.1 Operator Spaces and Operator Algebras

A (concrete) *operator space* X is a norm closed subspace of $B(H)$, for some Hilbert space H. If X is an operator space then each $M_n(X)$ has a canonical norm via the identification $M_n(X) \subset M_n(B(H)) \cong B(H^n)$, isometrically, where $H^n = H \oplus H \oplus \ldots \oplus H$. The collection of these norms $\{\|.\|_n\}$ is called the *matrix norm structure* of X. An operator space is characterized by its matrix norm structure. A Banach space X with matrix norms $\{\|.\|_n\}$ is an operator space if and only if it satisfies the following two axioms (called *Ruan's axioms*):

(i) $\|\alpha x \beta\|_n \leq \|\alpha\| \, \|x\|_n \, \|\beta\|$, for all $n \in \mathbb{N}$ and all α, $\beta \in M_n$, and $x \in M_n(X)$.

(ii) $\|x \oplus y\|_{m+n} = \max\{\|x\|_n \,, \|y\|_m\}$ for all $x \in M_n(X)$ and $y \in M_m(X)$.

Here $\oplus$ denotes the diagonal direct sum of matrices.

The norms on the square matrices determine the norms on the rectangular matrix spaces $M_{m,n}(X)$, and with the norms induced by the canonical algebra isomorphisms $M_p(M_{m,n}(X)) \cong M_{pm,pn}(X)$, $M_{m,n}(X)$ becomes an operator space. In particular, $C_n(X) = M_{n,1}(X)$ and $R_n(X) = M_{1,n}(X)$ are operator spaces. We write $\mathbb{M}_{I,J}$ for the set of $I \times J$ matrices whose finite submatrices have uniformly bounded norms, where I, J are cardinals. Such a matrix is normed by the supremum of the norms of its finite submatrices, and it is an operator space with the canonical matrix norm given by the identification $M_n(\mathbb{M}_{I,J}(X)) \cong \mathbb{M}_{I,J}(M_n(X))$. If $I = \aleph_0$, then we write $\mathbb{M}_{I,I}(X) = M_\infty(X)$, $\mathbb{M}_{I,1}(X) = C_\infty^w(X)$ and $\mathbb{M}_{1,I}(X) = R_\infty^w(X)$. The closure of the span of the finitely supported matrices in $\mathbb{M}_{I,J}(X)$ is denoted by $\mathbb{K}_{I,J}(X)$. If $I = \aleph_0$, then we write $\mathbb{K}_{I,I}(X) = K_\infty(X)$, $\mathbb{K}_{I,1}(X) = C_\infty(X)$ and $\mathbb{K}_{1,I}(X) = R_\infty(X)$.

Let X and Y be operator spaces, and $u : X \longrightarrow Y$ be a linear map. For each n, we write $u_n : M_n(X) \longrightarrow M_n(Y)$ for the associated map $[x_{ij}] \mapsto [u(x_{ij})]$, also called the *nth amplification* of u. Define $\|u\|_{cb} = \sup_n\{\|u_n\|\}$. Then u is *completely bounded* (resp. completely contractive) if $\|u\|_{cb} < \infty$ (resp. $\|u\|_{cb} \leq 1$). The map u is a *complete isometry* (resp. *complete quotient*) if each u_n is an isometry (resp. quotient).

Every Banach space E may be given a canonical operator space structure via the identification $E \hookrightarrow C(\Omega)$, where $\Omega = \mathrm{Ball}(E^*)$. Thus we can define a matrix norm structure on E via the inclusion $M_n(E) \subset M_n(C(\Omega))$. This operator space structure is called the *minimal operator space structure* and we write the operator space as $\mathrm{Min}(E)$. This is the smallest operator space structure on E. For any bounded linear u from an operator space Y into E, we have

$$\|u : Y \longrightarrow \mathrm{Min}(E)\|_{cb} = \|u : Y \longrightarrow E\| .$$

There also exists a largest operator space structure on E, denoted Max(E). The matrix norms on Max(E) are defined as

$$\|[x_{ij}]\|_n = \sup\{\|[u(x_{ij})]\| : \ u \in \mathrm{Ball}(B(E,Y)), \text{ all operator spaces Y}\}.$$

If X and Y are operator spaces, then $CB(X,Y)$ denotes the space of completely bounded linear maps from X to Y. With the matrix norms determined via the canonical isomorphism between $M_n(CB(X,Y))$ and $CB(X, M_n(Y))$, $CB(X,Y)$ is an operator space. The dual of the operator space X is defined to be $CB(X,\mathbb{C})$. The latter is the same as $B(X,\mathbb{C}) = X^*$ isometrically. Thus the dual of X, X^*, is an operator space. The *adjoint or dual* u^* of a completely bounded map $u : X \longrightarrow Y$, is completely bounded from Y^* to X^* with $\|u\|_{cb} = \|u^*\|_{cb}$. Furthermore, u is a complete quotient if and only if u^* is a complete isometry. Thus u is a complete isometry if and only if u^{**} is a complete isometry. Every operator space is completely isometrically embedded in its second dual X^{**}, via the canonical map $i_X : X \hookrightarrow X^{**}$ (see e.g. [14, Proposition 1.4.1]).

A (concrete) *operator algebra* A is a norm closed subalgebra of $B(H)$. A (concrete) *dual operator algebra* is a w^*-closed subalgebra of $B(H)$. We say that A is unital if A contains the unit I_H of $B(H)$. An *approximately unital operator algebra* A is an operator algebra which contains a contractive approximate identity (cai). A contractive approximate identity is a net $\{e_t\}$ such that $\|e_t\| \leq 1$ and $e_t a \longrightarrow a$ and $a e_t \longrightarrow a$ for all $a \in A$. Let $X \subset B(H)$ be an operator space. For each $x \in X$ we denote the adjoint of x in X by $x^\star$, and the *adjoint* of X is $X^\star = \{x^\star \ | \ x \in X\}$. If X is an operator algebra A then define the *diagonal* of A to be

$$\Delta(A) = A \cap A^\star = \{a \in A : \ a^* \in A\}.$$

If A is an operator algebra then ΔA is a C^*-algebra. Furthermore, if A is a dual operator algebra then ΔA is a von Neumann algebra (see e.g. [14, 2.1.2]).

A TRO is a closed subspace X of a C^*-algebra such that $XX^\star X \subset X$. A WTRO is a w^*-closed subspace of a von Neumann algebra with $XX^\star X \subset X$. A TRO is essentially the same as a Hilbert C^*-module (see e.g. [14, 8.1.19]). If X is a TRO, then X is a Hilbert C^*-bimodule over $XX^\star$-$X^\star X$ (see e.g. [14, 8.1.2]).

2.2 Tensor Products of Operator Spaces

We denote the operator space injective, projective, and Haagerup tensor products of operator spaces X and Y by $X \overset{\smile}{\otimes} Y$, $X \widehat{\otimes} Y$, and $X \otimes_{\mathrm{h}} Y$, respectively. We begin by stating (without proof) some identifications which will be used in Chapter 3 and Chapter 4. For the definitions and basic properties of these tensor products, and the proof of the following identifications, we refer the reader to [14, 27].

We have the completely isometric identification

$$M_{m,n} \overset{\smile}{\otimes} X \cong M_{m,n}(X) \cong C_m \otimes_{\mathrm{h}} X \otimes_{\mathrm{h}} R_n,$$

for all $m, n \in \mathbb{N}$. In particular,

$$C_n \overset{\smile}{\otimes} X \cong C_n(X) \cong C_n \otimes_{\mathrm{h}} X$$

and

$$R_n \overset{\smile}{\otimes} X \cong R_n(X) \cong X \otimes_{\mathrm{h}} R_n.$$

We write

$$C_n[X] \cong C_n \widehat{\otimes} X \text{ and } R_n[X] \cong R_n \widehat{\otimes} X,$$

for all $n \in \mathbb{N}$. We have the completely isometric identifications

$$C_n(X)^* \cong R_n[X^*],\ R_n(X)^* \cong C_n[X^*]$$

and

$$C_n[X]^* \cong R_n(X^*),\ R_n[X]^* \cong C_n(X^*),$$

where in each case the following duality pairings are used,

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} f_1 & f_2 & \dots & f_n \end{bmatrix} \right\rangle = \sum_{i=1}^{n} f_i(x_i) = \left\langle \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \right\rangle .$$

If X_i and Y_i are operator spaces for $i = 1, 2$ and if $u_i : X_i \longrightarrow Y_i$ are completely bounded, then the map $x \otimes y \mapsto u_1(x) \otimes u_2(y)$ has a unique continuous extension to a map $u_1 \otimes u_2$ from $X_1 \otimes_\beta X_2$ to $Y_1 \otimes_\beta Y_2$, where $\otimes_\beta$ is an operator space tensor product. We say that $\otimes_\beta$ is *functorial* if $\|u_1 \otimes u_2\|_{cb} \leq \|u_1\|_{cb} \|u_2\|_{cb}$. If u_i complete isometry implies that $u_1 \otimes u_2$ is a complete isometry, then we say that $\otimes_\beta$ is *injective.* If u_i complete quotient implies that $u_1 \otimes u_2$ is a complete quotient, then we say that $\otimes_\beta$ is *projective.* The injective tensor product, $\check{\otimes}$, is injective in the above sense, and the projective tensor product, $\widehat{\otimes}$, is projective. The Haagerup tensor product is both injective and projective. For operator spaces X and Y, there is an ordering on the various tensor norms on $X \otimes Y$, namely $\|.\|_{\smile} \leq \|.\|_{\mathrm{h}} \leq \|.\|_{\frown}$. Indeed the 'identity' map $X \widehat{\otimes} Y \longrightarrow X \otimes_{\mathrm{h}} Y \longrightarrow X \check{\otimes} Y$ is a complete contraction [14, Proposition 1.5.13].

2.3 One-Sided Multipliers

For the proofs and more details in this section, see e.g. [9, 10, 11, 14]. Let X be an operator space. We say a map $T : X \longrightarrow X$ is a *left multiplier* of X if there exists a linear complete isometry $\sigma : X \longrightarrow B(H)$ and an operator $S \in B(H)$ such that

$$\sigma(Tx) = S\sigma(x)$$

for all $x \in X$. We denote the set of all left multipliers of X by $\mathcal{M}_\ell(X)$. Then $\mathcal{M}_\ell(X)$ is a unital operator algebra such that $\mathcal{M}_\ell(X) \subset CB(X)$ as sets, and $\|T\|_{cb} \leq \|T\|_{\mathcal{M}_\ell(X)}$ for all $T \in \mathcal{M}_\ell(X)$. We define a *left adjointable map* of X to be a linear map $T : X \longrightarrow X$ such that there exists a linear complete isometry $\sigma : X \longrightarrow B(H)$ and an operator $A \in B(H)$ such that

$$\sigma(Tx) = A\sigma(x) \text{ for all } x \in X, \text{ and } A^*\sigma(X) \subset \sigma(X).$$

The collection of all left adjointable maps of X is denoted by $\mathcal{A}_\ell(X)$. Every left adjointable map of X is a left multiplier of X, that is, $\mathcal{A}_\ell(X) \subset \mathcal{M}_\ell(X)$. For $T \in \mathcal{A}_\ell(X)$

$$\|T\|_{\mathcal{M}_\ell(X)} = \|T\|_{cb} = \|T\|.$$

Also, $\mathcal{A}_\ell(X)$ is a C^*-algebra, in fact $\mathcal{A}_\ell(X) = \mathcal{M}_\ell(X) \cap \mathcal{M}_\ell(X)^* = \Delta(\mathcal{M}_\ell(X))$. If X is a dual operator space, then $\mathcal{M}_\ell(X)$ is a dual operator algebra and $\mathcal{A}_\ell(X)$ is a von Neumann algebra. Furthermore, every element in $\mathcal{M}_\ell(X)$ is weak*-continuous. Similar definitions and results hold for the right multiplier algebra, $\mathcal{M}_r(X)$, and the right adjointable multiplier algebra, $\mathcal{A}_r(X)$.

Remark. It is not enough to take any one embedding in the definition of $\mathcal{M}_\ell(X)$. For instance, let $X = l_2^\infty$, and ϵ_k be any decreasing sequence of real numbers such that $\epsilon_k \searrow 0$. Define $\phi : \ell_2^\infty \longrightarrow \ell^\infty(\ell_2^\infty)$ as $\phi(a_1,\ a_2) = (\phi_k(a_1,\ a_2))$ where $\phi_k : \ell_2^\infty \longrightarrow \ell_2^\infty$ such that

$$\phi_k(a_1, a_2) = (a_1(1 - 1/k) + 1/ka_2,\ a_2(1 - 1/k) + 1/ka_1)).$$

Then it is easy to check that ϕ is an isometry. Let $\vec{c} = ((c_k,\ d_k)) \in \ell^\infty(\ell_2^\infty)$ such that $\vec{c}\phi((a_1,\ a_2)) \in \phi(\ell_2^\infty)$, then $\vec{c}\phi((a_1,\ a_2)) = \phi(b_1,\ b_2)$ for some b_1, b_2. So,

$$c_ka_1(1 - 1/k) + 1/kc_ka_2 = b_1(1 - 1/k) + 1/kb_2,$$

$$d_k a_2(1-1/k)+1/kd_k a_1) = b_2(1-1/k)+1/kb_1.$$

In particular, let $a_1 = 0, a_2 = 1$, then

$$c_k = (k-1)b_1 + b_2, \text{ and } d_k = b_1/(k-1) + b_2.$$

If $((k-1)b_1+b_2,\ b_1/(k-1)+b_2) \in \ell^\infty(\ell_2^\infty)$ then $b_1 = 0$. Thus $\mathcal{M}_\ell{}^\phi(X) = \{((a,a)) : a \in \mathbb{C}\}$ which is not isomorphic to $\ell^\infty(\ell_2^\infty)$.

2.4 One-Sided M-Ideals and L-Ideals

The theory of one-sided M-ideals and one-sided L-ideals can be found in [11, 17, 18, 15, 62].

We begin with the definition of a one-sided complete M-projection. A *complete left M-projection* on X is an orthogonal projection in the C^*-algebra $\mathcal{A}_\ell(X)$. There are several equivalent characterizations of a complete left M-projection which are often easier to verify in practice:

Theorem 2.4.1. *Let X be an operator space and $P : X \longrightarrow X$ be a linear idempotent map. Then the following are equivalent:*

(i) *P is a complete left M-projection.*

(ii) *The map $\tau_P^c : C_2(X) \longrightarrow C_2(X) : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} P(x) \\ y \end{bmatrix}$ is a complete contraction.*

(iii) *The map $\nu_P^c : X \to C_2(X) : x \mapsto \begin{bmatrix} P(x) \\ x - P(x) \end{bmatrix}$ is a complete isometry.*

(iv) *The maps ν_P^c and $\mu_P^c : C_2(X) \longrightarrow X : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto P(x) + (\mathrm{Id} - P)(y)$ are completely contractive.*

(v) *P is an selfadjoint element in $\mathcal{A}_\ell(X)$, i.e., $P \in \mathcal{A}_\ell(X)_{sa}$.*

(vi) *P is an element in the unit ball of $\mathcal{M}_\ell(X)$.*

A linear subspace J of X is a *right M-summand* of X if it is the range of a complete left M-projection P on X. The kernel of P, which equals the range of $I - P$, is also a right M-summand. Since complete left M-projections are (completely) contractive, right M-summands are automatically closed. Furthermore, since complete left M-projections on a dual operator space are weak*-continuous, right M-summands on such spaces are automatically weak*-closed.

A closed linear subspace J of an operator space X is a right M-ideal of X if its second annihilator $J^{\perp\perp}$, is a right M-summand of X^{**}. Every right M-summand is a right M-ideal of X, but the converse is false.

Dual to the one-sided M-structure of an operator space X is its one-sided L-structure. A linear idempotent map $P : X \longrightarrow X$ is a complete right L-projection on X if $P^* : X^* \longrightarrow X^*$ is a complete left M-projection on X^*. There are several alternative characterizations of complete right L-projections as well. For example:

Proposition 2.4.2. *Let X be an operator space and $P : X \longrightarrow X$ be a linear idempotent map. Then the followings are equivalent:*

(i) *P is a complete right L-projection.*

(ii) *The map $\nu_P^r : X \to R_2[X] : x \mapsto \begin{bmatrix} P(x) \\ x - P(x) \end{bmatrix}$ is a complete isometry.*

(iii) *The maps ν_P^r and $\mu_P^r : R_2[X] \longrightarrow X : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto P(x) + (\mathrm{Id} - P)(y)$ are completely contractive.*

A linear subspace J of X is a *left L-summand* of X if it is the range of a complete right L-projection. There is no need to define the concept of a *left L-ideal*, since a closed linear subspace J of X is a left L-summand if and only if $J^{\perp\perp}$ is a left L-summand of X^{**} [11, Proposition 3.9].

Now we state some more facts which will be used frequently, and often without explicitly mentioning them. For the proofs see [11].

(i) Let $P : X \longrightarrow X$ be a bounded linear idempotent map. Then P is a complete left M-projection if and only if P^* is a complete right L-projection on X^*.

(ii) A closed linear subspace J of X is a right M-ideal of X if and only if $J^{\perp}$ is a left L-summand of X^*. A closed linear subspace J of X is a left L-summand if and only if $J^{\perp}$ is a right M-summand of X^*.

(iii) Every right M-summand (resp. left L-summand) is the range of a unique complete left M-projection (resp. complete right L-projection).

(iv) If a right M-ideal J is the range of a contractive projection P, then it is in fact a right M-summand and P is the unique complete left M-projection onto J.

2.5 u-Ideals and h-Ideals

We now give some definitions and terminology from the u-ideal theory of Godefroy, Kalton, and Saphar [31], which will be used in Chapter 3. Let X be a Banach space, then $J \subset X$ is called a *u-summand* if there is a contractive projection P on X, mapping onto J, such that $\|I - 2P\| = 1$. This norm condition is equivalent to the condition

$$\|(I-P)(x) + P(x)\| = \|(I-P)(x) - P(x)\| \text{ for all x} \in \text{X}.$$

We call such a projection, a *u-projection.* A subspace J of X is an *h-summand* if there is a contractive projection P from X onto J, such that $\|(I-P)-\lambda P\|=1$ for all scalars λ with $|\lambda|=1$. This norm condition is equivalent to the condition

$$\|(I-P)(x)-\lambda P(x)\|=\|(I-P)(x)+P(x)\| \text{ for all x} \in \text{X}.$$

Such a projection is called an *h-projection.* Clearly every h-projection is a u-projection and hence every h-summand is a u-summand.

The norm condition for an h-summand is equivalent to saying that P is hermitian in $B(X)$, that is, $\left\|e^{itP}\right\|=1$ for all $t \in \mathbb{R}$. We say that J is a *u-ideal* in X if $J^{\perp}$ is a u-summand in X^*, and J is an *h-ideal* if $J^{\perp}$ is an h-summand in X^*. So clearly every h-summand (resp. u-summand) is an h-ideal (resp. u-ideal). We refer the reader to [30] for further details on the above topics. We now show that one-sided M-summands (M-ideals) and one-sided L-summands are h-summands (h-ideals). Alternatively, the lemma below also follows from [15, Lemma 4.4].

Lemma 2.5.1. *One-sided M-summands (or M-ideals) and one-sided L-summands are h-summands and hence u-summands (*u ideals*).*

Proof. First let J be a right M-summand in X, and $\|\lambda\| = 1$, then

$$\begin{aligned}
\left\| \begin{bmatrix} x - P(x) \\ \lambda P(x) \end{bmatrix} \right\| &= \left\| \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x - P(x) \\ P(x) \end{bmatrix} \right\| \\
&\leq \left\| \begin{bmatrix} x - P(x) \\ P(x) \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x - P(x) \\ P(x) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/\lambda \end{bmatrix} \right\| \\
&\leq |1/\lambda| \left\| \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x - P(x) \\ P(x) \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} x - P(x) \\ \lambda P(x) \end{bmatrix} \right\|.
\end{aligned}$$

So, $\|x - P(x) - \lambda P(x)\| = \left\| \begin{bmatrix} x - P(x) \\ \lambda P(x) \end{bmatrix} \right\| = \left\| \begin{bmatrix} x - P(x) \\ P(x) \end{bmatrix} \right\| = \|x - P(x) - \lambda P(x)\|$.

Now if J is a right L-summand, then since the identity map $Id : C_2[X] \longrightarrow C_2(X)$ is a complete contraction, by the properties of the tensor product, we have

$$\begin{aligned}
\|x - P(x) - \lambda P(x)\| = \left\| \begin{bmatrix} x - P(x) \\ \lambda P(x) \end{bmatrix} \right\|_{C_2(X)} &= \left\| \begin{bmatrix} x - P(x) \\ P(x) \end{bmatrix} \right\|_{C_2(X)} \\
&\leq \left\| \begin{bmatrix} x - P(x) \\ P(x) \end{bmatrix} \right\|_{C_2[X]} \\
&= \|x - P(x) + \lambda P(x)\|.
\end{aligned}$$

By symmetry, we get the other inequality. Hence J is a h-summand in the underlying Banach space X. From this it is easy to see that one-sided M-ideals are h-ideals, and hence also u-ideals. Indeed, if J is a right M-ideal in X, then $J^{\perp}$ is a left L-summand in

X^*. So $J^\perp$ is a h-summand in X^*, and thus an h-ideal. By a similar argument we get the other assertion. $\square$

2.6 r-Ideals and l-Ideals

Let A be an approximately unital operator algebra. A *left (right) contractive approximate identity* is a net $\{e_t\}$ such that $\|e_t\| \leq 1$ and $e_t a \longrightarrow a$ ($ae_t \longrightarrow a$) for all $a \in A$. A subspace J of A is an *r-ideal* if J is a closed right ideal with a left contractive approximate identity. Similarly J is an *ℓ-ideal* in A if J is a closed left ideal with a right contractive approximate identity. In an approximately unital operator algebra, the right (left) M-ideals are precisely the r-ideals (ℓ-ideals) (see e.g. [11]).

Chapter 3

Operator Spaces and One-Sided M-Ideal Structure

Every Banach space X can be realized as a subspace of the second dual X^{**}, via the canonical embedding $i : X \longrightarrow X^{**}$ given by $i(x)(f) = f(x)$. A Banach space X is called M-embedded if it is an M-ideal in X^{**}. In the non-commutative setting, if X is an operator space, then the embedding i, defined above, is a complete isometry (see e.g. [14, Proposition 1.4.1]). Thus we can generalize the notion of M-embedded spaces to operator spaces, as the operator spaces which are one-sided M-ideals in their bidual. In this chapter, we present the non-commutative theory of these one-sided M-embedded operator spaces. Most of the work in this chapter is published in [56].

For the one-sided M-ideal theory of operator spaces, the reader can refer to [11, 18, 62], and for notation see Chapter 2.

3.1 One-Sided M-Embedded Spaces

Definition 3.1.1. An operator space X is called a *right M-embedded* operator space if X is a right M-ideal in X^{**}. We say X is *left L-embedded* if X is a left L-summand in X^{**}. Similarly we can define *right L-embedded* and *left M-embedded* spaces. If X is both right and left M-embedded, then X is called *completely M-embedded*. An operator space X is *completely L-embedded* if X is both a right and a left L-embedded operator space.

Remark. 1) X is completely M-embedded if and only if X is a complete M-ideal in its bidual (see e.g. [11, Lemma 3.1] and [18, Chapter 7]).

2) Reflexive spaces are automatically completely M-embedded. Let X be a right M-summand in X^{**}. Since X^{**} is a dual operator space, X is w^*-closed, by the discussion following [11, Theorem 2.3.1]. So $X = X^{**}$. Hence a non-reflexive operator space cannot be a non-trivial one-sided M-summand in its second dual. So non-reflexive right M-embedded spaces are proper right M-ideals. Henceforth we will assume all our operator spaces to be non-reflexive.

We state an observation of David Blecher which provides an alternative definition of completely L-embedded operator spaces. To explain the notation here, $M_n(X^*)_*$ is the 'obvious' predual of $M_n(X^*)$, namely the operator space projective tensor product of the predual of M_n and X.

Lemma 3.1.2. *Let X be an operator space. Then there exists a complete L-projection from X^{**} onto X if and only if for each n, there exists a L-projection from $M_n(X^*)^*$ onto $M_n(X^*)_*$.*

Proof. We are going to use the well known principle that if J is a subspace of X, then

J is an L-summand (resp. left L-summand, complete L-summand) of X iff $J^\perp$ is an M-summand (resp. right M-summand, complete M-summand) of X^*. See for example the proof of [11, Proposition 3.9].

By the above principle, X is a complete L-summand of X^{**} iff $X^\perp$ is a complete M-summand of X^{***}. By [26, Proposition 4.4], this happens iff $M_n(X^\perp)$ is an M-summand of $M_n(X^{***})$ for each n. Now $M_n(X^{***})$ is the dual of the operator space projective tensor product of the predual of M_n and X^{**}. Moreover, $M_n(X^\perp)$ is easily seen to be the 'perp' of the operator space projective tensor product of the predual of M_n and $i_X(X)$. That is, $M_n(X^\perp) = (M_n(X^*)_*)^\perp$. (We are using facts from [26, Proposition 7.1.6]). By the above principle, we deduce that X is a complete L-summand of X^{**} iff $M_n(X^*)_*$ is a L-summand of $M_n(X^*)^*$ for each n. $\square$

Let $i_X : X \longrightarrow X^{**}$ be the canonical map given by $i_X(x)(x^*) = x^*(x)$ for all $x^* \in X^*$ and $x \in X$. Then i_X is a complete isometry (see e.g. [14, Proposition 1.4.1]). We will denote $\hat{x} = i_X(x)$.

Lemma 3.1.3. *The canonical map* $\pi_{X^*} = i_{X^*} \circ {i_X}^* : X^{***} \longrightarrow X^{***}$ *is a completely contractive projection onto* $i_{X^*}(X^*)$ *with kernel* $(i_X(X))^\perp$.

Proof. We first show that ${i_X}^* \circ i_{X^*} = Id_{X^*}$. Let $x^* \in X^*$, and $x \in X$, then $\widehat{x^*}(i_X(x)) = \widehat{x^*}(\hat{x}) = \hat{x}(x^*) = x^*(x)$, so $\widehat{x^*} \circ i_X = x^*$, and hence ${i_X}^* \circ i_{X^*}(x^*) = {i_X}^*(\widehat{x^*}) = \widehat{x^*} \circ i_X = x^*$. This shows that $(\pi_{X^*})^2 = (i_{X^*} \circ {i_X}^*) \circ (i_{X^*} \circ {i_X}^*) = i_{X^*} \circ ({i_X}^* \circ i_{X^*}) \circ {i_X}^* = i_{X^*} \circ {i_X}^* = \pi_{X^*}$. Clearly, being the composition of completely contractive maps, π_{X^*} is completely contractive and $\mathrm{Ran}(\pi_{X^*}) \subset \mathrm{Ran}(i_{X^*}) = i_{X^*}(X^*)$. For the other containment, let $x^* \in X^*$, then $\pi_{X^*}(i_{X^*}(x^*)) = i_{X^*}((i_X)^*(i_{X^*})(x^*)) = i_{X^*}(x^*)$. Finally, it is clear that $\mathrm{Ker}(\pi_{X^*}) \supset \mathrm{Ker}((i_X)^*) = \mathrm{Ran}((i_X)^\perp) = (i_X(X))^\perp$. Let $x^{***} \in X^{***}$ such that $\pi_{X^*}(x^{***}) = 0$. Since

i_{X^*} is one to one, $0 = (i_X)^*(x^{***})(\hat{x}) = x^{***}(i_X(x))$, which implies that $x^{***} \in (i_X(X))^\perp$, and hence, $\operatorname{Ker}(\pi_{X^*}) = i_X(X)^\perp$. □

Proposition 3.1.4. *Let X be an operator space, then the following are equivalent:*

(i) *X is a right M-ideal in X^{**}.*

(ii) *The natural projection π_{X^*} is a complete right L-projection.*

Proof. (i)⇒(ii) Let $X \cong i_X(X)$ be a right M-ideal in X^{**}, then $i_X(X)^\perp$ is a complete left L-ideal in X^{***}. Let P be a complete right L-projection onto $i_X(X)^\perp$, then $i_X(X)^\perp$ is the kernel of the complementary right L-projection, namely $I - P$. Now $\operatorname{Ker} \pi_{X^*} = (i_X(X))^\perp = \operatorname{Ker}(I - P)$. So by [11, Theorem 3.10(b)], $\pi_{X^*} = I - P$. Hence π_{X^*} is a complete right L-projection.

(ii)⇒(i) If π_{X^*} is a complete right L-projection, then so is $I - \pi_{X^*}$. Now

$$\operatorname{Ran}(I - \pi_{X^*}) = \operatorname{Ker}(\pi_{X^*}) = (i_X(X))^\perp.$$

So $(i_X(X))^\perp$ is a left L-summand in X^{***}, and hence $i_X(X)$ is a right M-ideal in X^{**}. □

Corollary 3.1.5. *If X is a right M-embedded operator space, then X^* is a left L-embedded operator space.*

Proof. Since $i_{X^*}(X^*)$ is the range of π_{X^*}, and by Proposition 3.1.4, π_{X^*} is a complete right L-projection on X^{***}, the result follows. □

Remark. It is not true that if X is a left L-summand in its bidual then X^* is a right M-summand in its bidual. For example, let $X = S^1(H)$, the trace class operators on H. Then since $\mathbb{K}(H)$ is a closed two-sided ideal in $B(H)$, it is a complete M-ideal in $B(H)$. So by Corollary 3.1.5, $S^1(H)$ is complete L-summand in $B(H)^*$. But by the previous Remark, $B(H)$ is not a right (or left) M-summand in $B(H)^{**}$ since $B(H)$ is non-reflexive.

Proposition 3.1.6. *If X is a M-embedded Banach space, then* $\mathrm{Min}(X)$ *is a completely M-embedded operator space. If X is L-embedded, then* $\mathrm{Max}(X)$ *is completely L-embedded.*

Proof. Let X be a M-ideal in X^{**}, then $\mathrm{Min}(X)$ is a two-sided M-ideal in $\mathrm{Min}(X^{**})$. Indeed if Z is a Banach space, then the right M-ideals, as well as the left M-ideals, of $\mathrm{Min}(Z)$, coincide with the M-ideals of Z (see e.g. [11]). But $\mathrm{Min}(X^{**}) = \mathrm{Min}(X)^{**}$ completely isometrically. So $\mathrm{Min}(X)$ is a right M-ideal in $\mathrm{Min}(X)^{**}$, and hence $\mathrm{Min}(X)$ is M-embedded. The second assertion follows similarly, using the fact that L-ideals of any Banach space Z coincide with the left, as well as the right, L-ideals of $\mathrm{Max}(Z)$. $\square$

Theorem 3.1.7. *Let X be a right M-embedded space and Y be a subspace of X, then both Y and X/Y are right M-embedded.*

Proof. We first show that Y is right M-embedded. By Proposition 3.1.4, we need to show that π_{Y^*} is a complete right L-projection. Let $i : Y \longrightarrow X$ be the inclusion map, then i^{***} is a complete quotient map. So for every $[v_{ij}] \in M_n(Y^{***})$ we can find $[w_{ij}] \in M_n(X^{***})$ such that, $i_n^{***}([w_{ij}]) = [v_{ij}]$ and $\|[w_{ij}]\| \leq \|[v_{ij}]\|$. Also note that $\pi_{Y^*} \circ i^{***} = i^{***} \circ \pi_{X^*}$. For $[v_{ij}]$ and $[w_{ij}]$ as above, we have

$$\begin{aligned}
& \left\|[\pi_{Y^*}(v_{ij}) \quad v_{ij} - \pi_{Y^*}(v_{ij})]\right\|_{M_n(R_2[Y^{***}])} \\
= \; & \left\|[\pi_{Y^*} i^{***}(w_{ij}) \quad i^{***}(w_{ij}) - \pi_{Y^*} i^{***}(w_{ij})]\right\|_{M_n(R_2[Y^{***}])} \\
= \; & \left\|[i^{***}\pi_{X^*}(w_{ij}) \quad i^{***}(w_{ij}) - i^{***}\pi_{X^*}(w_{ij})]\right\|_{M_n(R_2[Y^{***}])} \\
\leq \; & \|i^{***}\|_{cb} \left\|[\pi_{X^*}(w_{ij}) \quad w_{ij} - \pi_{X^*}(w_{ij})]\right\|_{M_n(R_2[X^{***}])} \\
= \; & \|[w_{ij}]\| \\
\leq \; & \|[v_{ij}]\|.
\end{aligned}$$

This shows that the map $\mu^r_{\pi_{Y^*}} : Y^{***} \longrightarrow R_2[Y^{***}]$ given by

$$\mu^r_{\pi_{Y^*}}(y) = [\pi_{Y^*}(y) \qquad y - \pi_{Y^*}(y)],$$

is a complete contraction. Now since $\widehat{\otimes}$ is projective, and i^{***} is a complete quotient map, then so is $i^{***} \otimes Id : R_2[X^{***}] \longrightarrow R_2[Y^{***}]$. For each $[\ y_{ij} \qquad \acute{y}_{ij}\ ] \in R_2[Y^{***}]$ we can find $[x_{ij} \qquad \acute{x}_{ij}] \in R_2[X^{***}]$, such that $(i^{***} \otimes Id)([\ x_{ij} \qquad \acute{x}_{ij}\ ]) = [y_{ij} \qquad \acute{y}_{ij}]$ and $\|[x_{ij} \quad \acute{x}_{ij}]\| \leq \|[\ y_{ij} \quad \acute{y}_{ij}\ ]\|$. Consider

$$\begin{aligned}
& \|[\pi_{Y^*}(y_{ij}) + \acute{y}_{ij} - \pi_{Y^*}(\acute{y}_{ij})]\|_{M_n([Y^{***}])} \\
= \ & \|[\pi_{Y^*}(i^{***}(x_{ij})) + i^{***}(\acute{x}_{ij}) - \pi_{Y^*}(i^{***}(\acute{x}_{ij}))]\|_{M_n([Y^{***}])} \\
= \ & \|[i^{***}\pi_{X^*}(x_{ij}) + i^{***}(\acute{x}_{ij}) - i^{***}\pi_{X^*}(\acute{x}_{ij})]\|_{M_n([Y^{***}])} \\
\leq \ & \|[\pi_{X^*}(x_{ij}) + (\acute{x}_{ij}) - \pi_{X^*}(\acute{x}_{ij})]\|_{M_n([X^{***}])} \\
\leq \ & \|[x_{ij} \qquad \acute{x}_{ij}]\|_{M_n(R_2[X^{***}])} \\
\leq \ & \|[y_{ij} \qquad \acute{y}_{ij}]\|_{M_n(R_2[Y^{***}])}\,.
\end{aligned}$$

This shows that the map $\nu^r_{\pi_{Y^*}} : R_2[Y^{***}] \longrightarrow Y^{***}$ given by

$$\nu^r_{\pi_{Y^*}}([\ y \quad \acute{y}\ ]) = \pi_{Y^*}(y) + \acute{y} - \pi_{Y^*}(\acute{y}),$$

is a complete contraction. Hence by [11, Proposition 3.4], π_{Y^*} is a complete L-projection.

Consider the canonical complete quotient map $q : X \longrightarrow X/Y$, then $q^{***} : (X/Y)^{***} \longrightarrow X^{***}$ is a complete isometry. We also have that $\pi_{(X/Y)^*} \circ q^{***} = q^{***} \circ \pi_{(X/Y)^*}$. Since $R_2[(X/Y)^{***}] = R_2 \otimes_{\mathrm{h}} (X/Y)^{***}$ and $R_2[X^{***}] = R_2 \otimes_{\mathrm{h}} X^{***}$, and $\otimes_{\mathrm{h}}$ is injective, the map $Id \otimes q^{***} : R_2[(X/Y)^{***}] \longrightarrow R_2[X^{***}]$ will be a complete isometry. We need to show that $\pi_{(X/Y)^*}$ is a complete right L-projection on $(X/Y)^{***}$. For the sake of convenience we will write π for $\pi_{(X/Y)^*}$. Let $[v_{ij}] \in M_n((X/Y)^{***})$, then by using the above facts we get

$$\begin{aligned}
& \|[\pi(v_{ij}) \quad v_{ij} - \pi(v_{ij})]\|_{M_n(R_2[(X/Y)^{***}])} \\
= \;& \|[(q^{***} \circ \pi)(v_{ij}) \quad q^{***}(v_{ij}) - (q^{***} \circ \pi)(v_{ij})]\|_{M_n(R_2[X^{***}])} \\
= \;& \|[(\pi \circ q^{***})(v_{ij}) \quad q^{***}(v_{ij}) - (\pi \circ q^{***})(v_{ij})]\|_{M_n(R_2[X^{***}])} \\
= \;& \|[q^{***}(v_{ij})]\|_{M_n(X^{***})} \\
= \;& \|[v_{ij}]\|_{M_n((X/Y)^{***})}\,.
\end{aligned}$$

This shows that $\pi_{(X/Y)^*}$ is a left L-projection. Since $\mathrm{Ran}(\pi_{(X/Y)^*}) = (X/Y)^*$, X/Y is right M-embedded. $\square$

Remark. The property of one-sided "M-embeddedness" of subspaces and quotients does not pass to extensions, i.e., if Y is a subspace of X such that Y and X/Y are right M-embedded spaces, then X need not be right M-embedded. Consider $X = c_0 \oplus_1 c_0$ and $Y = c_0 \times \{0\}$, both with minimal operator space structure. Since Y and X/Y are M-embedded, $\mathrm{Min}(Y)$ and $\mathrm{Min}(X/Y)$ are completely M-embedded. Let P be the contractive projection from X onto Y, then $I - P$ is completely contractive, and hence a complete quotient map, from $\mathrm{Min}(Y)$ onto $\mathrm{Min}(\mathrm{Ran}(I-P))$. Thus $\mathrm{Min}(Y)/\mathrm{Ker}(P) \cong \mathrm{Min}(\mathrm{Ran}(I-P))$, completely isometrically. But $\mathrm{Ran}(I-P) = Y/X$ isometrically, so $\mathrm{Min}(X)/\mathrm{Min}(Y) \cong \mathrm{Min}(X/Y)$, completely isometrically. Now if $\mathrm{Min}(X)^{**}$ has a non-trivial right M-ideal, then since $\mathrm{Min}(X)^{**} = \mathrm{Min}(X^{**})$, X^{**} has a nontrivial M-ideal. But this is not possible, since X^{**} has a non-trivial L-summand, and by [36, Theorem I.1.8], a Banach space cannot contain nontrivial M-ideals and nontrivial L-summands simultaneously, unless it is two dimensional.

Proposition 3.1.8. *Let X be a left (right) M-embedded space, then*

(i) *$M_{m,n}(X)$ is left (right) M-embedded for all m and n. In particular, $C_n(X)$ (resp. $R_n(X)$) is left (right) M-embedded in $C_n(X^{**})$ (resp. $R_n(X^{**})$).*

(ii) *$C_\infty(X)$ (resp. $R_\infty(X)$) is a left (right) M-ideal in $C_\infty(X^{**})$ (resp. $R_\infty(X^{**})$).*

Proof. (i) If $J \subset X$ is a right M-ideal then $M_{m,n}(J)$ is a right M-ideal in $M_{m,n}(X)$ (see e.g. [11]). Now the result follows from the fact that $M_{m,n}(X^{**}) = M_{m,n}(X)^{**}$ completely isometrically.

(ii) If X is a left M-ideal in X^{**}, then by the left-handed version of Theorem 5.38 from [18], $C_\infty \otimes_{\mathrm{h}} X$ is a left M-ideal in $C_\infty \otimes_{\mathrm{h}} X^{**}$. But $C_\infty \otimes_{\mathrm{h}} X = C_\infty \overset{\smile}{\otimes} X = C_\infty(X)$ and $C_\infty \otimes_{\mathrm{h}} X^{**} = C_\infty \overset{\smile}{\otimes} X^{**} = C_\infty(X^{**})$. For the second assertion, use [18, Theorem 5.38] and that $Y \otimes_{\mathrm{h}} R_\infty = R_\infty(Y)$, for any operator space Y. ☐

It would be interesting to know when is $C_\infty(X)$ a right M-embedded space, that is, whether one can replace $C_\infty(X^{**})$ by $C_\infty(X)^{**} = C_\infty^w(X^{**})$ in Proposition 3.1.8 (ii). We will see in the remark after Proposition 3.1.14 that this is true in case of TRO. Also note that, if X is a WTRO then a routine argument shows that $C_\infty(X)$ is a right M-ideal in $C_\infty^w(X)$ (see the proposition below).

Proposition 3.1.9. *If X is a WTRO, then $C_\infty(X)$ is a one-sided M-ideal in $C_\infty^w(X)$.*

Proof. Let X be a WTRO, then clearly $C_\infty^w(X)$ is a TRO. Let $Z = C_\infty^w(X)$ and $N = \overline{X^\star X}^{w^*}$, then Z is a right Hilbert C^*-module over the C^*-algebra N. Indeed,

$$Z^\star Z = (C_\infty^w(X))^\star C_\infty^w(X) = R_\infty^w(X^\star) C_\infty^w(X) \subset \overline{X^\star X}^{w^*} = N,$$

and a right M-ideal of Z is the same as a right $Z^\star Z$-submodule (see [11, Theorem 6.6]).

Let

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \in C_\infty(X) \text{ and } w \in N.$$

Now since X is a $WTRO$, each $x_i w \in X$ and,

$$\left\| \begin{bmatrix} x_{k+1}w \\ x_{k+2}w \\ \vdots \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} x_{k+1} \\ x_{k+2} \\ \vdots \end{bmatrix} \right\| \|w\| .$$

The latter tends to zero, as $k \longrightarrow \infty$. This shows that

$$\begin{bmatrix} x_1 w \\ x_2 w \\ \vdots \end{bmatrix} \in C_\infty(X),$$

and hence $C_\infty(X)N \subset C_\infty(X)$. So $C_\infty(X)$ is a right M-ideal in $C_\infty^w(X)$. □

Proposition 3.1.10. *Every right M-embedded C^*-algebra is left M-embedded.*

Proof. Suppose A is a right M-ideal in A^{**}, then A is a closed right ideal in A^{**} and $A^\star$ is a closed left ideal in A^{**}. But A is self-adjoint, i.e., $A = A^\star$, hence A is a two-sided M-ideal in A^{**} (see e.g. [18, Section 4.4]). □

Remark. A complete M-ideal in an operator space is an M-ideal in the underlying Banach space. So by the above proposition, a one-sided M-embedded C^*-algebra is a M-embedded C^*-algebra in the classical sense. Hence by [36, Proposition 2.9], it has to be $*$-isomorphic to $\oplus_i^0 \mathbb{K}(H_i)$ (a c_0-sum), for Hilbert spaces H_i. These are Kaplansky's dual C^*-algebras, consequently, one-sided M-embedded C^*-algebras satisfy a long list of equivalent conditions which can be found for instance in the works of Dixmier and Kaplansky (see e.g. Exercise 4.7.20 from [23]). To mention a few:

(i) Every closed right ideal J in A is of the form eA for a projection e in the multiplier algebra of A.

(ii) There is a faithful $*$-representation $\pi : A \longrightarrow \mathbb{K}(H)$ as compact operators on some Hilbert space H.

(iii) The sum of all minimal right ideals in A is dense in A.

We imagine that several of these have variants that are valid for general one-sided M-embedded spaces (see e.g. Theorem 3.2.4).

Theorem 3.1.11. *Let Z be a non-reflexive operator space which is right M-embedded and if X is any finite dimensional operator space, then $Z \otimes_{\mathrm{h}} X$ is right M-embedded. Further, if $\mathcal{A}_{\mathrm{r}}(Z^{(4)} \otimes_{\mathrm{h}} X) \cong \mathbb{C}I$ then $Z \otimes_{\mathrm{h}} X$ is not left M-embedded.*

Proof. Since Z is a right M-ideal in Z^{**}, by [18, Proposition 5.38], $Z \otimes_{\mathrm{h}} X$ is a right M ideal in $Z^{**} \otimes_{\mathrm{h}} X$. Since X is finite dimensional, $(Z \otimes_{\mathrm{h}} X)^{**} = Z^{**} \otimes_{\mathrm{h}} X$ (see e.g. [14, 1.5.9]). Hence $Z \otimes_{\mathrm{h}} X$ is a right M-ideal in its bidual. Suppose that $Z \otimes_{\mathrm{h}} X$ is also left M-embedded and P be a projection in $\mathcal{A}_{\mathrm{r}}(Z^{(4)} \otimes_{\mathrm{h}} X)$ such that $(Z \otimes_{\mathrm{h}} X)^{\perp\perp} = P(Z^{(4)} \otimes_{\mathrm{h}} X)$. Since $\mathcal{A}_{\mathrm{r}}(Z^{(4)} \otimes_{\mathrm{h}} X) \cong \mathbb{C}I$, so $(Z \otimes_{\mathrm{h}} X)^{\perp\perp} = Z^{(4)} \otimes_{\mathrm{h}} X$. Now note that for any operator space X, if $E = i_X(X) \subset X^{**}$, then by basic functional analysis, $i_{X^{**}}(X^{**}) \cap E^{\perp\perp} = i_{X^{**}}(E)$. So if $i_X(X)^{\perp\perp} = X^{(4)}$, then $i_{X^{**}}(E) = i_{X^{**}}(X^{**})$, hence $X^{**} = E = i_X(X)$. This implies that $Z^{**} \otimes_{\mathrm{h}} X \cong Z \otimes X$, which is not possible since Z is non-reflexive. $\square$

Example. It is known that if J is a right M-ideal in an operator space X then $J \otimes_h E$ is a right M-ideal in $X \otimes_h E$, for any operator space E. By symmetry, if J is a left M-ideal in X, then $E \otimes_h J$ is a left M-ideal in $E \otimes_h X$. But it is not necessarily true that $J \otimes_h E$ is also a left M-ideal in $X \otimes_h E$ if J is a left M-ideal in X. To see this, let $X = M_3$,

$J = C_3$, and $E = M_5$. Then C_3 is a closed left ideal in the C^*-algebra M_3, and hence is a left M-ideal. Suppose $C_3 \otimes_h M_5$ is a left M-ideal in $M_3 \otimes_h M_5$. Since $M_3 \otimes_h M_5$ is finite dimensional, a left M-ideal in $M_3 \otimes_h M_5$ is a left M-summand. So there is a right M-projection in $M_3 \otimes_h M_5$. By [18, Theorem 5.42], $A_r(M_3 \otimes_h M_5) = M_5$, so $P = I \otimes q$ for some idempotent q in M_5, and $C_3 \otimes_h M_5 = (I \otimes q)(M_3 \otimes_h M_5) = M_3 \otimes_h M_5 q$. But this is not possible, since the dimension of $M_3 \otimes_h M_5 q$ is divisible by 9, while the dimension of $C_3 \otimes_h M_5$ is not. We can similarly show that, if J is a left M-ideal in X then $E \otimes_h J$ need not be a right M-ideal in $E \otimes_h X$.

As a result, we can generate many concrete examples of right M-embedded spaces which are not left M-embedded. If A is any algebra of compact operators, e.g. a nest algebra of compact operators, then we know that it is two-sided M-embedded. Hence, $Z = A \otimes_{\mathrm{h}} X$ is right M-embedded for all finite dimensional operator spaces X, as are all subspaces of Z. Almost all of these will, surely, not be left M-embedded. In Chapter 4, we will show that if A and B are approximately unital operator algebras, then $\mathcal{A}_\ell(A \otimes_{\mathrm{h}} B) \cong \Delta(M(A))$. As a consequence, using a similar argument as in Theorem 3.1.11, we can show that if A and B are approximately unital operator algebras such that A is completely M-embedded and B is finite dimensional with $B \neq \mathbb{C}1$, then $A \otimes_{\mathrm{h}} B$ is a right M-embedded operator space which is not left M-embedded.

For an operator space X, the *density character* of X is the least cardinal m such that there exists a dense subset Y of X with cardinality m. We denote the density character by *dens*(X). So if X is separable, then *dens*$(X) = \aleph_\circ$. A Banach space X is an *Asplund space* if every separable subspace has a separable dual. Also X is an Asplund space if and only if X^* has the RNP. For more details see [19, p.91, p.132], [22, p.82, p.195, p.213] and [51, p.34, p.75]. Using an identical argument to the classical case (see [36, Theorem III.3.1]), we can show the following.

Theorem 3.1.12. *If X is right M-embedded and Y is a subspace of X, then dens(Y) $=$ dens(Y^*). In particular, separable subspaces of X have a separable dual. So right M-embedded spaces are Asplund spaces, and X^* has the Radon-Nikodým Property.*

Proof. By Proposition 3.1.7, a subspace of a right M-embedded space is also right M-embedded, so WLOG we can assume that $Y = X$. Suppose that K is a dense subset of X. Then by Hahn-Banach theorem, for each $x \in K$ we can choose $x^* \in X^*$ such that, $\|x^*\| = 1$ and $x^*(x) = \|x\|$. Then the subset $N = \overline{\text{Span}}\{x^* \ : \ x \in K\}$ is norming for X^*, but by Corollary 3.2.12, X^* has no nontrivial norming subsets. So $N = X^*$ and hence dens(X^*) =dens(X). $\square$

Lemma 3.1.13. *If Z is a TRO which is isometrically isomorphic to $\mathbb{K}(H, K)$, the compact operators from H to K, then Z is completely isometrically isomorphic to either $\mathbb{K}(H, K)$ or $\mathbb{K}(K, H)$.*

Proof. Let $\theta : Z \longrightarrow \mathbb{K}(H, K)$ be an isometric isomorphism. Then $\theta^{**} : Z^{**} \longrightarrow B(H, K)$ is an isometric isomorphism. We use [59, Theorem 2.1] to prove this result. Take $M = B(K \oplus H)$ and $N = \mathcal{L}(Z)^{**}$, where $\mathcal{L}(Z)$ denotes the linking C^*-algebra of Z. Then by Lemma 3.1 from [59], there exists a projection $q \in \mathcal{L}(Z)^{**}$ such that both q and $I - q$ have central support equal to I, and $qN(I - q) \cong Z^{**}$. Let $p = P_K \in B(K)$, then $pM(I - p) \cong B(H, K)$. Thus by [59, Theorem 2.1], there exist central projections e_1, e_2 in M and f_1, f_2 in N with $e_1 + e_2 = I_{K \oplus H}$ and $f_1 + f_2 = I_{\mathcal{L}(Z)^{**}}$. Since there are no central projections in $B(K \oplus H)$, either $e_1 = I_{K \oplus H}$ or $e_1 = 0$. By [59, Theorem 2.1], either there exists a $*$-isomorphism $\psi : B(K \oplus H) \longrightarrow f_1 N f_1$ such that $(\theta^{**})^{-1} = \psi|_{B(H,K)}$, or there exists a $*$-'anti'-isomorphism $\phi : B(K \oplus H) \longrightarrow f_2 N f_2$ such that $(\theta^{**})^{-1} = \phi|_{B(H,K)}$.

In the first case, ψ is a complete isometry and hence so is $(\theta^{**})^{-1}$. Thus Z is completely isometrically isomorphic to $\mathbb{K}(H,K)$. We claim that the second case implies that Z is completely isometrically isomorphic to $\mathbb{K}(K,H)$. Let $\{e_i\}$ and $\{f_j\}$ be orthonormal bases for K and H resp., then $S = \{e_i\} \cup \{f_j\}$ is an orthonormal basis for $K \oplus H$. For each $T \in B(K \oplus H)$, define $\tilde{T} \in B(K \oplus H)^{\mathrm{op}}$ to be the transpose of T given by $\tilde{T}\eta = \sum_i \langle Te_i, \eta\rangle\, e_i + \sum_j \langle Tf_j, \eta\rangle\, f_j$, for every $\eta \in S$. Then $t : B(K \oplus H) \longrightarrow B(K \oplus H)^{\mathrm{op}}$ defined as $t(T) = \tilde{T}$, is a $*$-'anti'-isomorphism and $t(B(K,H)) = B(H,K)$. So $\tilde{\phi} = \phi \circ t$ is a $*$-isomorphism, and hence a complete isometry, such that $\tilde{\phi}(B(K,H)) = \phi(B(H,K)) = (\theta^{**})^{-1}(B(H,K))$. Thus restriction of $\tilde{\phi}$ to $B(K,H)$ is a complete isometry onto Z^{**}. $\square$

Proposition 3.1.14. *A one-sided M-embedded TRO is completely isometrically isomorphic to the c_0-sum of compact operators between some Hilbert spaces.*

Proof. Let X be a right M-embedded TRO, then by Theorem 3.1.12, X^* has the RNP. Also since X is a TRO, it is a JB^*-triple. From [5] we know that if X is a JB^*-triple and X^* has the Radon-Nykodým property, then X^{**} is isometrically an l^∞-sum of type-I triple factors, i.e., $X^{**} \cong \oplus_i^\infty B(H_i, K_i)$ isometrically, for some Hilbert spaces H_i and K_i. By Proposition 3.2.3 (ii), there exists a surjective isometry $\rho : X \longrightarrow \oplus_i^0 \mathbb{K}(H_i, K_i)$. Let $\mathbb{K}_i = \mathbb{K}(H_i, K_i)$, $\rho_i = \rho^{-1}|_{\mathbb{K}_i}$ and $Z_i = \rho_i(\mathbb{K}_i)$, then $X \cong \oplus_i^0 Z_i$, isometrically. So each Z_i is a M-summand in X. Every M-summand in a TRO is a complete M-summand, and hence each Z_i is a sub-TRO of X (see e.g. [14, 8.5.20]). Also since the Z_i are orthogonal, there is a ternary isomorphism between $\oplus_i^0 Z_i$ and X, given by $(x_i) \mapsto \sum_i x_i$. Hence $X \cong \oplus_i^0 Z_i$ completely isometrically (see e.g. [14, Lemma 8.3.2]). Thus by Lemma 3.1.13, for each i, either ρ_i^{-1} is a complete isometry or there exists a complete isometry $\tilde{\rho}_i : Z_i \longrightarrow \mathbb{K}(K_i, H_i)$. Define $\theta = \oplus \pi_i : \oplus_i^\infty Z_i^{**} \longrightarrow \oplus_i^\infty B_i$, where π_i is either $(\rho_i^{-1})^{**}$ or $(\tilde{\rho}_i)^{**}$ and B_i is either $B(H_i, K_i)$ or $B(K_i, H_i)$. Since each Z_i is a WTRO, and each π_i a complete isometry, θ is a complete isometry. So X^{**} is completely isometrically isomorphic to $\oplus_i^\infty B_i$. Hence by Proposition 3.2.3 (ii), X is completely isometrically isomorphic to $\oplus_i^0 \mathbb{K}_i$ where $\mathbb{K}_i$ is either $\mathbb{K}(H_i, K_i)$ or $\mathbb{K}(K_i, H_i)$. $\square$

Remark. As we stated earlier, if X is a right M-embedded TRO then $C_\infty(X)$ is also right M-embedded. Indeed, by Proposition 3.1.14, X is completely isometrically isomorphic to $\oplus_i^0 \mathbb{K}(H_i, K_i)$, which implies that $C_\infty(X)$ is completely isometrically isomorphic to $\oplus_i^0 \mathbb{K}(H_i, K_i^\infty)$, and the latter is a complete M-ideal in its bidual. More generally, if X is a right M-embedded TRO, then every $\mathbb{K}_{I,J}(X)$ is completely M-embedded.

Proposition 3.1.15. *Let X be the predual of a WTRO. Then X has the RNP if and only if X is the dual of a completely M-embedded space.*

Proof. Let X have the RNP, then by a similar argument as above, $X^* \cong \oplus^\infty B(H_i, K_j) \cong (\oplus^0 \mathbb{K}(H_i, K_j))^{**}$. So X is the dual of $\oplus^0 \mathbb{K}(H_i, K_j)$, which is M-embedded. The other direction follows by Theorem 3.1.12. $\square$

3.2 Properties of One-Sided M-Embedded Spaces

In this section we show that a number of nice properties from the classical setting are retained in the non-commutative setting of operator spaces. We start by stating two theorems from u-ideal theory which will allow us to draw some useful conclusions about one-sided M-embedded spaces. For proof of these theorems see [31, Theorem 6.6] and [31, Theorem 5.7], respectively. A u-ideal J of X is a *strict u-ideal* if $\mathrm{Ker}(P)$ is a norming subspace in X^*, where P is a u-projection onto $J^\perp$. By a *norming subspace* of X^* we mean a subspace N of X^* such that for each $x \in X$, $\|x\| = \sup\{|\phi(x)| : \phi \in N, \ \|\phi\| \leq 1\}$.

Theorem 3.2.1. *Let X be a Banach space which is an h-ideal in its bidual. Then the following are equivalent:*
(a) X is a strict h-ideal. (b) X^* is an h-ideal. (c) X contains no copy of ℓ^1.

Theorem 3.2.2. *Let X be a Banach space which contains no copy of ℓ^1 and is a strict u-ideal, then*

(a) *if $T : X^{**} \longrightarrow X^{**}$ is a surjective isometry, then $T = S^{**}$, for some surjective isometry $S : X \longrightarrow X$.*

(b) *X is the unique isometric predual of X^* which is a strict u-ideal.*

Proposition 3.2.3. *Let X be a non-reflexive right M-ideal in its bidual, then*

(i) *X do not contain a copy of ℓ^1 and X is a strict u-ideal.*

(ii) *If $T : X^{**} \longrightarrow X^{**}$ is a $($completely$)$ isometric surjection, then T is a bitranspose of some $($completely$)$ isometric surjective map on X.*

(iii) *X is the unique isometric predual of X^*.*

Proof. (i) Suppose that X is a right M-ideal in X^{**}, then being the range of a complete right L-projection, X^* is a right L-summand. So X and X^* are both h-ideals (see Section 1). Hence, by Theorem 3.2.1, X is a strict u-ideal and X does not contain a copy of ℓ^1.

(ii) By Theorem 3.2.2 and (i), $T = S^{**}$ for some isometric surjection S on X. Further, if T is a complete isometry then it is not difficult to see that S is also a complete isometry.

(iii) This follows from Theorem 3.2.2 and (i). $\square$

Theorem 3.2.4. *Suppose that X is a left M-embedded operator space. Then*

(i) *Every right M-ideal of X is a right M-summand.*

(ii) *Every complete left M-projection P in X^{**} is the bitranspose of a complete left M-projection Q on X.*

(iii) *Suppose that X is also a right M-embedded operator space and it has no nontrivial right M-summands. Then for every nontrivial right M-ideal J of X^{**}, either J contains X or $J \cap X = \{0\}$.*

(iv) $\mathcal{A}_\ell(X) \cong \mathcal{A}_\ell(X^{**})$.

(v) *If X is a completely M-embedded space, then $Z(X) = Z(X^{**})$, where $Z(X)$ is the centralizer algebra of X, in the sense of [18, Chapter 7].*

Proof. (i) Let J be a right M-ideal of X and suppose that P is a projection in $\mathrm{Ball}(\mathcal{M}_\ell(X^{**}))$ such that $P(X^{**}) = J^{\perp\perp}$. Since X is a left M-ideal in X^{**} and $P \in \mathcal{M}_\ell(X^{**})$, by [18, Proposition 4.8] we have that $P(X) \subset X$. Then $Q := P|_X \in \mathcal{M}_\ell(X)$ and $\|Q\| = \|P|_X\|_{\mathcal{M}_\ell(X)} \leq 1$. Also $J^{\perp\perp} \cap X = J$ is the range of Q. Hence by [11, Theorem 5.1], J is a complete right M-summand.

(ii) If P is an complete left M-projection in X^{**}, then $T := 2P - Id_{X^{**}}$ is a complete isometric surjection of X^{**}. Hence by Proposition 3.2.3, $T = S^{**}$, for some complete surjective isometry on X. So, $2P = T + Id_{X^{**}} = (S + Id_X)^{**}$, and since by [18, Section 5.3] $\mathcal{A}_\ell(X) \subset \mathcal{A}_\ell(X^{**})$, $S + Id_X$ must be a complete left M-projection in X.

(iii) Let P be a complete two-sided M-projection from $X^{(4)}$ onto $X^{\perp\perp}$ and Q be a complete left M-projection from $X^{(4)}$ onto $J^{\perp\perp}$. Then by [11, Theorem 5.1], $P \in \mathrm{Ball}(\mathcal{M}_\ell(X^{(4)}))$ and $Q \in \mathrm{Ball}(\mathcal{M}_r(X^{(4)}))$, which implies that $PQ = QP$. Hence by [18, Theorem 5.30 (ii)], $J \cap X$ is a right M-ideal in X^{**}. But $J \cap X \subset X$, so by [18, Theorem

5.3], $J \cap X$ is a right M-ideal in X. Hence by (i), $J \cap X$ is a right M-summand. By the hypothesis, either $J \cap X = \{0\}$ or $J \cap X = X$, i.e., $J \cap X = \{0\}$ or $X \subset J$.

(iv) We know that $\mathcal{A}_\ell(X) \subset \mathcal{A}_\ell(X^{**})$, completely isometrically, via the map $\phi : T \longrightarrow T^{**}$ (see e.g. [18, Section 5.3]). By (i), ϕ is surjective and maps onto the set of complete left M-projections. But the left M-projections are exactly the contractive projections in $\mathcal{A}_\ell(X^{**})$, and since $\mathcal{A}_\ell(X^{**})$ is a von Neumann algebra, the span of these projections is dense in it. So ϕ maps onto $\mathcal{A}_\ell(X^{**})$.

(v) If X is right M-embedded then we can show similarly to (iv) that $\mathcal{A}_r(X) \cong \mathcal{A}_r(X^{**})$. By definition, $Z(X) = \mathcal{A}_\ell(X) \cap \mathcal{A}_r(X)$, hence it follows that $Z(X) = Z(X^{**})$. $\square$

Remarks (from [3]). 1) Theorem 3.2.4 (i) can be improved in the case that X is an approximately unital operator algebra A. Theorem 3.2.4, is valid for all one-sided M-ideals, both right and left. This follows from [3, Proposition 2.12] and [3, Proposition 2.9].

2) Theorem 3.2.4 (iii) can also be improved in the case that X is an operator algebra A. If A is an operator algebra with right cai which is a left ideal in A^{**} (or equivalently, if A is a left M-ideal in its bidual), and if J is a right ideal in A^{**}, then $JA \subset J \cap A$. Hence if $J \cap A = (0)$ then $JA = (0)$. Thus $JA^{**} = (0)$, and hence $J = (0)$, since A^{**} has a right identity. Thus the case $J \cap A = (0)$ will not occur in the conclusion of Theorem 3.2.4 (iii), in the case that X is an approximately unital operator algebra.

Corollary 3.2.5. *Let X be a left M-embedded operator space. Suppose that J is a complete right M-ideal of X, and $\otimes_\beta$ is any operator space tensor product with the following properties:*

(i) $- \otimes_\beta Id_Z$ *is* functorial. *That is, if $T : X_1 \longrightarrow X_2$ is completely contractive, then*

$T \otimes_\beta Id_Z : X_1 \otimes_\beta Z \longrightarrow X_2 \otimes_\beta Z$ *is completely contractive,*

(ii) *the canonical map* $C_2(X) \otimes Z \longrightarrow C_2(X \otimes Z)$*extends to a completely isometric isomorphism* $C_2(X) \otimes_\beta Z \longrightarrow C_2(X \otimes_\beta Z)$*,*

(iii) *the span of elementary tensors* $x \otimes z$ *for* $x \in X$*,* $z \in Z$ *is dense in* $X \otimes_\beta Z$*.*

Then $J \otimes_\beta E$ *is a complete right* M*-summand of* $X \otimes_\beta E$*. In particular,* $J \overset{\smile}{\otimes} E$ *is a complete right* M*-summand of* $X \overset{\smile}{\otimes} E$*.*

Proof. Since J is a complete right M-ideal of X, by Theorem 3.2.4, it is also a right M-summand. Hence by the argument in [18, Section 5.6], $J \otimes_\beta E$ is a right M-summand of $X \otimes_\beta E$. □

Following is a Banach space result stated for operator spaces. We give a proof for completion which is along similar lines to the Banach space proof.

Proposition 3.2.6. *Let* X *be an operator space and* π_{X^*} *be the canonical projection from* X^{***} *onto* X^**. Then the following are equivalent:*

(i) π_{X^*} *is the only completely contractive projection on* X^{***} *with kernel* $X^\perp$*.*

(ii) *The only completely contractive operator on* X^{**} *which restricts to identity on* X *is* $Id_{X^{**}}$*.*

(iii) *If* U *is a surjective complete isometry on* X*, then the only completely contractive operator on* X^{**} *which restricts to* U *on* X *is* U^{**}*.*

Proof. (i) $\Rightarrow$ (ii) Suppose that T is as in (i). Define $P = T^* \circ \pi_{X^*}$. Then since $T(x) = x$ for all $x \in X$, $(T^*x^* - x^*)(x) = x^*(Tx) - x^*(x) = 0$. So $(T^*(x^*) - x^*) \in X^\perp$ for all $x^* \in X^*$.

Using this we now show that P is a completely contractive projection with $\mathrm{Ker}(P) = X^{\perp}$. It is clear that P is completely contractive. Also since $\mathrm{Ker}(\pi_{X^*}) = X^{\perp}$, $X^{\perp} \subset \mathrm{Ker}(P)$. For the other containment, since $P(y) = 0$ for all $y \in X^{\perp}$, we can assume that $P(x^*) = 0$ for some $x^* \in X^*$. Now $0 = T^* \circ \pi_{X^*}(x^*) = T^*(x^*)$ and $T^*(x^*) - x^* \in X^{\perp}$, and so $x^* \in X^{\perp}$. Hence $\mathrm{Ker}(P) = X^{\perp}$. Define $I = \pi_{X^*} \circ T^* \circ \pi_{X^*} - \pi_{X^*}$, then clearly $I(y) = 0$ for all $y \in X^{\perp}$. If $x^* \in X^*$ then $T^* \circ \pi_{X^*}(x^*) - x^* = T^*(x^*) - x^* \in X^{\perp}$, so that $I(x^*) = 0$. Hence $I = 0$ and $\pi_{X^*} T^* \pi_{X^*} = \pi_{X^*}$. Thus, $P^2 = (T^* \circ \pi_{X^*})(T^* \circ \pi_{X^*}) = T^* \circ (\pi_{X^*} \circ T^* \pi_{X^*}) = T^* \circ \pi_{X^*} = P$. By (i), $P = \pi_{X^*}$, so that $x^* = \pi_{X^*}(x^*) = T^* \pi_{X^*}(x^*) = T^*(x^*)$. Hence $\widehat{T(x^{**})} = \widehat{x^{**}}$, that is, $T = Id_{X^{**}}$. Indeed, if $x^* \in X^*$, then $\widehat{T(x^{**})}(x^*) = \widehat{x^*}(T(x^{**})) = T(x^{**})(x^*) = x^{**}(T^*(x^*)) = \widehat{x^{**}}(x^*)$.

(ii) ⇒ (i) With P as in (i), define $T = (P|_X)^* \circ i_{X^{**}}$. Then clearly $\|T\|_{cb} \leq 1$. Since $\mathrm{Ker}(P) = X^{\perp}$, $P(x^*|_X) = x^*|_X$, and hence $T|_X = Id_X$. By the assumption, $Id_{X^{**}} = T = (P|_X)^* \circ i_{X^{**}}$, so that $P(x^*) = x^*$, and $\mathrm{Ran}(P) \subset X^*$. This shows that $P = \pi_{X^*}$, since they have the same kernel and $\mathrm{Ran}(\pi_{X^*}) \subset \mathrm{Ran}(P)$.

(iii) ⇒ (ii) Obvious.

(ii) ⇒ (iii) Suppose that $T|_X = U$, where T and U are as in (iii). Let $V = (U^{**})^{-1} T$, then $\|V\| \leq 1$ and $V|_X = Id_X$. Hence by the assumption, $V = id_{X^{**}}$. □

The property in (ii) above is sometimes called the *unique extension property.*

Corollary 3.2.7. *Every right M-embedded space has the unique extension property.*

Proof. If X is a right M-ideal in X^{**}, then by Proposition 3.1.4, π_{X^*} is a complete right L-projection with kernel $X^{\perp}$. By [11, Theorem 3.10(b)], it is the only completely contractive projection with kernel $X^{\perp}$. Hence X satisfies all the equivalent conditions in Proposition 3.2.6. In particular, it has the unique extension property. □

An operator space X has the *completely bounded approximation property* (respectively, *completely contractive approximation property*) if there exists a net of finite-rank mappings $\phi_\nu : X \longrightarrow X$ such that $\|\phi_\nu\|_{cb} \leq K$ for some constant K (respectively, $\|\phi_\nu\|_{cb} \leq 1$) and $\|\phi_\nu(x) - x\| \to 0$, for every $x \in X$.

Corollary 3.2.8. *Let X be a right M-embedded operator space. If X has the completely bounded approximation property then X^* has the completely bounded approximation property.*

Proof. Let T_λ be a net of finite rank operators in $CB(X)$, such that $\|T_\lambda\|_{cb} \leq K$ for some $K > 0$, and $\|T_\lambda(x) - x\| \longrightarrow 0$. We first show that there exists a subnet of $\{T_\lambda^*\}$ which converges to Id_{X^*}, in the point-weak topology. We know that $CB(X^{**})$ is a dual operator space with $CB(X^{**}) = (X^* \widehat{\otimes} X^{**})^*$, so the closed ball of radius K in $CB(X^{**})$, $K\text{Ball}(CB(X^{**}))$, is w^*-compact. Since $T_\lambda^{**} \in K\text{Ball}(CB(X^{**}))$, there exists a subnet $\{T_{\lambda_\nu}^{**}\}$ and T in $K\text{Ball}(CB(X^{**}))$, such that $T_{\lambda_\nu}^{**} \xrightarrow{w^*} T$. That is, $T_{\lambda_\nu}^{**}(\phi)(f) \longrightarrow T(\phi)(f)$ for all $\phi \in X^{**}$ and $f \in X^*$. In particular for $\hat{x} \in X \subset X^{**}$, the latter convergence implies that $f(T_{\lambda_\nu} x) \longrightarrow T(\hat{x})(f)$ for all $f \in X^*$ and $x \in X$. Now since $T_\lambda \longrightarrow Id_X$ in the point-norm topology, it also converges in the point-weak topology. So $f(T_{\lambda_\nu} x) \longrightarrow f(x)$ for all $x \in X$ and $f \in X^*$. Hence $T|_X = Id_X$. By Corollary 3.2.7, X has the unique extension property. Hence $T = Id_{X^{**}}$, so $(T_{\lambda_\nu}^{**}\phi)(f) \longrightarrow \phi(f)$. Equivalently, $\phi(T_{\lambda_\nu}^* f) \longrightarrow \phi(f)$ for all $\phi \in X^{**}$ and $f \in X^*$, which proves the claim. Thus Id_{X^*} is in the point-weak closure of the convex hull of $\{T_\lambda^*\}$. But since the norm and the weak topologies coincide on a convex set [24, p.477], Id_{X^*} is in the point-norm closure of the convex hull of $\{T_\lambda^*\}$. □

Along similar lines, we can prove that if a right M-embedded space has the completely contractive approximation property then so does its dual. We are grateful to Z. J. Ruan for the following result. Since we could not find this in the literature, we include his proof.

Lemma 3.2.9. *Suppose X^* has the completely bounded approximation property and X is a locally reflexive (or C-locally reflexive) operator space. Then X has the completely bounded approximation property.*

Proof. We prove the locally reflexive case, the C-locally reflexive case is similar. Suppose that X is locally reflexive. Since X^* has the completely bounded approximation property, there exists a net of finite rank maps $T_\lambda : X^* \longrightarrow X^*$ such that $\|T_\lambda\|_{cb} \leq K < \infty$ and $T_\lambda \longrightarrow \mathrm{Id}$ in the point-norm topology. Then $\phi_\lambda := (T_\lambda)^*|_X : X \longrightarrow X^{**}$ is a net of finite rank maps such that $\langle \phi_\lambda(x) - x, f\rangle \to 0$ for all $x \in X$ and $f \in X^*$. Let $Z_\lambda = \phi_\lambda(X)$ and ρ_λ be the inclusion map from Z_λ to X^{**}. Since $\phi_\lambda(X)$ is a finite dimensional subspace of X^{**} and X is locally reflexive, for each λ we can find a net of completely contractive maps $\rho_t^\lambda : Z_\lambda \longrightarrow X$ such that ρ_t^λ converges to ρ_λ in the point-weak* topology. Then the maps $\psi_{\lambda,t} = \rho_t^\lambda \circ \phi_\lambda$ are finite rank maps from X to X such that $\|\psi_{\lambda,t}\|_{cb} \leq K$. Now using a reindexing argument based on [8, Lemma 2.1], we show that there exists a net γ such that $\lim_\gamma \langle \psi_\gamma(x) - x, f\rangle = 0$ for all $x \in X$ and $f \in X^*$. Define Γ to be a set of 4-tuples $(\lambda, t, Y, \epsilon)$, where Y is a finite subset of $X \times X^*$ and where $\epsilon > 0$ is such that $|\psi_{\lambda,t}(x)(f) - \phi_\lambda(x)(f)| < \epsilon$ for all $(x, f) \in Y$. Then it is easy to check that Γ is a directed set with ordering $(\lambda, t, Y, \epsilon) \leq (\lambda', t', Y', \epsilon')$ iff $\lambda \leq \lambda'$, $Y \subset Y'$ and $\epsilon' \leq \epsilon$. Let $\psi_\gamma = \psi_{\lambda,t}$ if $\gamma = (\lambda, t, Y, \epsilon)$. If $\epsilon > 0$ choose λ_o such that for all $\lambda \geq \lambda_o$ we have $|\phi_\lambda(x)(f) - \hat{x}(f)| < \epsilon$. Choose t_o such that $\gamma_o = (\lambda_o, t_o, \{x, f\}, \epsilon) \in \Gamma$. Now if $\gamma = (\lambda, t, Y, \epsilon') \geq \gamma_o$ then

$$|\psi_\gamma(x)(f) - \hat{x}(f)| \leq |\psi_{\lambda,t}(x)(f) - \phi_\lambda(x)(f)| + |\phi_\lambda(x)(f) - \hat{x}(f)| \leq \epsilon' + \epsilon < 2\epsilon.$$

Hence $\psi_\gamma \to \mathrm{Id}_X$ in the point-weak topology and thus, Id_X is in the point-weak closure of $K\mathrm{Ball}(CB(X))$. But the point-weak and the point-norm closures of $K\mathrm{Ball}(CB(X))$ coincide [24, p.477], thus there exist a net $\{\eta_p\} \subset K\mathrm{Ball}(CB(X))$ such that $\eta_p \to \mathrm{Id}_X$ in the point-norm topology. $\square$

Remark. A natural question is whether right M-embedded or completely M-embedded spaces are locally reflexive? Also note that if X has the completely bounded approximation property then by [27, Theorem 11.3.3], X has the strong operator space approximation property. Hence by [27, Corollary 11.3.2], X has the slice map property for subspaces of $B(\ell^2)$. There seems some hope that the argument in [27, Theorem 14.6.6] can be made to imply that X is 1-exact, and hence is locally reflexive.

The following lemma is a well known Banach space result (see [36, Lemma III.2.14] for proof).

Lemma 3.2.10. *For a Banach space X and $x^* \in X^*$ with $\|x^*\| = 1$, the following are equivalent:*

(i) *x^* has a unique norm preserving extension to a functional on X^{**}.*

(ii) *The relative w- and w^*-topologies on the ball of X^*, B_{X^*} agree at x^*, i.e., the map $Id_{B_{X^*}} : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$ is continuous at x^*.*

Corollary 3.2.11. *If X is a one-sided M-ideal in its bidual, then the relative w- and w^*-topologies on B_{X*} agree on the unit sphere.*

Proof. This is an immediate consequence of the fact that the one-sided M-ideals are Hahn-Banach smooth (see e.g. [18, Chapter 2]) and the above lemma. □

The following result follows from Corollary 3.2.11 (see the argument in Corollary III.2.16 [36]). By a *norming subspace* we mean a subspace N of X^* such that for each $x \in X$, $\|x\| = \sup\{|\phi(x)| : \phi \in N, \ \|\phi\| \leq 1\}$.

Corollary 3.2.12. *If X is a one-sided M-ideal in its bidual, then X^* contains no proper norming subspace.*

Remark. The above corollary combined with Proposition 2.5 in [32], immediately gives a second proof of the unique extension property for one-sided M-embedded operator spaces. We note that the Proposition 2.5 in [32] is proved for a real Banach space since it uses a lemma ([32, Lemma 2.4]) on real Banach spaces. However, it is easy to see, using the fact that $(E_{\mathbb{R}})^* = (E^*)_{\mathbb{R}}$, isometrically (see [43, Proposition 1.1.6]), that the lemma is also true for any complex Banach space E. Here $E_{\mathbb{R}}$ denotes the underlying real Banach space.

Proposition 3.2.13. *Let Y be a completely contractively complemented operator space in Y^{**}, i.e., $Y \oplus Z = Y^{**}$, and $\|[y_{ij}]\| \leq \|[\phi_{ij}]\|$ for all $\phi_{ij} = y_{ij} + z_{ij}$ where $y_{ij} \in Y$, $z_{ij} \in Z$ and $\phi_{ij} \in Y^{**}$ for all i,j. Then Y cannot be a proper right M-ideal in any other operator space.*

Proof. Let X be an operator space with Y a complete right M-ideal in X. Suppose that P is a complete left M-projection from X^{**} onto $Y^{\perp\perp}$. By the hypothesis, there is a completely contractive projection $Q : Y^{**} \longrightarrow Y^{**}$ mapping onto Y. Let R be the restriction of $(Q \circ P)$ to X. Then since $Y^{**} \cong Y^{\perp\perp}$ completely isometrically, R is a completely contractive projection onto Y. Hence by the uniqueness of a left M-projection (see e.g. [11, Theorem 3.10]), R has to be a complete left M-projection, and thus, Y is a right M-summand. $\square$

Proposition 3.2.14. *Every non-reflexive right M-embedded operator space contains a copy of c_0. Moreover, every subspace and every quotient of a right M-embedded space, which is not reflexive, contains a copy of c_0.*

Proof. Suppose that X is a non-reflexive right M-ideal in its bidual, and suppose that X does not contain a copy of c_0. Since X is a u-ideal, by [31, Theorem 3.5] it is a

u-summand. Since u-summands are contractively complemented, X is the range of a contractive projection. But this implies that X is a right M-summand (see the discussion at the end in [18, Section 2.3]). Since X is non-reflexive, it cannot be a non-trivial M-summand in X^{**}. Hence X has to contain a copy of c_0. The rest follows from Theorem 3.1.7. $\square$

Let X be an operator space. Then $\pi_{X^{**}} := i_{X^{**}} \circ (i_{X^*})^*$ is a completely contractive projection onto X^{**} with kernel $(X^*)^\perp$. So $X^{(4)} = X^{**} \oplus (X^*)^\perp$. The following may be used to give an alternative proof of some results above. Let K be a convex set in X, then $x \in K$ is called *exposed point* of K if there is an $f \in X^*$ such that attains its maximum on K at x and only at x, and f is said to expose x. An $x \in K$ is a *strongly exposed point* of K if there exists $f \in X^*$ which exposes x and so that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in K$ and $\|y - x\| \geq \delta$ then $Re(f(y)) \leq Re(f(x)) - \epsilon$.

Proposition 3.2.15. *If X is a right M-embedded operator space, then $\pi_{X^{**}}$ is the only contractive projection from $X^{(4)}$ onto X^{**}.*

Proof. Since X is right M-embedded, then by Theorem 3.1.12, X^* has the RNP, i.e., $(X^*)_{\mathbb{R}}$ has the RNP, where $X_{\mathbb{R}}$ denotes the underlying real Banach space. Then by [22, p.202], $\text{Ball}(X^*)_{\mathbb{R}}$ is the closure of the convex hull of its strongly exposed points. If ψ is a strongly exposed point in $\text{Ball}(X^*)_{\mathbb{R}}$, then it is a denting point (see e.g. [39]). Hence ψ is a point of continuity of $\text{Id} : ((X^*)_{\mathbb{R}}, w) \longrightarrow ((X^*)_{\mathbb{R}}, \|.\|)$. Thus by [30, p.144], $(X^*_{\mathbb{R}})^*_{\mathbb{R}}$ satisfies the assumptions of [30, Theorem II.1]. Hence there is a unique contractive $\mathbb{R}$-linear projection from $(X^*_{\mathbb{R}})^{***}$ onto $(X^*_{\mathbb{R}})^*$. Since $(X^*)_{\mathbb{R}} = (X_{\mathbb{R}})^*$ ([43, Proposition 1.1.6]), there is a unique $\mathbb{R}$-linear contractive projection from $(X^{(4)})_{\mathbb{R}}$ to $(X^{**})_{\mathbb{R}}$, and hence a unique $\mathbb{C}$-linear contractive projection from $X^{(4)}$ onto X^{**}. $\square$

Remark. Note that the above result also holds for Banach spaces X such that X^* has the RNP, and in particular for h-ideals which are strict in the sense of [31]. It also holds for separable strict u-ideals by the proof of Theorem 5.5 from [31].

3.3 One-Sided L-Embedded Spaces

In this section we discuss the dual notion of L-embedded spaces. For the definition of a one-sided L-embedded and a completely L-embedded operator space, see Section 3.1.

Examples. We list a few examples of right L-embedded spaces:

(a) Duals of left M-embedded spaces.

(b) Preduals of von Neumann algebras.

(c) Preduals of subdiagonal operator algebras, in the sense of Arveson [4].

We have already noted (a) in Corollary 3.1.5. For (b), note that it is well known that $(M_*)^\perp$ is a w^*-closed two-sided ideal in M^{**}, for any von Neumann algebra M. So $(M_*)^\perp$ is a complete M-ideal in M^{**}. Hence by [11, Proposition 3.10(e)] and [11, Proposition 3.9], M_* is a complete L-summand in M^*. For (c), let $A = H^\infty(M, \tau)$, where M is a von Neumann algebra and τ a faithful normal tracial state. Then by [61, Theorem 3.1], A has a unique predual, namely $A_* = M_*/A_\perp$. Also, each $M_n(A)$ is a subdiagonal operator algebra, so applying [61, Corollary 3.3] to $M_n(A)^*$ we have that each $M_n(A)_*$ is an L-summand in $M_n(A)^*$. Thus by Lemma 3.1.2, A_* is a complete L-summand in A^*.

In Chapter 4, we will look at algebras which will provide natural examples of spaces which are right but not left M-ideals in their second dual. Their duals will be left but not right L-summands in their second dual, by the next result, which is from the joint paper [3].

Lemma 3.3.1. *If an operator space X is a right but not a left M-ideal in its second dual, then X^* is a left but not a right L-summand in its second dual.*

Proof. We first remark that a subspace J of operator space X is a complete L-summand of X if and only if it is both a left and a right L-summand. This follows e.g. from the matching statement for M-ideals [14, Proposition 4.8.4], and the second 'bullet' on p. 8 of [18]. By Proposition 3.1.4, X^* is a left L-summand in X^{***}, via the canonical projection $i_{X^*} \circ (i_X)^*$. Thus if X^* is both a left and a right L-summand in its second dual, then $i_{X^*} \circ (i_X)^*$ is a left L-projection by the third 'bullet' on p. 8 of [18]. Hence by Proposition 3.1.4, X is a left M-ideal in its second dual, a contradiction. $\square$

Definition 3.3.2. Let X be left L-embedded. Then a closed subspace Y of X is a *left L-subspace* if Y is left L-embedded and for the right L-projection Q from Y^{**} onto Y, we have that $\operatorname{Ker}(Q) = \operatorname{Ker}(P) \cap Y^{\perp\perp}$, where P is a right L-projection from X^{**} onto X.

Theorem 3.3.3. *Let X be a left L-summand in X^{**} and let Y be a subspace of X. Let $P : X^{**} \longrightarrow X^{**}$ be a complete right L-projection onto X. Then the following conditions are equivalent:*

(i) *Y is a left L-subspace of X.*

(ii) $P(\overline{Y}^{w^*}) = Y$.

(iii) $P(\overline{B_Y}^{w^*}) = B_Y$.

Proof. If Y is a left L-subspace then since, $\overline{Y}^{w^*} = Y^{\perp\perp} = Y \oplus (Y^{\perp\perp} \cap \operatorname{Ker}(P))$, it is clear that $P(\overline{Y}^{w^*}) = Y$. Hence (i) implies (ii). Also since,

$$B_Y = P(B_Y) \subset P(\overline{B_Y}^{w^*}) = P(B_{Y^{\perp\perp}}) = P(B_{\overline{Y}^{w^*}}) \subset B_Y,$$

it is clear that (ii) and (iii) are equivalent. We now show that (ii) implies (i). Since $P(Y^{\perp\perp}) = Y \subset Y^{\perp\perp}$, the restriction of P to $Y^{\perp\perp} = Y^{**}$, say Q, is a completely contractive projection from Y^{**} onto Y. Also we have that $P \in \mathcal{C}_r(X^{**})$ (for notation see [18, Chapter 2]) and $P = P^\star$, and $P(Y^{\perp\perp}) \subset Y^{\perp\perp}$, so by [18, Corollary 5.12] we have $Q \in \mathcal{C}_r(Y^{**})$. Thus Q is a right L-projection and clearly since $Q = P|_{Y^{\perp\perp}}$, $\mathrm{Ker}(Q) = \mathrm{Ker}(P) \cap Y^{\perp\perp}$. Hence Y is a left L-subspace of X. □

Corollary 3.3.4. *Let X be a left L-embedded operator space and Y be a left L-subspace of X, then X/Y is left L-embedded.*

Proof. Let $P : X^{**} \longrightarrow X^{**}$ be a complete right L-projection onto X, then by Theorem 3.3.3, P maps $Y^{\perp\perp}$ onto Y. Consider the map

$$P/Y^{\perp\perp} : X^{**}/Y^{\perp\perp} \longrightarrow X^{**}/Y^{\perp\perp}$$

given by $(P/Y^{\perp\perp})(x^{**}+Y^{\perp\perp}) = P(x^{**})+Y^{\perp\perp}$. Then since $P \in \mathcal{C}_r(X^{**})$ (see [18, Chapter 2] for the notation) with $P(Y^{\perp\perp}) = P^\star(Y^{\perp\perp}) \subset Y^{\perp\perp}$, by [18, Corollary 5.12] we have that $P/Y^{\perp\perp} \in \mathcal{C}_r(X^{**}/Y^{\perp\perp})$. So $P/Y^{\perp\perp}$ is a complete right L-projection onto $(X + Y^{\perp\perp})/Y^{\perp\perp}$. Since $(X/Y)^{**}$ is completely isometrically isomorphic to $X^{**}/Y^{\perp\perp}$ and under this isomorphism X/Y is mapped onto $(X + Y^{\perp\perp})/Y^{\perp\perp}$, it is clear that X/Y is left L-embedded. □

Proposition 3.3.5. *If an operator space X is a left L-summand in its bidual, then any left L-summand Y of X is a left L-summand in Y^{**}.*

Proof. Indeed if X is the range of a right L-projection P on X^{**}, and if Y is the range of a right L-projection Q on X, then Q^{**} and P are in the right Cunningham algebra of X^{**} [18, p. 8–9]. Note that $Q^{**}P = PQ^{**}P$ (since $\mathrm{Ran}(Q^{**}P) \subset Y \subset X$). Since we are dealing with projections in a C^*-algebra, we deduce that $PQ^{**} = Q^{**}P$. It follows that $P(Y^{\perp\perp}) \subset Y$, and so Y is a left L-subspace of X. By Theorem 3.3.3, Y is a left L-summand in its bidual. $\square$

The following corollary can also be proved using Proposition 3.3.3 (see [36, Proposition IV.1.6]). We include a proof for completeness.

Corollary 3.3.6. *Let X be a left L-embedded space and let Y_1, Y_2, $\{Y_i\}_{i\in I}$ be left L-subspaces of X. Then*

(i) $\cap_{i\in I} Y_i$ *is a left L-subspace.*

(ii) $Y_1 + Y_2$ *is closed if and only if $Y_1 + Y_2$ is a left L-subspace of X.*

Proof. (i) Let $P : X^{**} \longrightarrow X^{**}$ be a complete right L-projection onto X, then by Theorem 3.3.3, $P(\overline{Y_i}^{w*}) = Y_i$ for all i. Since $P(x) = x$ for all $x \in X$, and $\cap_{i\in I} Y_i \subset \overline{\cap_i Y_i}^{w^*}$ and $\overline{\cap_i Y_i}^{w^*} \subset \cap_i \overline{Y_i}^{w^*}$,

$$\cap_i Y_i = P(\cap_i Y_i) \subset P(\overline{\cap_i Y_i}^{w^*}) \subset P(\cap_i \overline{Y_i}^{w^*}) \subset \cap_i P(\overline{Y_i}^{w^*}) = \cap_i P(Y_i) = \cap_i Y_i.$$

Hence $P(\overline{\cap_i Y_i}^{w^*}) = \cap_i Y_i$, and now use Theorem 3.3.3.

(ii) If $Y_1 + Y_2$ is left L-embedded then it is closed in X, since $Y_1 + Y_2 = P(\overline{Y_1 + Y_2}^{w^*})$. Conversely, suppose that $Y_1 + Y_2$ be closed. By a standard application of the open mapping theorem, we get a $c > 0$ such that $B_{Y_1+Y_2} \subset c(B_{Y_1} + B_{Y_2})$. Hence

$$P(\overline{B_{Y_1+Y_2}})^{w^*} \subset cP(\overline{Y_1}^{w^*} + \overline{Y_2}^{w^*}) = c(B_{Y_1} + B_{Y_2}) \subset Y_1 + Y_2.$$

So, if $y \in \overline{(Y_1+Y_2)}^{w^*}$, then $y/(\|y\|+1) \in B_{\overline{Y_1+Y_2}^{w^*}} \subset Y_1+Y_2$, and hence $y \in Y_1+Y_2$. Therefore,

$$Y_1+Y_2 = P(Y_1+Y_2) \subset P(\overline{Y_1+Y_2}^{w^*}) \subset Y_1+Y_2.$$

Hence Y_1+Y_2 is a left L-subspace of X. $\square$

We omit the proofs of the proposition below, because it is identical to the classical version (see [36, Proposition IV.1.12]).

Proposition 3.3.7. *Let X be a left L-embedded space and let Y be a left L-subspace of X. Then Y is proximinal in X and the set of best approximations to x from Y is weakly compact for all x in X.*

The following two results are non-commutative version of some of Godefroy's results.

Proposition 3.3.8. *Let X be left L-embedded and let P be a complete right L-projection from X^{**} onto X. Then*

- (i) *there is at most one predual of X, up to complete isometric isomorphism, which is right M-embedded,*
- (ii) *there is a predual of X which is a right M-ideal in its bidual if and only if* $\operatorname{Ker}(P)$ *is w^*-closed in X^{**}.*

Proof. (i) Let Y_1 and Y_2 be two preduals of X, that is $Y_1^* \cong X \cong Y_2^*$ completely isometrically via a map $I : Y_1^* \longrightarrow Y_2^*$. Let $P = {I^{**}}^{-1}\pi_{Y_2{}^*}I^{**}$, then $P : Y_1^{***} \longrightarrow Y_1^{***}$ is a completely contractive projection onto Y_1^*. Thus by [11, Theorem 3.10(a)] $\pi_{Y_1{}^*} = P = {I^{**}}^{-1}\pi_{Y_2{}^*}I^{**}$. By basic functional analysis, this is equivalent to the w^*-continuity of I, which implies that $I = J^*$ for some complete isometric isomorphism $J : Y_2 \longrightarrow Y_1$. Thus the predual is unique up to complete isometry.

(ii) Suppose that Y is a right M-embedded operator space such $Y^* = X$. Then π_{Y^*} is a complete right L-projection from Y^{***} onto $Y^* = X$, so $\pi_{Y^*} = P$. The kernel of π_{Y^*} is $i_Y(Y)^\perp \subset Y^{***}$, which is clearly w^*-closed in X^{**}. Conversely, suppose that $\mathrm{Ker}(P)$ is w^*-closed in X^{**}. Let $Y = (\mathrm{Ker}(P))_\perp \subset X^*$, then $Y^\perp = \overline{\mathrm{Ker}(P)}^{w^*} = \mathrm{Ker}(P) = \mathrm{Ran}(I - P)$, which means that $Y^\perp$ is a left L-summand. Hence Y is a right M-ideal in X^*. Since $P : X^{**} \longrightarrow X^{**}$ is a complete quotient map onto X, $X^{**}/\mathrm{Ker}(P) \cong X$. But $X^{**}/\mathrm{Ker}(P) \cong ((\mathrm{Ker}(P))_\perp)^*$, so $X \cong Y^*$. □

Corollary 3.3.9. *Let X be a right M-ideal in its bidual and Y be a w^*-closed subspace of X^*. Then*

(i) *Y is the dual of a space which is a right M-ideal in its bidual.*

(ii) *Y is a left L-summand in its bidual.*

Proof. If Y is w^*-closed, then $Y = (X/Y_\perp)^*$. Now since X is right M-embedded, by Theorem 3.1.7, $X/Y_\perp$ is right M-embedded. This proves (i). It is easy to see that (ii) follows by Corollary 3.1.5. □

Remark. In connection with the last results, in general, if $P = (i_X)^*$ is the natural projection from X^{***} onto X^*, then for every subspace Y of X^*, the following are equivalent:

(i) $P(Y^{\perp\perp}) = Y$

(ii) $Y^{\perp\perp} = Y \oplus (Y^{\perp\perp} \cap X^\perp)$

(iii) Y is w^*-closed in X^*.

It is fairly easy to prove these implications. Notice that (i) ⇒ (iii) follows by a variant of the Krein-Smulian Theorem.

Analogues of many classical results about the RNP are also true for the right L-embedded spaces. For instance, suppose that X is right L-embedded and P is a left L-projection from X^{**} onto X. Then, if the ball of $\mathrm{Ker}(P)$ is w^*-dense in the ball of X^{**}, X fails to have the RNP. This is because the unit ball of X does not have any strongly exposed points (see e.g. [36, Remark IV.2.10 (a)]).

Proposition 3.3.10. *Let X be a left L-embedded operator space and $Y \subset X$ be a left L-subspace. Let Z be an operator space such that Z^* is an injective Banach space* (*resp. injective operator space*)*. Then for every contractive* (*resp. completely contractive*) *operator $T : Z \longrightarrow X/Y$ there exists a contractive* (*resp. completely contractive*) *map $S : Z \longrightarrow X$ such that $qS = T$, where $q : X \longrightarrow X/Y$ is the canonical quotient map.*

The above proposition can be proved by routine modifications to the argument in [36, Proposition IV.2.12]. For the following corollaries see arguments in [36, Corollary IV.2.13] and [36, Corollary IV.2.14], respectively.

Corollary 3.3.11. *If X is a right L-embedded space with Y a left L-subspace of X, and if X/Y contains a subspace W isometric* (*resp. completely isometric*) *to $L^1(\mu)$, then there is a subspace Z of X such that $q(Z) = W$ and $q|_Z$ is an isometric* (*resp. completely isometric*) *isomorphism. If also $X/Y \cong L^1(\mu)$, then there is a contractive* (*resp. completely contractive*) *projection P on X with $Y = \mathrm{Ker}(P)$.*

Corollary 3.3.12. *Let X be a right L-embedded space with Y a left L-subspace of X. Then if X has the RNP then X/Y has the RNP.*

Chapter 4

Operator Algebras and One-Sided M-Ideal Structure

In the first part of this chapter, we study the Haagerup tensor product of operator algebras and their one-sided M-ideal structure. In Section 4.2, we consider the 1-matricial algebras defined in [3], and construct interesting examples of one-sided M-embedded operator algebras. We end the chapter with a discussion of some results related to the "Wedderburn-Artin type" structure theorems for operator algebras. The "Wedderburn-Artin type" theorems can be found in [3, 37, 41]. Most of the work in this chapter is joint work with M. Almus and D. P. Blecher, and appears in [3]. The parts which do not appear in [3] are joint work with D. P. Blecher.

4.1 Tensor Products of Operator Algebras

In this section we extend several known results about the Haagerup tensor products of C^*-algebras (mainly from [7, 18]), to general operator algebras, and give some applications. For example, we investigate the one-sided M-ideal structure of the Haagerup tensor products of nonselfadjoint operator algebras.

We will write $M \otimes^{\sigma h} N$ for the *σ-Haagerup tensor product* (see e.g. [28, 25, 13, 14]). We will repeatedly use the fact that for operator spaces X and Y, we have $(X \otimes_h Y)^{**} \cong X^{**} \otimes^{\sigma h} Y^{**}$ (see e.g. 1.6.8 in [14]). We recall from [13, Section 3] that the Haagerup tensor product and σ-Haagerup tensor product of unital operator algebras is a unital operator space (in the sense of [13]), and also is a unital Banach algebra. We write $\mathrm{Her}(D)$ for the hermitian elements in a unital space D (recall that h is hermitian iff $\varphi(h) \in \mathbb{R}$ for all $\varphi \in \mathrm{Ball}(D^*)$ with $\varphi(1) = 1$).

We first prove a 'two-sided' version of Proposition 5.42 from [18], for von Neumann algebras. We start by proving a few very useful lemmas.

Lemma 4.1.1. *If A and B are unital operator spaces then* $\mathrm{Her}(A \otimes_h B) = A_{\mathrm{sa}} \otimes 1 + 1 \otimes B_{\mathrm{sa}}$ *and* $\Delta(A \otimes_h B) = \Delta(A) \otimes 1 + 1 \otimes \Delta(B)$*. Similarly, if M and N are unital dual operator algebras, then* $\mathrm{Her}(M \otimes^{\sigma h} N) = M_{\mathrm{sa}} \otimes 1 + 1 \otimes N_{\mathrm{sa}}$ *and* $\Delta(M \otimes^{\sigma h} N) = \Delta(M) \otimes 1 + 1 \otimes \Delta(N)$*.*

Proof. If A and B are unital operator spaces then $A \otimes_h B$ is a unital operator space (see [13]), and $\mathrm{Her}(A \otimes_h B) \subset \mathrm{Her}(C^*(A) \otimes_h C^*(B))$. By a result in [7], it follows that if $u \in \mathrm{Her}(A \otimes_h B)$ then there exist $h \in C^*(A)_{\mathrm{sa}}, k \in C^*(B)_{\mathrm{sa}}$ such that $u = h \otimes 1 + 1 \otimes k$. It is easy to see that this forces $h \in A, k \in B$. For example if φ is a functional in $A^\perp$ then $0 = (\varphi \otimes I_B)(u) = \varphi(h)1$, so that $h \in (A^\perp)_\perp = A$. Conversely, it is obvious that $A_{\mathrm{sa}} \otimes 1 + 1 \otimes B_{\mathrm{sa}} \subset \mathrm{Her}(A \otimes_h B)$. Indeed the canonical maps from A and B into $A \otimes_h B$

must take hermitians to hermitians. This gives the first result, and taking spans gives the second.

Now let M and N be unital dual operator algebras. Again it is obvious that $M_{\mathrm{sa}} \otimes 1 + 1 \otimes N_{\mathrm{sa}} \subset \mathrm{Her}(M \otimes^{\sigma h} N)$. For the other direction, we may assume that $M = N$ by the trick of letting $R = M \oplus N$. It is easy to argue that $M \otimes^{\sigma h} N \subset R \otimes^{\sigma h} R$, since M and N are appropriately complemented in R. If $W^*_{\mathrm{max}}(M)$ is the 'maximal von Neumann algebra' generated by M, then by Theorem 3.1 (1) of [13] we have $M \otimes^{\sigma h} M \subset W^*_{\mathrm{max}}(M) \otimes^{\sigma h} W^*_{\mathrm{max}}(M)$. So (again using the trick in the first paragraph of our proof) we may assume that M is a von Neumann algebra. By a result of Effros and Kishimoto [25, Theorem 2.5], $\mathrm{Her}(M \otimes^{\sigma h} M)$ equals

$$\mathrm{Her}(CB_{M'}(B(H))) \subset \mathrm{Her}(CB(B(H))) = \{h \otimes 1 + 1 \otimes k : \; h, k \in B(H)_{\mathrm{sa}}\},$$

the latter by a result of Sinclair and Sakai (see e.g. [15, Lemma 4.3]). By a small modification of the argument in the first paragraph of our proof it follows that $h, k \in M$. The final result again follows by taking the span. □

Lemma 4.1.2. *Let V and W be dual operator spaces and $x \in V \otimes W \subset V \otimes^{\sigma \mathrm{h}} W$. If $\phi \otimes Id_W(x) = 0$ for all w^*-continuous functional ϕ on V, then $x = 0$. Likewise, if $(Id_V \otimes \psi)(x) = 0$ for all w^*-continuous functionals $\psi \in W^*$, then $\psi = 0$.*

Proof. We only prove the first assertion. We have

$$V \otimes W \subset V \otimes^{\sigma \mathrm{h}} W = (V \otimes^{eh} W)^*,$$

where $\otimes^{eh}$ denote the extended sigma Haagerup tensor product. Let $p_x \in (V \otimes^{eh} W)^*$ be the image of x under the above inclusion.

We have

$$p_x(\phi \otimes \psi) = (\phi \otimes \psi)(x) = \psi(\phi \otimes Id_W)(x) = 0$$

for all w^*-continuous functionals $\phi \in V^*$ and $\psi \in W^*$. It follows that $p_x = 0$, which implies that $x = 0$. ☐

Theorem 4.1.3. *Let M and N be von Neumann algebras, with neither M nor N equal to $\mathbb{C}$. Then $Z(M \otimes^{\sigma \mathrm{h}} N)$ is trivial.*

Proof. Let $T \in \mathrm{Her}(Z(M \otimes^{\sigma\mathrm{h}} N))$, with $\|T\|_{\mathcal{M}_\ell(M \otimes^{\sigma\mathrm{h}} N)} \leq 1$. We will use the fact that $M \otimes^{\sigma\mathrm{h}} N$ is a unital Banach algebra with product

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'.$$

From [18, Lemma 2.4] we know that $T(1 \otimes 1) \in \mathrm{Her}(M \otimes_{\mathrm{h}} N)$. Thus by Lemma 4.1.1 we have that

$$T(1 \otimes 1) = h \otimes 1 + 1 \otimes k \tag{4.1.1}$$

for some $h \in M_{sa}$, $k \in N_{sa}$. Since left and right multipliers of an operator space automatically commute, if $\rho : N \to \mathcal{A}_{\mathrm{r}}(M \otimes_{\mathrm{h}} N)$ be the canonical injective $*$-homomorphism, (see the discussion above Lemma 5.41 in [18]), then $\rho(N)$ commutes with T. Thus

$$T(a \otimes b) = T(\rho(b)(a \otimes 1)) = \rho(b)(T(a \otimes 1)) = T(a \otimes 1)(1 \otimes b) \tag{4.1.2}$$

for $a \in M$, $b \in N$. We next will prove the identity

$$(\mathrm{Id}_M \otimes \psi)(T(a \otimes w)) = (\mathrm{Id}_M \otimes \psi)(T(1 \otimes w))a \tag{4.1.3}$$

for all $\psi \in N^*$ and $w \in N$. First suppose that w is a unitary in N, and that $\psi \in N^*$ is a normal functional satisfying $\psi(w) = 1 = \|\psi\|$. Consider the operator $u(a) = (\mathrm{Id}_M \otimes \psi)(T(a \otimes w))$ on M. We have for any $a' \in M$ that

$$\left\| \begin{bmatrix} u(a) \\ a' \end{bmatrix} \right\| = \left\| \begin{bmatrix} (\mathrm{Id}_{\mathcal{A}} \otimes \psi)(T(a \otimes w)) \\ (\mathrm{Id}_{\mathcal{A}} \otimes \psi)(a' \otimes w) \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} T(a \otimes w) \\ a' \otimes w \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} a \otimes w \\ a' \otimes w \end{bmatrix} \right\|,$$

the last inequality by Theorem 5.1 in [11]. First inequality is due to the fact that ψ being a w^*-continuous completely contractive map on N induces a w^*-continuous completely contractive map $\mathrm{Id}_M \otimes N : M \otimes^{\sigma\mathrm{h}} N \longrightarrow M \otimes^{\sigma\mathrm{h}} \mathbb{C}$ (see [28] for more details). Since we clearly have

$$\left\| \begin{bmatrix} a \otimes w \\ a' \otimes w \end{bmatrix} \right\| = \left\| \begin{bmatrix} a \\ a' \end{bmatrix} \right\|,$$

we see using Lemma 4.1 in [15] that there exists an $a_{w,\psi} \in M$ such that

$$(\mathrm{Id}_M \otimes \psi)(T(a \otimes w)) = a_{w,\psi} a$$

for all $a \in M$. Setting $a = 1$ gives $a_{w,\psi} = (\mathrm{Id}_M \otimes \psi)(T(1 \otimes w))$, and this establishes (4.1.3) in this case.

If $g \in N^*$ is a normal functional, that is $g \in N_*$, then $g_w = g(\cdot w) \in N_*$. Since N_* is the closure of the span of the normal states on N, there exist a sequence $\{g_w^t\}$ such that each $g_w^t = \sum_{k=1}^4 \alpha_k^t f^{k,t}$ for α_k^t scalars and $f^{k,t}$ normal states on N. Then $g = \lim \sum_{k=1}^4 \alpha_k^t f_{w^*}^{k,t}$. Setting $\psi = f_{w^*}^{k,t}$ in (4.1.3), and using the continuity of the map in Equation (4.1.3) and the fact that Equation (4.1.3) is linear in ψ we now have (4.1.3) with w unitary. By the well-known fact that the unitary elements span a C^*-algebra, and the linearity of Equation (4.1.3) in w, we obtain (4.1.3) for any $w \in N$. Thus we have proved (4.1.3) in general.

Combining (4.1.3) and (4.1.2) we obtain

$$(\mathrm{Id}_M \otimes \psi)(T(a \otimes b)) = (\mathrm{Id}_A \otimes \psi)(T(1 \otimes 1)(1 \otimes b))a.$$

Writing $T(1 \otimes 1)$ as in (4.1.1), we have that

$$(\mathrm{Id}_M \otimes \psi)(T(a \otimes b)) = \psi(b)ha + \psi(kb)a = (\mathrm{Id}_M \otimes \psi)((h \otimes 1 + 1 \otimes k)(a \otimes b)).$$

For any $x \in M \otimes^{\sigma\mathrm{h}} N$, denote by L_x the operator of left multiplication by x on $M \otimes^{\sigma\mathrm{h}} N$ and by R_y denote the operator of right multiplication by y on $M \otimes^{\sigma\mathrm{h}} N$. Now by Theorem

I.3.10 in [36], T commutes with $L_{a\otimes 1}$ and $R_{1\otimes b}$. So

$$T(a\otimes b) = T(L_{a\otimes 1}R_{1\otimes b}(1\otimes 1)) = L_{a\otimes 1}R_{1\otimes b}T(1\otimes 1) = (a\otimes 1)(h\otimes 1 + 1\otimes k)(1\otimes b).$$

This shows that $T(a\otimes b)$ is a finite rank tensor, hence $T(a\otimes b)\in M\otimes_h N$. Since ψ is arbitrary, we deduce by Lemma 4.1.2 that

$$T(a\otimes b) = (h\otimes 1 + 1\otimes k)(a\otimes b)$$

for all $a\in M$, $b\in N$.

By using the argument in the end of the proof of [18, Theorem 5.42], we can show that k is a scalar multiple of 1. So $k=\alpha 1$ for some $\alpha\in\mathbb{C}$, which implies that $L_{1\otimes k}\in Z(M\otimes^{\sigma\mathrm{h}} N)$. We claim that h is also a scalar multiple of 1. Suppose not. Since $T\in\mathcal{A}_r(M\otimes^{\sigma\mathrm{h}} N)_{sa}$ and $L_{1\otimes k}\in Z(M\otimes^{\sigma\mathrm{h}} N)$, we deduce that $L_{h\otimes 1}\in Z(M\otimes^{\sigma\mathrm{h}} N)_{sa}$. Let $d\in N_{sa}$. Then $R_{1\otimes d}\in\mathcal{A}_r(M\otimes^{\sigma\mathrm{h}} N)$. It follows that $S = L_{h\otimes 1}R_{1\otimes d}\in\mathcal{A}_r(M\otimes^{\sigma\mathrm{h}} N)_{sa}$. By the right hand variant of [18, Lemma 2.4],

$$h\otimes d = S(1\otimes 1)\in Her(M\otimes^{\sigma\mathrm{h}} N)$$

and so by Lemma 4.1.1,

$$h\otimes d = h_d\otimes 1 + 1\otimes k_d$$

for some $h_d\in M_{sa}$, $k_d\in N_{sa}$. Now let $\phi\in N^*$ be a state. Then

$$\phi(d)h = (\phi\otimes\mathrm{Id}_N)(h\otimes d) = (\phi\otimes\mathrm{Id}_N)(h_d\otimes 1 + 1\otimes k_d) = \phi(h_d)1 + k_d,$$

so that

$$h_d = \phi(d)h - \phi(k_d)1.$$

But then

$$h\otimes d = (\phi(d)h - \phi(k_d)1)\otimes 1 + 1\otimes k_d,$$

which implies that

$$h \otimes (d - \phi(d)1) = 1 \otimes (k_d - \phi(k_c)1).$$

Since h and 1 are linearly independent, $d = \phi(d)1$. Since the choice of d was arbitrary, $N = \mathbb{C}$, a contradiction. So $T = cI$ for some scalar c, on $M \otimes N$. Since T is w^*-continuous and $M \otimes N$ is w^*-dense in $M \otimes^{\sigma \mathrm{h}} N$, by density argument, $T = cI$ on $M \otimes^{\sigma \mathrm{h}} N$. $\square$

Unfortunately, the argument in the above theorem cannot be easily modified to work for a more general setting of dual operator algebras. Nevertheless, the result does extend to this general setting, which is the next result.

Theorem 4.1.4. *Let M and N be unital dual operator algebras. If $\Delta(M)$ is not one-dimensional then $\Delta(M) \cong \mathcal{A}_\ell(M \otimes^{\sigma h} N)$. If $\Delta(N)$ is not one-dimensional then $\Delta(N) \cong \mathcal{A}_r(M \otimes^{\sigma h} N)$. If $\Delta(M)$ and $\Delta(N)$ are one-dimensional then*

$$\mathcal{A}_\ell(M \otimes^{\sigma h} N) = \mathcal{A}_r(M \otimes^{\sigma h} N) = \mathbb{C}\, I.$$

Proof. We just prove the first and the last assertions. Let M and N be unital dual operator algebras, and let $X = M \otimes^{\sigma h} N$. The map $\theta : \mathcal{A}_\ell(X) \to X$ defined by $\theta(T) = T(1)$ is a unital complete isometry (see the end of the notes section for 4.5 in [14]). Hence, by [14, Corollary 1.3.8] and Lemma 4.1.1, it maps into $\Delta(X) = \Delta(M) \otimes 1 + 1 \otimes \Delta(N)$. The last assertion is now clear. For the first, if we can show that $\mathrm{Ran}(\theta) \subset \Delta(M) \otimes 1$, then we will be done. There is a copy of $\Delta(M)$ in $\mathcal{A}_\ell(X)$ via the embedding $a \mapsto L_{a \otimes 1}$, and this is a C^*-subalgebra. Note that θ restricts to a $*$-homomorphism from this C^*-subalgebra into the free product $M * N$ discussed in [13]. Let $T \in \mathcal{A}_\ell(X)_{\mathrm{sa}}$, then $\theta(T) \in X_{\mathrm{sa}}$. By Lemma 4.1.1, $T(1 \otimes 1) = h \otimes 1 + 1 \otimes k$, with $h \in \Delta(M)_{\mathrm{sa}}, k \in \Delta(N)_{\mathrm{sa}}$. It suffices to show that $\theta(T - L_{h \otimes 1}) = 1 \otimes k \in \Delta(M) \otimes 1$. So let $S = T - L_{h \otimes 1}$. By [14, Proposition 1.3.11] we have for $a \in \Delta(M)_{\mathrm{sa}}$ that

$$S(a \otimes 1) = \theta(SL_{a\otimes 1}) = \theta(S) * (a \otimes 1) = (1 \otimes k) * (a \otimes 1).$$

The involution in $M * N$, applied to the last product, yields $a * k = a \otimes k \in M \otimes^{\sigma h} N$. Hence

$$S(a \otimes 1) \in \Delta(M \otimes^{\sigma h} N) = \Delta(M) \otimes 1 + 1 \otimes \Delta(N) \subset \Delta(M) \otimes \Delta(N).$$

Since left and right multipliers of an operator space automatically commute, $\rho(N)$ commutes with S, where $\rho : N \to \mathcal{A}_r(M \otimes^{\sigma h} N)$ is the canonical injective $*$-homomorphism. Thus

$$S(a \otimes b) = S(\rho(b)(a \otimes 1)) = \rho(b)(S(a \otimes 1)) = S(a \otimes 1)(1 \otimes b) \in \Delta(M) \otimes \Delta(N)$$

By linearity this is true for any $a \in \Delta(M)$ too. It follows that $\Delta(M) \otimes_h \Delta(N)$ is a subspace of $M \otimes^{\sigma h} N$ which is invariant under S. Since S is selfadjoint, it follows from [18, Proposition 5.2] that the restriction of S to $\Delta(M) \otimes_h \Delta(N)$ is adjointable, and selfadjoint. Hence by [18, Theorem 5.42] we have that there exists an $m \in \Delta(M)$ with $S(1 \otimes 1) = m \otimes 1 = 1 \otimes k$. Thus $1 \otimes k \in \Delta(M) \otimes 1$ as desired. ☐

Corollary 4.1.5. *Let A and B be approximately unital operator algebras. If $\Delta(A^{**})$ is not one dimensional then $\Delta(M(A)) \cong \mathcal{A}_\ell(A \otimes_h B)$. If $\Delta(B^{**})$ is not one dimensional then $\Delta(M(B)) \cong \mathcal{A}_r(A \otimes_h B)$. If $\Delta(A^{**})$ and $\Delta(B^{**})$ are one dimensional then $\mathcal{A}_\ell(A \otimes_h B) = \mathcal{A}_r(A \otimes_h B) = \mathbb{C}\, I$.*

Proof. We just prove the first and last relations. Let $\rho : \Delta(LM(A)) \to \mathcal{A}_\ell(A \otimes_h B)$ be the injective $*$-homomorphism given by $S \mapsto S \otimes I_B$. If $T \in \mathcal{A}_\ell(A \otimes_h B)_{\mathrm{sa}}$, then by Proposition 5.16 from [18], we have $T^{**} \in \mathcal{A}_\ell(A^{**} \otimes^{\sigma h} B^{**})_{\mathrm{sa}}$. By the last theorem, $T^{**}(a \otimes b) = T(a \otimes b) = L_{h\otimes 1}(a \otimes b)$, for some $h \in A^{**}_{\mathrm{sa}}$ and for all $a \in A$, $b \in B$. Since

$T(a \otimes b)$ is in $A \otimes_h B$, so is $L_{h\otimes 1}(a \otimes b)$ for all $a \in A$, $b \in B$. Also $L_{h\otimes 1}(a \otimes 1) = ha \otimes 1 \in A \otimes_h B$ for all $a \in A$. So $ha \in A$ for all $a \in A$. Thus $L_h \in \Delta(LM(A)_{\mathrm{sa}})$. This shows that ρ is surjective, since selfadjoint elements span $\mathcal{A}_\ell(A \otimes_h B)$. Thus $\Delta(LM(A)) \cong \mathcal{A}_\ell(A \otimes_h B)$. By the proof of [12, Proposition 5.1], we have $\Delta(LM(A)) = \Delta(M(A))$. This proves the first relation. If $\Delta(A^{**})$ and $\Delta(B^{**})$ are one dimensional, then so is $\Delta(M(A))$, and so is $\mathcal{A}_\ell(A^{**} \otimes^{\sigma h} B^{**})$, by the theorem. Hence the T above is in $\mathbb{C}\, I$, and this proves the last assertion. ☐

Remark. For A, B, M, N as in the last results, it is probably true that $\Delta(M(A)) \cong \mathcal{A}_\ell(A \otimes_h B)$, and similarly that $\Delta(M) \cong \mathcal{A}_\ell(M \otimes^{\sigma h} N)$, if A and M are not one-dimensional, with no other restrictions. We are able to prove this if $B = N$ is a finite dimensional C^*-algebra.

The following is a complement to [18, Theorem 5.38]:

Theorem 4.1.6. *Let A and B be approximately unital operator algebras, with $\Delta(A^{**})$ not one-dimensional. Then the right M-ideals (resp. right summands) in $A \otimes_h B$ are precisely the subspaces of the form $J \otimes_h B$, where J is a closed right ideal in A having a left cai (resp. having form eA for a projection $e \in M(A)$).*

Proof. The summand case follows immediately from Corollary 4.1.5. The one direction of the M-ideal case is [18, Theorem 5.38]. For the other, suppose that I is a right M-ideal in $A \otimes_h B$. View $(A \otimes_h B)^{**} = A^{**} \otimes^{\sigma h} B^{**}$. Then $I^{\perp\perp}$ is a right M-summand in $A^{**} \otimes^{\sigma h} B^{**}$. By Theorem 4.1.4 we have $I^{\perp\perp} = eA^{**} \otimes^{\sigma h} B^{**}$ for a projection $e \in A^{**}$. Let $J = eA^{**} \cap A$, a closed right ideal in A. We claim that $I = J \otimes_h B$. Since $I = I^{\perp\perp} \cap (A \otimes_h B)$, we need to show that $(eA^{**} \otimes^{\sigma h} B^{**}) \cap (A \otimes_h B) = (eA^{**} \cap A) \otimes_h B$. By injectivity of $\otimes_h$, it is clear that $(eA^{**} \cap A) \otimes_h B \subset (eA^{**} \otimes^{\sigma h} B^{**}) \cap (A \otimes_h B)$. For the other containment we use a slice map argument. By [57, Corollary 4.8], we need to show that for all $\psi \in B^*$, $(1 \otimes \psi)(u) \in eA^{**} \cap A = J$. Let $\psi \in B^*$, then $\langle \tilde{u}, 1 \otimes \psi \rangle = (1 \otimes \psi)(u) \in A$, where $\tilde{u}$ is u regarded as an element in $eA^{**} \otimes^{\sigma h} B^{**}$. Since $u \in eA^{**} \otimes^{\sigma h} B^{**}$, we have $\langle \tilde{u}, 1 \otimes \psi \rangle \in eA^{**}$. So $(1 \otimes \psi)(u) \in eA^{**} \cap A = J$, and so $u \in J \otimes_h B$ as desired.

Next we show that J has a left cai. It is clear that $J^{\perp\perp} = \overline{J}^{w^*} \subset eA^{**}$. Suppose that there is $x \in eA^{**}$ such that $x \notin J^{\perp\perp}$. Then there exists $\phi \in J^{\perp}$ such that $x(\phi) \neq 0$. Since $I = J \otimes_h B$ and $\phi \in J^{\perp}$, we have $\phi \otimes \psi \in I^{\perp}$ for all states ψ on B. So $I^{\perp\perp}$ annihilates $\phi \otimes \psi$, and in particular $0 = (x \otimes 1)(\phi \otimes \psi) = x(\phi)$, a contradiction. Hence $J^{\perp\perp} = eA^{**}$, and it follows from basic principles about approximate identities that J has a left cai. $\square$

Theorem 4.1.7. *Let M and N be unital (resp. unital dual) operator algebras, with neither M nor N equal to $\mathbb{C}$. Then the operator space centralizer algebra $Z(M \otimes_h N)$ (resp. $Z(M \otimes^{\sigma h} N)$) (see* [18, Chapter 7]*) is one-dimensional.*

Proof. First we consider the dual case. If $\Delta(M)$ and $\Delta(N)$ are both one-dimensional then $Z(M \otimes^{\sigma h} N) \subset \mathcal{A}_{\ell}(M \otimes^{\sigma h} N) = \mathbb{C}\, I$, and we are done. If $\Delta(M)$ and $\Delta(N)$ are both not one-dimensional, let P be a projection in $Z(M \otimes^{\sigma h} N)$. By the theorem, $Px = ex = xf$, for all $x \in M \otimes^{\sigma h} N$, for some projections $e \in M$ and $f \in N$. Then

$$e^{\perp} \otimes f = e^{\perp} \otimes ff = P(e^{\perp} \otimes f) = ee^{\perp} \otimes f = 0,$$

which implies that either $e^{\perp} = 0$ or $f = 0$. Hence $P = 0$ or $P = I$. So $Z(M \otimes^{\sigma h} N)$ is a von Neumann algebra with only trivial projections, hence it is trivial.

Suppose that $\Delta(N)$ is one-dimensional, but $\Delta(M)$ is not. Again it suffices to show that any projection $P \in \mathcal{A}_{\ell}(M \otimes^{\sigma h} N)$ is trivial. By Theorem 4.1.4, P is of the form $Px = ex$ for a projection $e \in M$. Assume that e is not 0 or 1. If $D = \text{Span}\{e, 1 - e\}$, and X is the copy of $D \otimes N$ in $M \otimes^{\sigma h} N$, then P leaves X invariant. Note that $X = D \otimes_h N$, since $\otimes_h$

is known to be completely isometrically contained in $\otimes^{\sigma h}$ (see [28]). Hence by Section 5.2 in [18] we have that the restriction of P to X is in $\mathcal{A}_\ell(X) \cap \mathcal{A}_r(X) = Z(X)$. Thus we may assume without loss of generality that $M = D = \ell^\infty_2$, and P is left multiplication by e_1, where $\{e_1, e_2\}$ is the canonical basis of ℓ^∞_2. Thus $\|e_1 \otimes x + e_2 \otimes y\|_h = \max\{\|x\|, \|y\|\}$, for all $x, y \in N$. Set $x = 1_N$, and let $y \in N$ be of norm 1. Then $\|e_1 \otimes 1 + e_2 \otimes y\|_h = 1$. If we can show that $y \in \mathbb{C}\,1_N$ then we will be done: we will have contradicted the fact that N is not one-dimensional, hence e, and therefore P, is trivial. By the injectivity of the Haagerup tensor product, we may replace N with $\text{Span}\{1, y\}$. By basic facts about the Haagerup tensor product, there exist $z_1, z_2 \in \ell^\infty_2$ and $v, w \in N$ with $e_1 \otimes 1 + e_2 \otimes y = z_1 \otimes v + z_2 \otimes w$, and with $\|[z_1\ z_2]\| = \|v^*v + w^*w\| = 1$. Multiplying by $e_1 \otimes 1$ we see that $z_1(1)v + z_2(1)w = 1$, so that

$$1 \leq (|z_1(1)|^2 + |z_2(1)|^2)\|v^*v + w^*w\| = |z_1(1)|^2 + |z_2(1)|^2 \leq \|[z_1\ z_2]\|^2 = 1.$$

From basic operator theory, if a pair of contractions have product I, then the one is the adjoint of the other. Thus v, w, and hence y, are in $\mathbb{C}\,1$.

A similar argument works if $\Delta(M)$ is one-dimensional, but $\Delta(N)$ is not.

In the 'non-dual case', use [18, Theorem 7.4 (ii)] to see that $Z(M \otimes_h N) \subset Z(M^{**} \otimes^{\sigma h} N^{**}) = \mathbb{C}I$. □

Corollary 4.1.8. *Let A and B be approximately unital operator algebras, with neither being one-dimensional. Then $A \otimes_h B$ contains no non-trivial complete M-ideals.*

Proof. Suppose that J is a complete M-ideal in $A \otimes_h B$. The complete M-projection onto $J^{\perp\perp}$ is in $Z((A \otimes_h B)^{**}) = Z(A^{**} \otimes^{\sigma h} B^{**})$, and hence is trivial by Theorem 4.1.7. □

Remark. The ideal structure of the Haagerup tensor product of C^*-algebras has been studied in [2] and elsewhere.

Proposition 4.1.9. *Let A and B be approximately unital operator algebras with A non-reflexive, B finite dimensional and $B \neq \mathbb{C}$. If A is a right ideal in A^{**}, then $A \otimes_h B$ is a right M-ideal in its second dual, and it is not a left M-ideal in its second dual.*

Proof. Since A is a right M-ideal in A^{**}, $A \otimes_h B$ is a right M-ideal in $A^{**} \otimes_h B$ by [18, Proposition 5.38]. Since B is finite dimensional, $(A \otimes_h B)^{**} = A^{**} \otimes_h B$ (see e.g. [14, 1.5.9]). Hence $A \otimes_h B$ is a right M-ideal in its bidual. Suppose that it is also a left M-ideal. Then it is a complete M-ideal in its bidual, and therefore corresponds to a projection in $Z(A^{(4)} \otimes_h B)$. However, the latter is trivial by Theorem 4.1.7. This forces $A \otimes_h B$, and hence A, to be reflexive, which is a contradiction. So $A \otimes_h B$ is not a left M-ideal in its bidual. $\square$

Remark. Note that the above proposition is a generalized version of Proposition 3.1.11.

4.2 1-Matricial Algebras

We now define a class of operator algebras which will provide natural examples of one-sided M-embedded operator algebras.

Definition 4.2.1. We say that an operator algebra A is *matricial* if it has a set of matrix units $\{T_{ij}\}$, whose span is dense in A. Thus $T_{ij}T_{kl} = \delta_{jk}T_{il}$, where δ_{jk} is the Kronecker delta. Define $q_k = T_{kk}$. We say that a matricial operator algebra A is 1-*matricial* if $\|q_k\| = 1$ for all k, that is, iff the q_k are orthogonal projections. Our main focus is on 1-matricial algebras.

We are only interested in separable (or finite dimensional) algebras, and in this case we prefer the following equivalent description of 1-matricial algebras. Consider a (finite or infinite) sequence $T_1, T_2, \cdots$ of invertible operators on a Hilbert space K, with $T_1 = I$. Set $H = \ell^2 \otimes^2 K = K^{(\infty)} = K \oplus^2 K \oplus^2 \cdots$, (in the finite sequence case, $H = K^{(n)}$). Define $T_{ij} = E_{ij} \otimes T_i^{-1} T_j \in B(H)$ for $i, j \in \mathbb{N}$, and let A be the closure of the span of the T_{ij}. Then $T_{ij} T_{kl} = \delta_{jk} T_{il}$, so that these are matrix units for A. Then A is a 1-matricial algebra, and all separable or finite dimensional 1-matricial algebras arise in this way. Let $q_k = T_{kk}$, then $\sum_k q_k = 1$ strictly. A *σ-matricial* algebra is a c_0-direct sum of 1-matricial algebras. Since we only care about the separable case these will all be countable direct sums. It would certainly be better to call these σ-1-matricial algebras, or something similar, but since we shall not consider any other kinds, we drop the '1' for brevity.

For the proof of the following results see [3, Proposition 4.2] and [3, Lemma 4.4], respectively.

Proposition 4.2.2. *If A is an Arens regular Banach algebra with idempotents $(q_k)_{k=1}^{\infty}$ with $\sum_k Aq_k$ or $\sum_k q_k A$ dense in A (for example, if $\sum_k q_k = 1$ left or right strictly), then A is a right ideal in A^{**} iff $q_k A$ is reflexive for all k. If A is topologically simple, then A is a right ideal in A^{**} iff eA is reflexive for some idempotent $e \in A$.*

Lemma 4.2.3. *Any* 1*-matricial algebra A is approximately unital, topologically simple, hence semisimple and semiprime, and is a compact modular annihilator algebra. It is an HSA in its bidual, so has the unique Hahn-Banach extension property in* [12, Theorem 2.10]. *It also has dense socle, with the q_k algebraically minimal projections with $A = \oplus_k^c q_k A = \oplus_k^r Aq_k$. The canonical representation of A on Aq_1 is faithful and irreducible, so that A is a primitive Banach algebra.*

Corollary 4.2.4. *A* 1*-matricial algebra A is a right (resp. left, two-sided) ideal in its* *bidual iff q_1A (resp. Aq_1, q_1A and Aq_1) is reflexive.*

Remark. It is known that semisimple (and many semiprime) annihilator algebras are ideals in the bidual [46, Corollary 8.7.14]. In particular, a 1-matricial annihilator algebra is an ideal in its bidual.

Definition 4.2.5. Let X be a Banach space. Then (x_k) in X is a *Schauder basis* for X if every $x \in X$ can be uniquely written as $x = \sum \alpha_k x_k$. Equivalently, if $\overline{\mathrm{Span}\{x_k\}} = X$ and $x_k \neq 0$ for all k and

$$\|\sum_{k=1}^{m_1} \alpha_k x_k\| \leq C \|\sum_{k=1}^{m_2} \alpha_k x_k\|,$$

for all $m_1 \leq m_2$ in $\mathbb{N}$. If $C = 1$, then (x_k) is called a *monotone Schauder basis.*

Let $T_1, T_2, \ldots, T_k, \ldots$ be invertible operators on a Hilbert space K such that $T_1 = I$. Suppose that A is the 1-matricial algebra generated by T_k. Then $(T_{1k}) = (E_{1k} \otimes T_k)$ is a monotone Schauder basis for q_1A. Indeed, clearly the closure of the span of the T_{1k} equals q_1A, and if $n < m$ then

$$\| \sum_{k=1}^{n} \alpha_k T_{1k} \|^2 = \| \sum_{k=1}^{n} |\alpha_k|^2 T_k T_k^* \| \leq \| \sum_{k=1}^{m} |\alpha_k|^2 T_k T_k^* \| = \| \sum_{k=1}^{m} \alpha_k T_{1k} \|^2.$$

Let

$$Y = \{\alpha = (\alpha_1, \alpha_2, \ldots) : \sum_{k}^{\infty} |\alpha_k|^2 \, T_k T_k^* \text{ converges in norm in B(K)}\},$$

and $|||\alpha||| = \left\| \sum_k^\infty |\alpha_k|^2 \, T_k T_k^* \right\|^{\frac{1}{2}}$. Then $(Y, |||.|||)$ is a Banach space which is isometric to q_1A via $\rho : q_1A \to \mathbb{C}^\infty : (\sum_{k=1}^\infty \alpha_k T_{1k}) \mapsto (\alpha_1, \alpha_2, \ldots)$.

From [3, Lemma 4.7] we know that if A is an infinite dimensional 1-matricial algebra, then A is completely isomorphic to $\mathbb{K}(\ell^2)$ iff $\|T_k\| \left\|T_k^{-1}\right\|$ is bounded. If the Hilbert space K in the definition of a 1-matricial algebra A is finite dimensional, then we shall say that A

is *subcompact*. The following lemma gives a characterization of the subcompact 1-matricial algebras.

Lemma 4.2.6. *If A is a subcompact 1-matricial algebra, then A is completely isometrically isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$, and q_kA (resp. Aq_k) is linearly completely isomorphic to a row (resp. column) Hilbert space. Conversely, if a 1-matricial algebra A is isometrically (resp. completely isometrically) isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$, then A is isometrically (resp. completely isometrically) isomorphic to a subcompact 1-matricial algebra. In the latter case, A is an ideal in its bidual and q_kA and Aq_k are isomorphic (resp. completely isomorphic) to a Hilbert space.*

Proof. The first statement follows from the definition. If θ was an isometric (resp. completely isometric) homomorphism from A onto a subalgebra of $\mathbb{K}(\ell^2)$, then $e_k = \theta(q_k)$ is a finite rank projection. Hence $e_k\,\mathbb{K}(\ell^2)$ is isomorphically Hilbertian. Thus q_kA is isomorphically Hilbertian, and similarly Aq_k is isomorphically Hilbertian. These are reflexive, and so A is an ideal in its bidual by Corollary 4.2.4. If H_0 is the closure of $\theta(A)(\ell^2)$, then the compression of θ to H_0 is a nondegenerate isometric (resp. completely isometric) homomorphism, with range easily seen to be inside $\mathbb{K}(H_0)$. So we may assume that θ is nondegenerate from the start. Now appeal to [3, Theorem 4.6] and its proof to see that A is isometrically (resp. completely isometrically) isomorphic to a subcompact 1-matricial algebra. The statement about the bidual follows from the above, or note that in this case $q_1A \subset B(K, K^{(\infty)})$, which is reflexive. □

Now we look at few explicit constructions of 1-matricial algebras which are M-ideal in their second dual.

Example 4.2.7. Let $K = \ell^2_2$ and $T_k = \mathrm{diag}(1, \sqrt{\frac{1}{k}})$. Let A be the 1-matricial algebra generated by T_k. Since $\|T_k\|\,\|T_k^*\| = \sqrt{k}$, A is not isometrically isomorphic to $\mathbb{K}(\ell^2)$. Let $x = [\alpha_{ij}T_{ij}] \in A$. Then by canonical shuffling, if $a = [\alpha_{ij}]$ and $S = \operatorname{diag}(1, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, ...)$

$$A = \left\{ \begin{bmatrix} a & 0 \\ 0 & S^{-1}aS \end{bmatrix} \ : \ a,\ SaS^{-1} \in \mathbb{K}(\ell^2) \right\}.$$

It is now clear that A is subcompact. Thus by Lemma 4.2.6, q_1A and Aq_1 are Hilbertian, and hence A is a two-sided ideal in its bidual, by Proposition 4.2.2.

Example 4.2.8. Let $K = \ell^2_2$, and $T_k = \operatorname{diag}\{k, 1/k\}$. In this example also the 1-matricial algebra constructed from T_k, A is two-sided ideal in its bidual, but is not topologically isomorphic to $\mathbb{K}(\ell^2)$ as a Banach algebra. Here q_1A is a row Hilbert space and Aq_1 is a column Hilbert space. Note that A is not an annihilator algebra by [46, Theorem 8.7.12], since $(q_1A)^*$ is not isomorphic to Aq_1 via the canonical pairing.

Example 4.2.9. Let $K = \ell^2$, and $T_k = E_{kk} + \frac{1}{k}I$. Claim: q_1A is not reflexive. Indeed the Schauder basis (T_{1k}) (see Remark 2 after Corollary 4.2.4) fails the first part of the well known two part test for reflexivity [44], because $\sum_{k=1}^{\infty} T_kT_k^*$ converges weak* but not in norm. Or one can see that $q_1A \cong c_0$ by Lemma 4.12 from [3]. Here T_k^{-1} has k in all diagonal entries but one, which has a positive value $< k$. It follows that Aq_1 is a column Hilbert space. By Corollary 4.2.4, A is a left ideal in its bidual, but is not a right ideal in its bidual. This is interesting since any C^*-algebra which is a left ideal in its bidual is also a right ideal in its bidual (see Proposition 3.1.10).

Note that this is not an annihilator algebra by [46, Theorem 8.7.12], since $(q_1A)^*$ is not isomorphic to Aq_1. It is also not bicontinuously isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$ by the last part of Lemma 4.2.6.

Lemma 4.2.10. *A 1-matricial algebra is subcompact if and only if the C^*-envelope of A, $C^*_e(A)$, is an annihilator C^*-algebra.*

Proof. Let $A \subset \mathbb{K}(\ell^2)$. Let B be the C^*-algebra generated by A in $\mathbb{K}(\ell^2)$, then B is an annihilator C^*-algebra. By the universal property of a C^*-envelope, $C^*_e(A) \cong B/I$, for some closed two-sided ideal I in B. Since B is an annihilator C^*-algebra, then so is B/I (since if $B = \oplus^0_{i\in D} K_i$, then $I = \oplus^0_{i\in E} K_i$, for some $E \subset D$ and thus $B/I = \oplus^0_{i \in D\setminus E} K_i$). Conversely, suppose that $C^*_e(A)$ is an annihilator algebra. It is clear, since $A \hookrightarrow C^*_e(A) \subset \mathbb{K}(\ell^2)$, that A is completely isometrically isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$. Hence A is subcompact by Lemma 4.2.6. $\square$

Example 4.2.11. Let $K = \ell^2$ and $T_k = I - \sum_{i=1}^k (1 - \sqrt{\frac{i}{k}})E_{ii}$. Then $\|T_k\| = 1$ and $\left\|T_k^{-1}\right\| = \sqrt{k}$. Thus since the set $\{\|T_k\| \left\|T_k^{-1}\right\|\}$ is unbounded, by [3, Lemma 4.7], A is not isomorphic to $\mathbb{K}(\ell^2)$. Consider $q_1 A$, which is a Banach space with norm given by $\|(\alpha_k T_k)\|^2 = \left\|\sum |\alpha_k|^2 T_k T_k^*\right\|$. Now

$$\sum_{k=1}^{p} |\alpha_k|^2 T_k T_k^* = \mathrm{diag}(d_1, d_2, \ldots, d_s, \ldots)$$

where $d_s = \sum_{i=1}^s |\alpha_i|^2 + \sum_{i=(s+1)}^p |\alpha_i|^2 \left(\frac{s}{i}\right)$ for all $s < k$ and if $s \geq k$ then $d_s = \sum_{i=1}^p |\alpha_i|^2$. It is easy to see that $\|(\alpha_k)\|^2 = \sum |\alpha_k|^2$. Hence $q_1 A$ is Hilbertian, i.e, $q_1 A \cong H$ isometrically, where $H = \ell^2$. By [3, Corollary 4.5], A is also an ideal in its bidual.

We next show that $q_1 A \cong H^c$ completely isometrically. Let $a = [\alpha_k^{ij}] \in M_n(l^2)$ correspond to

$$x = [\alpha_1^{ij} T_1 \;\; \alpha_2^{ij} T_2 \;\; \ldots \;\; \alpha_p^{ij} T_p \;\; 0 \;\; 0 \;\; \ldots]$$

in $M_n(q_1 A)$. Then by shuffling and using the C^*-identity, $\|x\|^2_{M_n(q_1 A)} = \|\sum S_k S_k^*\|$, where

$S_k = [\alpha_k^{ij} T_k]$. Hence

$$\begin{aligned}
\|x\|^2 = \|\sum_{k=1}^{p}\sum_{l=1}^{n} \alpha_k^{il}\overline{\alpha_k^{jl}} T_k (T_k)^*\| &= \|\sum_{k=1}^{p}\sum_{l=1}^{n} \alpha_k^{il}\overline{\alpha_k^{jl}} T_k^2\| \\
&= \|\sum_{r=1}^{k} \frac{r}{k} aa^* E_{rr} + \sum_{r=k+1}^{\infty} aa^* E_{rr}\| = \|aa^*\|\,.
\end{aligned}$$

Since $\sqrt{\|aa^*\|}$ is the column Hilbert norm of x, $q_1 A$ is a column Hilbert space. Proceeding on similar lines we can show that Aq_1 is row Hilbertian. In this case, we get a factor of k in the norm of $[\alpha_1^{ij} T_1]$ but we can always re-scale α_k^{ij}.

Let $x = [\alpha_{ij} T_{ij}]$ be any element in A. By shuffling, x can be viewed as a block diagonal matrix with blocks, $B_1, B_2, ..., B_k, ...$ where

$$B_1 = \begin{bmatrix}
\alpha_{11} 1 & \alpha_{12}\sqrt{\frac{1}{2}} & \alpha_{13}\sqrt{\frac{1}{3}} & \alpha_{14}\sqrt{\frac{1}{4}} & \cdots \\
\alpha_{21}\sqrt{\frac{2}{1}} & \alpha_{22} 1 & \alpha_{23}\sqrt{\frac{2}{3}} & \alpha_{24}\sqrt{\frac{2}{4}} & \cdots \\
\alpha_{31}\sqrt{\frac{3}{1}} & \alpha_{32}\sqrt{\frac{3}{2}} & \alpha_{33} 1 & \alpha_{34}\sqrt{\frac{3}{4}} & \cdots \\
\alpha_{41}\sqrt{\frac{4}{1}} & \alpha_{42}\sqrt{\frac{4}{2}} & \alpha_{43}\sqrt{\frac{4}{3}} & \alpha_{44} 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},$$

$$B_2 = \begin{bmatrix}
\alpha_{11} 1 & \alpha_{12} 1 & \alpha_{13}\sqrt{\frac{2}{3}} & \alpha_{14}\sqrt{\frac{2}{4}} & \cdots \\
\alpha_{21} 1 & \alpha_{22} 1 & \alpha_{23}\sqrt{\frac{2}{3}} & \alpha_{24}\sqrt{\frac{2}{4}} & \cdots \\
\alpha_{31}\sqrt{\frac{3}{2}} & \alpha_{32}\sqrt{\frac{3}{2}} & \alpha_{33} 1 & \alpha_{34}\sqrt{\frac{3}{4}} & \cdots \\
\alpha_{41}\sqrt{\frac{4}{2}} & \alpha_{42}\sqrt{\frac{4}{2}} & \alpha_{43}\sqrt{\frac{4}{3}} & \alpha_{44} 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},$$

$$B_3 = \begin{bmatrix} \alpha_{11}1 & \alpha_{12}1 & \alpha_{13}1 & \alpha_{14}\sqrt{\frac{3}{4}} & \cdots \\ \alpha_{21}1 & \alpha_{22}1 & \alpha_{23}1 & \alpha_{24}\sqrt{\frac{3}{4}} & \cdots \\ \alpha_{31}1 & \alpha_{32}1 & \alpha_{33}1 & \alpha_{34}\sqrt{\frac{3}{4}} & \cdots \\ \alpha_{41}\sqrt{\frac{4}{3}} & \alpha_{42}\sqrt{\frac{4}{3}} & \alpha_{43}\sqrt{\frac{4}{3}} & \alpha_{44}1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \text{etc.}$$

Thus we can write $B_k = S_k^{-1} a S_k$ where $S_k =$diag$(\overbrace{1, 1, \ldots 1}^{k}, \sqrt{\frac{k}{k+1}}\sqrt{\frac{k}{k+2}}, \ldots)$ and $a = [\alpha_{ij}] \in \mathbb{K}(\ell^2)$. Thus $A = \{\text{diag}(S_1 a S_1^{-1}, S_2 a S_2^{-1}, \ldots, S_k a S_k^{-1}, \ldots) : a \in \mathbb{K}(\ell^2)\}$.

Claim : A is not subcompact.

By the above lemma it is enough to show that the $C_e^*(A)$ is not an annihilator C^*-algebra. First note that A is generated by elements of the form $T_{ij} = T_i^{-1} T_j \otimes E_{ij}$, where E_{ij} are the matrix units for $B(\ell^2)$. Since $T_i^{-1} T_j \otimes E_{ij} \in B(K) \otimes \mathbb{K}(\ell^2) \subset B(K) \otimes_{\min} \mathbb{K}(\ell^2)$, $C^*(A) \subset B(K) \otimes_{\min} \mathbb{K}(\ell^2)$. Also,

$$T_k^{-1} T_l = \sum_{i=1}^{k} (\sqrt{\frac{k}{l}} - 1) E_{i,i} + \sum_{j=1}^{l-k} (\sqrt{\frac{k+j}{l}} - 1) E_{k+j,k+j} + I,$$

for $k \leq l$, which is clearly in c_0^1. Similarly, $T_k^{-1} T_l \in c_0^1$ for all $k > l$ and hence $C^*(A) \subset c_0^1 \otimes_{\min} \mathbb{K}(\ell^2)$. Now we show that $c_0^1 \otimes_{\min} \mathbb{K}(\ell^2) \subset C^*(A)$. Consider $T_{12} T_{12}^* = T_2 T_2^* \otimes E_{11}$. Now $T_{12} T_{12}^* - T_{11} = c_1 E_{11} \otimes E_{11}$, for some constant c_1, so $E_{11} \otimes E_{11} \in C^*(A)$. We get $E_{22} \otimes E_{11}$ by subtracting an appropriate constant multiple of $E_{11} \otimes E_{11}$ from $T_{13} T_{13}^* - T_{11}$, so $E_{22} \otimes E_{11} \in C^*(A)$. Continuing like this, we can show that $E_{nn} \otimes E_{11} \in C^*(A)$ for all n. Then $(E_{nn} \otimes E_{11}) T_{1j} = E_{nn} T_j \otimes E_{1j} = c_2 E_{nn} \otimes E_{1j} \in C^*(A)$ for some scalar c_2. So $E_{nn} \otimes E_{1j} \in C^*(A)$ for all j and n. Thus $(E_{nn} \otimes E_{1i})^* (E_{nn} \otimes E_{1j}) = E_{nn} \otimes E_{ij} \in C^*(A)$ for all n. Hence $c_0 \otimes_{\min} \mathbb{K}(\ell^2) \subset C^*(A)$. Since $1 \otimes E_{1j} = T_{1j} + \sum_{k=1}^{j} c_k E_{kk} \otimes E_{1j}$, for some scalars c_k, and $1 \otimes E_{ij} = (1 \otimes E_{1i})^* (1 \otimes E_{1j})$, we have $1 \otimes E_{ij} \in C^*(A)$. Hence

$C^*(A) = c_0^1 \otimes_{\min} \mathbb{K}(\ell^2)$. Now $c_0^1 \otimes_{\min} \mathbb{K}(\ell^2) \cong \mathbb{K}_\infty(c_0^1)$ completely isometrically, and closed ideals in $\mathbb{K}_\infty(c_0^1)$ are of the form $\mathbb{K}_\infty(J)$, for some closed ideal J in c_0^1. Thus the closed ideals in $C^*(A)$ correspond bijectively to the closed ideals in c_0^1. Also, c_0^1 is a commutative C^*-algebra which is $*$-isomorphic to $C(\mathbb{N}^1)$. Here $\mathbb{N}^1$ denotes the one-point compactification of $\mathbb{N}$, which is homeomorphic to the set $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. So closed ideals J of c_0^1 are precisely the ones which vanish on a closed subset $D \subset \mathbb{N}^1$. Thus if D is a closed subset of $\mathbb{N}^1$ then $J_D = \{(x_i) + \lambda \in c_0^1 : x_i = 0 \text{ for all } i \in D \setminus \{0\}\}$ if $0 \in D$, otherwise $J_D = \{(x_i) \in c_0 : x_i = 0 \text{ for all } i \in D\}$. If D is a finite set then c_0^1/J_D is a finite dimensional unital $*$-subalgebra of c_0^1, thus $c_0^1/J_D \cong \ell_\infty^n$ for some n. In the case D is infinite, $c_0^1/J_D \cong c_0^1$, since c_0^1/J_D is an infinite dimensional unital $*$-subalgebra of c_0^1. By definition, $C_e^*(A)$ equals $(C^*(A)/I, q_I|_A)$ for some closed ideal I in $C^*(A)$, where $q_I : C^*(A) \longrightarrow C^*(A)/I$ is the canonical quotient map. Thus $C_e^*(A) = \mathbb{K}(c_0^1)/\mathbb{K}(J) \cong \mathbb{K}(c_0^1/J) \cong (c_o^1/J) \otimes_{\min} \mathbb{K}(\ell^2)$, where J is a closed ideal in $C^*(A)$. By above, either $C_e^*(A) \cong \ell_\infty^n \otimes_{\min} \mathbb{K}(\ell^2)$ or $C_e^*(A) \cong c_0^1 \otimes_{\min} \mathbb{K}(\ell^2)$. In the first case, $C_e^*(A)$ is clearly an annihilator C^*-algebra. If $C_e^*(A)$ is an annihilator C^*-algebra in the latter case, then so are its subalgebras. But $c_0^1 \subset C_e^*(A)$ is not an annihilator C^*-algebra since it is unital. Thus $C_e^*(A)$ is an annihilator C^*-algebra only when c_0^1/J is finite dimensional. So $q_J|_A : A \longrightarrow c_0^1/J \otimes_{\min} \mathbb{K}(\ell^2)$ is a complete isometry, where $J = \{(x_i) + \lambda \in c_0^1 : x_i = 0 \text{ for all } i \in D \setminus \{0\}\}$ whenever $0 \in D$, otherwise $J = \{(x_i) \in c_0 : x_i = 0 \text{ for all } i \in D\}$, for some finite set $D \subset \mathbb{N}^1$. Thus, depending upon what J is, either the map on A which removes all rows and columns of each T_k corresponding to indices in $\mathbb{N} \setminus D$, or the map on A which removes all rows and columns of each T_k corresponding to indices in $\mathbb{N} \setminus D$ and adds a $m+1$ row and column which has just a 1 in the $(m+1, m+1)$ entry, is a complete isometry, where m is the cardinality of D. Let θ be such a map and $R_k = \theta(T_k)$. It is clear that in the first case θ cannot be an isometry since if we choose $k > i$ where $i = \max D$, then

$1 = \|T_k\| > \|R_k\| = \frac{1}{i}$. Let $D = \{1, \dots N\}$, and suppose that θ, in the latter case, is a complete isometry. This implies that A is completely isometrically isomorphic to the 1-matricial algebra generated by $\theta(T_k) = R_k = I_{N+1} - \sum_{i=1}^{N}(1 - \sqrt{\frac{i}{k}})E_{ii}$. The image of $[\alpha_{ij}T_{ij}] \in A$ under this map, after shuffling, can been viewed as a diagonal matrix with $N+1$ blocks $B_1, \dots B_{N+1}$, where $B_{N+1} = [\alpha_{ij}]$ and for $j = 1, \dots N$, $B_j = S_j^{-1}[\alpha_{ij}]S_j$, where $S_j = \text{diag}(\overbrace{1, 1, \dots 1}^{j}, \sqrt{\frac{j}{j+1}}\sqrt{\frac{j}{j+2}}, \dots)$. Let $k > N$ and choose α_{ij} to be all zeroes except 1 in the $(1, k)$ and $(k+1, k)$ entry. Then $\|B_{N+1}\| = \|[\alpha_{ij}]\| = \sqrt{2}$ and $\|[\alpha_{ij}T_{ij}]\| = \sqrt{\|T_{1,k}\|^2 + \|T_{k+1,k}\|^2} = \sqrt{\frac{2k+1}{k}}$. For $j = 1, \dots N$, if we compute $B_j = S_j^{-1}[\alpha_{ij}]S_j$, we see that B_j is a matrix with zeroes except $\sqrt{\frac{j}{k}}$ in the $(1, k)$ entry and $\sqrt{\frac{k+1}{k}}$ in the $(k+1, k)$ entry. Thus $\|B_j\| = \sqrt{\frac{j+k+1}{k}}$. Hence $\|[\alpha_{ij}T_{ij}]\| > \max_{j=1,\dots N+1}\{\|B_j\|\}$. Thus if D is any finite subset of $\mathbb{N}$, take N to be the maximum of the set D. By the above argument we can show that $\|[\alpha_{ij}T_{ij}]\| > \max_{j=1,\dots N+1}\{\|B_j\|\} > \max_{j \in D}\{\|B_j\|\}$, which is a contradiction. Thus $C_e^*(A)$ is not an annihilator C^*-algebra.

Remark. The last example shows that the conditions in Lemma 4.2.6 that q_kA and Aq_k are bicontinuously isomorphic to a Hilbert space, are necessary but not sufficient.

4.3 Wedderburn-Artin Type Theorems

We now give a characterization amongst the C^*-algebras, of C^*-algebras consisting of compact operators. There are many such characterizations in the literature, however we have not seen the following, in terms of the following notions introduced by Hamana. If X contains a subspace E then we say that X is an *essential extension* (resp. *rigid extension*) of E if any complete contraction with domain X (resp. from X to X) is completely isometric (resp. is the identity map) if it is completely isometric (resp. is the identity map) on E. If

X is injective then it turns out that it is rigid iff it is essential, and in this case we say X is an *injective envelope* of E, and write $X = I(E)$. See e.g. 4.2.3 in [14], or the works of Hamana; or [49] for some related topics.

Theorem 4.3.1. *If A is a C^*-algebra, the following are equivalent:*

(i) *A is an annihilator C^*-algebra.*

(ii) *A^{**} is an essential extension of A.*

(iii) *A^{**} is an injective envelope of A.*

(iv) *$I(A^{**})$ is an injective envelope of A.*

(v) *Every surjective complete isometry $T : A^{**} \to A^{**}$ maps A onto A.*

(vi) *A is nuclear and A^{**} is a rigid extension of A.*

Proof. Let $A = \oplus_i^0 \ \mathbb{K}(H_i)$, then by [35, Lemma 3.1 (ii)], $I(A) = \oplus_i^\infty \ I(\mathbb{K}(H_i)) = \oplus_i^\infty \ B(H_i) = A^{**}$. Hence (ii) implies (iii). Clearly (iii) $\Rightarrow$ (iv), since $I(A^{**}) = A^{**}$ if A^{**} is injective. Since A is nuclear iff A^{**} is injective, we have (vi) iff (iii).

Suppose that (iv) holds. Let $u : A^{**} \longrightarrow Z$ be a complete contraction into some operator space Z, such that the restriction of u to A is a complete isometry. Let $\tilde{u}$ be the extension of u to $I(A^{**})$ into $I(Z)$. So $\tilde{u}$ is completely contractive such that $\tilde{u}|_A = u$ is a complete isometry. Now $I(A^{**})$ being an injective envelope of A, is an essential extension of A. Thus $\tilde{u}$ is a complete isometry, and so is u. Hence (iv) implies (ii).

Item (ii) implies that every faithful $*$-representation of A is universal. This implies that A^{**} is injective, the latter since $\pi_a(A)'' \cong \oplus_i^\infty \ B(H_i)$ is injective (see [50, Lemma 4.3.8]), where π_a is the atomic representation of A. So (ii) $\Leftrightarrow$ (iii). Moreover, in the notation above, these imply that $\pi_a(A)''$ is an injective envelope of A since $A^{**} \cong \pi_a(A)'' \cong \oplus_i^\infty B(H_i)$ is an essential injective extension of A. Hence by [35, Lemma 3.1 (iii)], $\pi_a(A)$ contains $\oplus_i^0 K(H_i)$. Thus A has a subalgebra B with $\pi_a(B) = \oplus_i^0 K(H_i)$, and $\tilde{\pi}_a(B^{\perp\perp}) = \overline{\pi_a(B)}^{w^*} = \overline{\oplus_i^0 \mathbb{K}(H_i)}^{w^*} = \pi_a(A)'' = \pi_a(A^{**})$, where $\tilde{\pi}_a : A^{**} \longrightarrow B(K)$ is the w^*-extension of π_a. Hence $B^{\perp\perp} = A^{**}$, and so $A = B$. So (ii) $\Rightarrow$ (i).

By Proposition 3.2.3, (i) $\Rightarrow$ (v), and conversely, if p is a projection in A^{**} then $u = 1-2p$ is unitary, and so $(1-2p)A \subset A$ if (v) holds. Indeed clearly $p \in M(A)$, so that $A^{**} \subset M(A)$. Thus A is an ideal in A^{**}, which implies (i) [36]. □

Remark. In [34], Hamana defines the notion of a *regular* extension of a unital C^*-algebra. It is not hard to see that A^{**} is an essential extension of A iff it is a regular extension. This uses the fact that (ii) is equivalent to $A^{**} \subset I(A)$, and the fact that the regular monotone completion of A from [34], resides inside $I(A)$. In fact, as we see below, the result also holds for non-unital C^*-algebras.

Let A be a (non-unital) C^*-algebra. We define a *regular extension* of A to be the regular extension of A^1 [34], where A^1 is the unitization of A. Also, define the regular monotone completion of A, $\overline{A}$ to be the regular monotone completion [34] of A^1. Then we have

$$A \subset A^1 \subset \overline{A} \subset \tilde{A} \subset I(A),$$

where $\tilde{A}$ is the maximal regular extension of A (see e.g. [35, 34]).

Proposition 4.3.2. *Let A be a C^*-algebra. Then A^{**} is a regular extension of A if and only if A^{**} is an essential extension if and only if A is an annihilator C^*-algebra.*

Proof. We first show that A^{**} is an essential extension of A is equivalent to $A^{**} \subset I(A)$. If A^{**} is an essential extension of A, then a completely contractive completely positive extension of the canonical unital $*$-monomorphism from $A+\mathbb{C}1_{A^{**}}$ into $I(A)$, is completely isometric. So $A^{**} \subset I(A)$. Conversely, $A^{**} \subset I(A)$ implies A^{**} is an essential extension since $I(A)$ is an essential extension.

Now suppose that A^{**} is a regular extension of A. Then by the above, we need to show that $A^{**} \subset I(A)$. Since $A \subset A^1 \subset A^{**}$, by the discussion in the last paragraph of p. 169 in [34] and [34, Theorem 3.1], $A \subset A^{**} \subset \tilde{A} \subset I(A)$. Thus we have the desired embedding. On the other hand, if A^{**} is an essential extension of A, then by [3, Theorem 5.4], A^{**} is an injective envelope of A. But by [40, Lemma 1], $\overline{A}$ is the closure in the weak operator topology of A. Thus by the Kaplansky density theorem, $A^{**} = \overline{A}^{\mathrm{w}*} = \overline{A^1}^{\mathrm{w}*} = \overline{A^1}^{\mathrm{WOT}} = \overline{A^1}$, identifying A^1 with $\hat{A}+\mathbb{C}1_{A^{**}}$ in A^{**}. So $A^{**} = \overline{A^1}$, i.e., A^{**} is the regular monotone completion of A. Hence A^{**} is a regular extension. $\square$

The following illustrates the necessity of ‘row’ or ‘column’ sums in a Wedderburn type theory of operator algebras.

Lemma 4.3.3. *The compact operators, $\mathbb{K}(\ell^2)$, cannot be bicontinuously isomorphic to either $\oplus_i^\infty H_i$, $\oplus_i^1 H_i$ or $\oplus_i^0 H_i$ where H_i are Hilbert spaces.*

Proof. It is clear that $\mathbb{K}(\ell^2)$ cannot be bicontinuously isomorphic to either $\oplus_i^\infty H_i$ or $\oplus_i^1 H_i$, since both $\oplus_i^\infty H_i$ and $\oplus_i^1 H_i$ are dual spaces but $\mathbb{K}(\ell^2)$ is not a dual space. If $\mathbb{K}(\ell^2) \cong \oplus_i^0 H_i$ then $B(\ell^2) \cong \oplus_i^\infty H_i$, which is not possible. For instance, $B(\ell^2)$ has only one M-ideal, namely, $\mathbb{K}(\ell^2)$. On the other hand, $J_k = \oplus_i^\infty K_i$ where $K_i = \{0\}$ for all $i \neq k$, and $K_k = H_k$, is an M-ideal in $\oplus_i^\infty H_i$, for all indicies k. $\square$

Proposition 4.3.4. *Let $A = \mathbb{K}(\ell^2)$ and let e_{ij} be the matrix units in A. Then A cannot be A-isomorphic to $\oplus_i^\alpha e_{ii}A$, where $\alpha = 0, 1, 2$ or ∞.*

Proof. Let $\theta : A \longrightarrow \oplus^{\alpha} e_{ii}A$ be an A-isomorphism, then being a A-module map, θ is bounded (see e.g. [14, Corollary A.6.3]). Thus by the open mapping theorem, θ is bicontinuous. Now $\theta(Ae_{ii}) = \theta(A)e_{ii}$, so the restriction of θ to Ae_{ii}, is bicontinuous which maps onto $(\oplus_j^{\alpha} e_{jj}A)e_{ii} = \oplus_j^{\alpha} e_{jj}Ae_{ii}$. Since each $e_{jj}Ae_{ii}$ is a one-dimensional subspace of A, we can say $e_{jj}Ae_{ii} = \mathbb{C}e_{ji} \cong \mathbb{C}$. Thus $\oplus_j^{\alpha} e_{jj}Ae_{ii} \cong \ell^{\alpha}(\mathbb{C})$. We know that $\ell^{\alpha}(\mathbb{C})$ is not a Hilbert space except when $\alpha = 2$. But Ae_{ii} is a row-Hilbert space which contradicts that θ is a bicontinuous A-isomorphism. In the case when $\alpha = 2$, $\oplus_j^{2} e_{jj}A$ is a Hilbert space. But since A is not reflexive, A cannot be a Hilbert space, thus such a θ cannot exist. $\square$

Definition 4.3.5. We recall that a right ideal J of a normed algebra A is *regular* if there exists $y \in A$ such that $(1-y)A \subset J$. We shall say that J is *1-regular* if this can be done with $\|y\| \leq 1$. An element $x \in A$ is said to be *left (right) quasi-invertible* if there exists some $y \in A$ such that $yx = y + x - yx = 0$ ($xy = x + y - xy = 0$).

Proposition 4.3.6. *Let A be an operator algebra which contains nontrivial 1-regular ideals (or equivalently,* $\text{Ball}(A) \setminus \{1\}$ *is not composed entirely of quasi-invertible elements). Then proper maximal r-ideals of A exist. Indeed, if $y \in$* $\text{Ball}(A)$ *is not quasi-invertible then $(1-y)A$ is contained in a proper (regular) maximal r-ideal. The unit ball of the intersection of the 1-regular maximal r-ideals of A is composed entirely of quasi-invertible elements of A.*

Proof. We adapt the classical route. For $y \in \text{Ball}(A)$, let (J_t) be an increasing set of proper r-ideals, each containing $(1-y)A$. Then $J = \cup_t J_t$ is a right ideal which does not contain y, or else there is a t with $a = ya + (1-y)a \in J_t$ for all $a \in A$. The closure $\bar{J}$ of J is an r-ideal since it equals the closure of $\cup_t \bar{J}_t$. Also $\bar{J}$ is proper, since the closure of a proper regular ideal is proper [46]. Thus by Zorn's lemma, $(1-y)A$ is contained in a (regular) maximal r-ideal. Let I be the intersection of the proper 1-regular maximal r-ideals. If

$y \in \text{Ball}(I)$, but y is not quasi-invertible, then $y \notin \overline{(1-y)A}$. Let K be a maximal proper r-ideal containing $\overline{(1-y)A}$. Then $y \notin K$ (for if $y \in K$ then $a = (1-y)a + ya \in K$ for all $a \in A$), and K is regular. So $y \in I \subset K$, a contradiction. Hence every element of $\text{Ball}(I)$, is quasi-invertible. □

Remarks. 1) One may replace r-ideals in the last result with ℓ-ideals, or HSA's.

2) In connection with the last result we recall from algebra that the Jacobson radical is the intersection of all maximal (regular) one-sided ideals.

Chapter 5

One-Sided Real M-Ideals

This chapter is a part of an on-going project. We want to develop an analogous theory for real one-sided M-ideals and M-embedded operator spaces. In this chapter, we prove some basic results which are the beginnings of that process.

5.1 Real Operator Spaces

A (concrete) *real operator space* is a closed subspace of $B(H)$, for some real Hilbert space H. An abstract real operator space is a pair $(X, \|.\|_n)$, where X is a real vector space such that there is a complete isometry $u : X \longrightarrow B(H)$, for some real Hilbert space H. As in the case of complex operator spaces, Ruan's norm characterization hold for real operator spaces, and we say that $(X, \|.\|_n)$ is an abstract real operator space if and only if it satisfies

(i) $\|x \oplus y\|_{n+m} = \max\{\|x\|_n , \|y\|_m\}$,

(ii) $\|\alpha x \beta\|_n \leq \|\alpha\| \, \|x\|_n \, \|\beta\|$, for all $x \in M_n(X)$, $y \in M_m(X)$ and $\alpha, \beta \in M_n(\mathbb{R})$.

Let $X \subset B(H)$, then $X_c \subset B(H)_c$ and $B(H)_c \cong B(H_c)$ completely isometrically, where H_c is a complex Hilbert space (see e.g. discussion on page 1051 from [54]). Thus there is a canonical matrix norm structure on X_c inherited from $B(H_c)$, and X_c is a complex operator space with this canonical norm structure. The space $B(H_c)$ can be identified with a real subspace of $M_2(B(H))$ via

$$B(H_c) = B(H) + iB(H) = \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \ : \ x, y \in B(H) \right\} \in M_2(B(H)). \tag{5.1.1}$$

Thus the matrix norm on the complexification is given by

$$\|[x_{kl} + iy_{kl}]\| = \left\| \begin{bmatrix} x_{kl} & -y_{kl} \\ y_{kl} & x_{kl} \end{bmatrix} \right\|,$$

and we have the following complete isometric identification

$$X_c = \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \ : \ x, y \in X \right\} \in M_2(X).$$

This canonical complex operator space matrix norm structure on its complexification $X_c = X + iX$ which extends the original norm on X, i.e., $\|x + i0\|_n = \|x\|_n$ and satisfies the *reasonable* (in the sense of [54]) condition

$$\|x + iy\| = \|x - iy\|,$$

for all $x + iy \in M_n(X_c) = M_n(X) + iM_n(X)$ and $n \in \mathbb{N}$. By [54, Theorem 2.1], the operator space structure on X_c is independent of the choice of H. Moreover, by [54, Theorem 3.1], any other reasonable in the above sense operator space structure on X_c is completely isometric to the canonical operator space structure on X_c.

Let T be a (real linear) bounded operator between real operator spaces X and Y, then define the complexification of T, $T_c : X_c \longrightarrow Y_c$ as $T_c(x+iy) = T(x) + iT(y)$, a complex linear bounded operator. It is shown in [54, Theorem 3.1] that if T is a complete contraction (respectively, a complete isometry) then T_c is a complete contraction (respectively, complete isometry) with $\|T_c\|_{cb} = \|T\|_{cb}$. This is not true in the case of a Banach space, that is, the complexification of a contraction on a real Banach space is bounded, but is not necessarily a contraction, and $\|T\| \neq \|T_c\|$, in general. If $\pi : X_c \longrightarrow Z_c$ is linear, then as in [54], define a linear map $\overline{\pi} : X_c \longrightarrow Y_c$ as $\overline{\pi}(x+iy) = \overline{\pi(x-iy)}$. Let $\mathrm{Re}(\pi) = \frac{\pi+\overline{\pi}}{2}$ and let $\mathrm{Im}(\pi) = \frac{\pi-\overline{\pi}}{2i}$. Then $\mathrm{Re}(\pi)$ and $\mathrm{Im}(\pi)$ are (complex) linear maps which map X into Z such that $\overline{\mathrm{Re}(\pi)} = \mathrm{Re}(\pi)$, $\overline{\mathrm{Im}(\pi)} = \mathrm{Im}(\pi)$, and $\pi = \mathrm{Re}(\pi) + i\mathrm{Im}(\pi)$.

5.1.1 Minimal Real Operator Space Structure

A real C^*-algebra is a norm closed $*$-subalgebra of $B(H)$, where H is a real Hilbert space. By [43, Proposition 5.13], every real C^*-algebra A is a fixed point algebra of $(B, -)$, i.e., $A = \{b \in B \ : \ \overline{b} = b\}$, where B is a (complex) C^*-algebra, and "-" is a conjugate linear $*$-algebraic isomorphism of B with period 2. Moreover, $B = A + iA$ is the complexification of A.

Let A be a commutative real C^*-algebra. Then define the *spectral space* of A as,

$$\Omega = \{\rho|_A \ : \ \rho \text{ is nonzero multiplicative linear functional on } \mathrm{A_c}\}.$$

In other words, Ω is the set of all non-zero complex valued multiplicative real linear functionals on A. Then using the "$-$" on A_c, define "$-$" on Ω as,

$$\overline{\rho}(a) = \overline{\rho(a)}.$$

Then every commutative real C^*-algebra A is of the form

$$A \cong C_0(\Omega, -) = \{f \in C_0(\Omega) : f(\bar{t}) = \overline{f(t)} \; \forall \; t \in \Omega\},$$

where Ω is the spectral space of A and "$-$" is a conjugation on Ω defined above (see e.g. [43, 5.1.4]). Also $A_c = C_0(\Omega, -)_c = C_0(\Omega)$.

If Ω is any compact Hausdorff space then there is a canonical real C^*-algebra, $C(\Omega, \mathbb{R}) = \{f : \Omega \longrightarrow \mathbb{C} : \text{f is continuous}\}$. But not every commutative real C^*-algebra A is of the form $C(\Omega, \mathbb{R})$. To see this, let $\Omega = S^2 \subset \mathbb{R}^3$, the 3-dimensional sphere. Let $A = \{f : \Omega \longrightarrow \mathbb{C} : \; f(-t) = \overline{f(t)} \; \forall \; t \in \Omega\}$. Then $A_c = C(\Omega)$, so A is a real C^*-algebra. But A is not $*$-isomorphic to $C(K, \mathbb{R})$ since $A_{\text{sa}} = \{f : \Omega \longrightarrow \mathbb{R} : \; f(t) = f(-t)\} \ncong C(K, \mathbb{R})$.

We can define an operator space structure on $C(\Omega, -)$ by the canonical structure it inherits as a subspace of $C(\Omega)$. Then $C(\Omega)$ is the operator space complexification of $C(\Omega, -)$, in the sense defined above (see e.g., [43, Proposition 5.1.3]). Let E be a real Banach space. Then E can be embedded isometrically into a real commutative C^*-algebra A of the form $C(\Omega, \mathbb{R})$. For instance, take $\Omega = \text{Ball}(E^*)$ where $E^* = \{f : E \longrightarrow \mathbb{R}, \, f \text{ continuous}\}$. Since commutative real C^*-algebras are real operator spaces, there is an operator space matrix structure on E via the identification $M_n(E) \subseteq M_n(A)$. This operator space structure is called the minimal operator space structure since it is the smallest operator space structure on E. To see this, let E be the operator space sitting inside $C(\Omega, \mathbb{R})$ and F denote the Banach space E, with a different operator space structure. Let $u : F \longrightarrow E$ be the identity map. So u is an isometry, $\|u(x)\|_E = \|x\|_E = \|x\|_F$, and for any $[x_{ij}] \in M_n(E)$

and $\Omega = \mathrm{Ball}(E^*)$,

$$\begin{aligned}
\|u_n[x_{ij}]\|_{M_n(E)} &= \|[u(x_{ij})]\|_{M_n(E)} \\
&= \sup\{\|[u(x_{ij})(t)]\|_{M_n(\mathbb{R})} : t \in \Omega\} \\
&= \sup\{|\sum_{i,j} u(x_{ij})(t) w_j v_i| : \vec{v}, \vec{w} \in l_n^2(\mathbb{R}),\ t \in \Omega\} \\
&= \sup\{\|u(\sum_{i,j} x_{ij} w_j v_i)\|_E : \vec{v}, \vec{w} \in l_n^2(\mathbb{R})\} \\
&= \sup\{\|\sum_{i,j} x_{ij} w_j v_i\|_F : \vec{v}, \vec{w} \in l_n^2(\mathbb{R})\} \\
&\leq \|[x_{ij}]\|_{M_n(F)}.
\end{aligned}$$

This implies that $\|u : F \longrightarrow E\|_{cb} \leq 1$.

Let E be a Banach space and let $x, y \in E$. Define

$$\|x + iy\| = \sup\{\|\alpha x + \beta y\| :\ \alpha^2 + \beta^2 \leq 1,\ \alpha, \beta \in \mathbb{R}\}.$$

Then $\|x + iy\| = \|x - iy\|$ and $\|x + i0\| = \|x\|$, and thus with this new norm E_c is a complexification of the Banach space E. This norm is called the w_2-norm in [21]. Also note that for any $z + iw \in \mathbb{C}$,

$$|z + iw| = \sup\{|\alpha z + \beta w| :\ \alpha^2 + \beta^2 \leq 1,\ \alpha, \beta \in \mathbb{R}\}.$$

So,

$$\begin{aligned}
\|x + iy\| &= \sup\{|\alpha f(x) + \beta f(y)| :\ \alpha^2 + \beta^2 \leq 1,\ \alpha, \beta \in \mathbb{R} \text{ and } f \in \mathrm{Ball}(E^*)\} \\
&= \sup\{|f(x) + i f(y)|\ :\ f \in \mathrm{Ball}(E^*)\} \\
&= \sup \left\{ \left\| \begin{bmatrix} f(x) & -f(y) \\ f(y) & f(x) \end{bmatrix} \right\| : f \in \mathrm{Ball}(E^*) \right\}.
\end{aligned}$$

Proposition 5.1.1. *Let E be a real Banach space and E_c be the complexification of E with the norm defined above. Then* $(\mathrm{Min}(E))_c = \mathrm{Min}(E_c)$.

Proof. Let $\pi : E \longrightarrow C(\Omega, \mathbb{R})$ be the canonical isometry. Then $\mathrm{Min}(\pi) : \mathrm{Min}(E) \longrightarrow C(\Omega, \mathbb{R})$ is a complete isometry, and so, $\mathrm{Min}(\pi)_c : \mathrm{Min}(E)_c \longrightarrow C(\Omega)$ is a complete isometry. Further,

$$\begin{aligned} \|\pi_c(x+iy)\| &= \sup\{|\pi(x)(f) + i\pi(y)(f)| : f \in \Omega = \mathrm{Ball}(E^*)\} \\ &= \sup\{|f(x) + if(y)| : f \in \mathrm{Ball}(E^*)\} \\ &= \sup\{\|\alpha x + \beta y\| : \ \alpha^2 + \beta^2 \leq 1, \ \alpha, \beta \in \mathbb{R}\} \\ &= \|x + iy\|_{E_c} . \end{aligned}$$

So $\pi_c : E_c \longrightarrow C(\Omega)$ is an isometry, and hence $\mathrm{Min}(\pi_c) : \mathrm{Min}(E_c) \longrightarrow C(\Omega)$ is a complete isometry. So we have the following diagram which commutes.

$$\begin{tikzcd}
\mathrm{Min}(E)_c \arrow[r, "\text{c.i.}"] \arrow[d, "\mathrm{Id}"'] & C(\Omega) \arrow[d, "\mathrm{Id}"] \\
\mathrm{Min}(E_c) \arrow[r, "\text{c.i.}"'] & C(\Omega)
\end{tikzcd}$$

Hence $(\mathrm{Min}(E))_c = \mathrm{Min}(E_c)$, completely isometrically. □

Lemma 5.1.2. *Let A and B be real C^*-algebras, and let $\pi : A \longrightarrow B$ be a homomorphism. Then π is a $*$-homomorphism if and only if it is completely contractive. Further, π is a complete isometry if and only if it is one-one.*

Proof. Let $\pi : A \longrightarrow B$ be a $*$-homomorphism, then $\pi_c : A_c \longrightarrow B_c$ is a $*$-homomorphism. Hence π_c is a complete contraction, by [14, Proposition 1.2.4], so $\pi = \pi_c|_A$ is a complete contraction. A similar argument using the complexification proves the converse. The last assertion follows from [14, Proposition 1.2.4] and that π_c is one-one if π is. □

The following proposition has been noted in [54].

Proposition 5.1.3. *Let X be a real operator space, then $(X_c)^* = (X^*)_c$, completely isometrically.*

Proposition 5.1.4. *If X is a real operator space then $X \subset X^{**}$ completely isometrically via the canonical map i_X.*

Proof. Let X be a real operator space and let $\pi : X_c \hookrightarrow (X_c)^{**}$ be the canonical embedding. By Proposition 5.1.3, $(X_c)^{**} = (X^{**})_c$, completely isometrically via, say, θ. Then $\theta \circ \pi$ is a complete isometry such that $(\theta \circ \pi)(z) = \mathrm{Re}(\pi(z)) + i\mathrm{Im}(\pi(z))$, for all $z \in X_c$. So the restriction of $\theta \circ \pi$ to X is a complete isometry on X such that, for all $f \in X^*$ and $x \in X$, $\overline{(\theta \circ \pi)}(x) = (\theta \circ \pi)(x)$, and $((\theta \circ \pi)(x))(f) = (\mathrm{Re}(\pi(x)))(f) = f(x) = i_X(x)$. Thus i_X is a complete isometry. $\square$

The maximal operator space structure is the largest operator space structure that can be put on a real operator space, and its matrix norms are defined exactly as in the complex case.

$$\|[x_{ij}]\| = \sup\{\|[u(x_{ij})]\| : \ u \in \mathrm{Ball}(B(E,Y)), \ Y \text{ a real operator space}\}.$$

If we put the maximal operator space structure on E, then it has the universal property that for any real operator space Y, and $u : E \longrightarrow Y$ bounded linear, we have

$$\|u : E \longrightarrow Y\| = \|u : \mathrm{Max}(E) \longrightarrow Y\|_{cb}$$

i.e., $B(E,Y) = CB(\mathrm{Max}(E), Y)$

Lemma 5.1.5. *Let K be a compact Hausdorff space then $C(K,\mathbb{R})^{**}$ is a (real) commutative C^*-algebra of the form $C(\Omega,\mathbb{R})$.*

Proof. Let $u : C(K,\mathbb{R}) \longrightarrow C(K,\mathbb{R})_c = C(K)$ be the inclusion map. Then $u^{**} : C(K,\mathbb{R})^{**} \longrightarrow C(K)^{**}$ is a $*$-monomorphism. The second dual of a (real or complex) commutative C^*-algebra is a commutative C^*-algebra. Let $C(K)^{**} \cong C(\Omega)$, $*$-isomorphically. Then $C(\Omega,\mathbb{R})$ sits inside $C(\Omega)$ as a real space, in fact, as the real part such that, $C(\Omega,\mathbb{R})_c = C(\Omega)$. It is enough to show that $u^{**}(f) = \overline{(u^{**}(f))}$ for all $f \in C(K,\mathbb{R})^{**}$. We use a weak*-density argument. First, note that $u^{**}|_{C(K,\mathbb{R})} = u$ and u is selfadjoint, i.e., $u(g) = \overline{u(g)}\ \forall\ g \in C(K,\mathbb{R})$. Let $f \in C(K,\mathbb{R})^{**}$, then there exists a net $\{f_\lambda\}$ in $C(K,\mathbb{R})$ converging weak* to f. Then $u^{**}(f_\lambda) \overset{\text{weak}^*}{\longrightarrow} u^{**}(f)$. This implies that $u^{**}(f_\lambda)(\omega)$ converges pointwise to $u^{**}(f)(\omega)$ in $\mathbb{C}$ for all $\omega \in \Omega$. Hence $\overline{u^{**}(f_\lambda)(\omega)} \longrightarrow \overline{u^{**}(f)(\omega)}$ in $\mathbb{C}$ for all $\omega \in \Omega$. So $\overline{u^{**}(f_\lambda)} \overset{\text{weak}^*}{\longrightarrow} \overline{u^{**}(f)}$. But $u^{**}(f_\lambda) = u(f_\lambda) = \overline{u^{**}(f_\lambda)} = \overline{u(f_\lambda)}$. So $u^{**}(f_\lambda) \overset{\text{weak}^*}{\longrightarrow} \overline{u^{**}(f)}$. Hence, by uniqueness of limit, $u^{**}(f) = \overline{u^{**}(f)}$. This shows that the map u^{**} is real, and hence it maps into $C(\Omega,\mathbb{R})$. Let $f \in C(\Omega,\mathbb{R}) = (C(K)^{**})_{\text{sa}}$. Let $\{f_\lambda\} \in C(K)$ be a net which converges weak* to f. Then $\{\overline{f_\lambda}\}$ also converges weak* to f, and so does $g_\lambda = \frac{f_\lambda + \overline{f_\lambda}}{2} \in C(K,\mathbb{R})$. Thus $f \in \overline{\text{Ran}(u)}^{\text{weak}^*} \subset \text{Ran}(u^{**})$. Hence u^{**} maps onto $C(\Omega,\mathbb{R})$. $\square$

Proposition 5.1.6. *Let E be a real Banach space, then*

$$\text{Min}(E^*) = \text{Max}(E)^* \quad \text{and} \quad \text{Min}(E)^* = \text{Max}(E^*).$$

Proof. We have that $M_n(\text{Max}(E)^*) \cong CB(\text{Max}(E), M_n(\mathbb{R})) \cong B(E, M_n(\mathbb{R}))$, isometrically, for each n. On the other hand,

$$M_n(\text{Min}(E^*)) \cong M_n(\mathbb{R}) \check{\otimes} E^* \cong B(E, M_n(\mathbb{R})),$$

isometrically, where $\hat{\otimes}$ denotes the Banach space injective tensor product. Thus $\text{Min}(E^*) = \text{Max}(E)^*$.

Let K be a compact Hausdorff space, then by Lemma 5.1.5,

$$\text{Min}(C(K,\mathbb{R}))^{**} = C(K,\mathbb{R})^{**} = C(\Omega,\mathbb{R}).$$

On the other hand, $\mathrm{Min}(C(K,\mathbb{R})^{**}) = \mathrm{Min}(C(\Omega,\mathbb{R})) = C(\Omega,\mathbb{R})$. Hence $\mathrm{Min}(C(K,\mathbb{R}))^{**}$ $= \mathrm{Min}(C(K,\mathbb{R})^{**})$.

Let E be a real Banach space, and suppose that $\mathrm{Min}(E) \hookrightarrow C(K,\mathbb{R})$ completely isometrically. By taking the duals, we get the following commuting diagram

$$\begin{tikzcd}
C(K,\mathbb{R})^{**} & \mathrm{Min}(C(K,\mathbb{R})^{**}) \arrow[l] \\
\mathrm{Min}(E^{**}) \arrow[u] & \mathrm{Min}(E)^{**}. \arrow[l] \arrow[u]
\end{tikzcd}$$

Let u denote the map from $\mathrm{Min}(E)^{**}$ to $\mathrm{Min}(E^{**})$. Since all the maps except u, in the above diagram are complete isometries and since the diagram commutes, it forces u to be a complete isometry. Hence $\mathrm{Min}(E)^{**} = \mathrm{Min}(E^{**})$. Applying the first identity, we proved above, to E^*, we get $\mathrm{Min}(E^{**}) = \mathrm{Max}(E^*)^*$. Hence, $\mathrm{Max}(E)^{**} = \mathrm{Min}(E)^{**}$. Let $X = \mathrm{Max}(E^*)$ and $Y = \mathrm{Min}(E)^*$, then since $X^* = Y^*$, this implies $X^{**} = Y^{**}$ completely isometrically. By the commuting diagram below

$$\begin{tikzcd}
X \arrow[r, "Id"] \arrow[d, hook] & Y \arrow[d, hook] \\
X^{**} \arrow[r] & Y^{**}
\end{tikzcd}$$

it is clear that $X = Y$, completely isometrically. ☐

We write $\ell_2^1(\mathbb{R})$ for the two-dimensional real Banach space $\mathbb{R} \oplus_1 \mathbb{R}$, and $\ell_2^\infty(\mathbb{R})$ for $\mathbb{R} \oplus_\infty \mathbb{R}$. Then $\ell_2^1(\mathbb{R})$ is isometrically isomorphic to $\ell_2^\infty(\mathbb{R})$ via $(x, y) \mapsto (x+y, x-y)$. We also have that $(\ell_2^\infty(\mathbb{R}))^* \cong \ell_2^1(\mathbb{R})$ and $(\ell_2^1(\mathbb{R}))^* \cong \ell_2^\infty(\mathbb{R})$, isometrically. From [47] we know that there is a unique operator space structure on the two-dimensional complex Banach space, $\ell_2^1(\mathbb{C})$. We see next that this is not true in the case of real operator spaces.

Proposition 5.1.7. *The operator space structure on $l_2^1(\mathbb{R})$ is not unique.*

Proof. We consider the maximal and the minimal operator space structures on $l_1^2(\mathbb{R})$. Using the facts stated above and Proposition 5.1.6, we have that

$$\mathrm{Max}(l_2^1(\mathbb{R})) \cong \mathrm{Max}(l_2^\infty(\mathbb{R})^*) \cong \mathrm{Min}(l_2^\infty(\mathbb{R}))^* = l_2^\infty(\mathbb{R})^*,$$

completely isometrically. So the maximal operator space matrix norm on $l_2^1(\mathbb{R})$ is given by

$$\|[(a_{ij}, b_{ij})]\|_{\max} = \sup\{\|[a_{ij}d_{kl} + b_{ij}e_{kl}]\| : \ [d_{kl}], [e_{kl}] \in \mathrm{Ball}(M_m(\mathbb{R})), \ m \in \mathbb{N}\}.$$

On the other hand, $\mathrm{Min}(l_2^1(\mathbb{R})) \cong \mathrm{Min}(l_2^\infty(\mathbb{R})) = l_2^\infty(\mathbb{R})$, completely isometrically via the map $(x, y) \mapsto (x + y, x - y)$. So the matrix norm on $\mathrm{Min}(l_2^1(\mathbb{R}))$ is

$$\|[(a_{ij}, b_{ij})]\|_{\min} = \max\{\|[(a_{ij} + b_{ij})]\|, \|[(a_{ij} - b_{ij})]\|\}.$$

It is clear that $\|[(a_{ij}, b_{ij})]\|_{\min} \leq \|[(a_{ij}, b_{ij})]\|_{\max}$. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then

$$\begin{aligned}
\|(A,B)\|_{M_2(\mathrm{Min}(l_2^1))} &= \max\{\|A + B\|, \|A - B\|\} \\
&= \max\left\{\left\|\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right\|, \left\|\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\right\|\right\} \\
&= \sqrt{2}.
\end{aligned}$$

Now if we take $D = A$ and $E = B$, then for the norm of (A, B) in $M_2(\mathrm{Max}(l_2^1))$, we have

$$\|(A,B)\| \geq \left\| \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 0 \\ 0 & -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \end{bmatrix} \right\|$$

On adding and rearranging the rows and columns we see that this norm is the same as

$$\begin{aligned}\left\|\begin{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} & 0\\ 0 & \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{bmatrix}\right\| &= \max\left\{\left\|\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\|, \left\|\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right\|\right\}\\ &= 2.\end{aligned}$$

So $\|(A,B)\|_{M_2(\mathrm{Max}(l_2^1))} \geq 2 > \sqrt{2} = \|(A,B)\|_{M_2(\mathrm{Min}(l_2^1))}$. Hence, there are two different operator space structures on $l_2^1(\mathbb{R})$. $\square$

If X is a complex operator space then, it is also a real operator space, and hence we can talk about the dual of X both as a real operator space X_r^*, as well as a complex operator space X^*, and ask the question, whether these two spaces are the same real operator spaces. Then by [43], $(X^*)_r$ is isometrically isomorphic to $(X_r)^*$. We see next that these spaces need not be completely isometrically isomorphic.

Proposition 5.1.8. *Let $X = \mathbb{C}$, be a (complex) operator space with the canonical operator space structure. Then $(X_r)^*$ and $(X^*)_r$ are isometrically isomorphic but not necessarily completely isometrically isomorphic.*

Proof. It follows from [43, Proposition 1.1.6] that $(\mathbb{C}_r)^* \cong (\mathbb{C}^*)_r$, isometrically. Note that $\mathbb{C}^*$ is completely isometrically isomorphic to $\mathbb{C}$ via the map $\phi_z \longrightarrow z$. Consider the canonical map $\theta : \mathbb{C}^* \longrightarrow \mathbb{C}_r^*$ given by $\theta(\phi) = \mathrm{Re}(\phi)$. By the identification $\mathbb{C} \cong \mathbb{C}^*$, we can view the above map as $\theta(z)(y) = \mathrm{Re}(y\bar{z})$. If there is any complete isometric isomorphism, say ψ, then since ψ is an onto isometry between 2-dimensional real Hilbert spaces, it is unitarily equivalent to θ. Any unitary from $\mathbb{C}$ to $\mathbb{C}$, is a rotation by an angle α. So, u is multiplication by $e^{i\alpha}$, which is a complete isometry with the canonical operator space matrix norm structure on $\mathbb{C}$. Then $\theta = u^{-1}\psi u$ is a complete isometry. Thus ψ is a complete isometry if and only if θ is a complete isometry. Hence it is enough to show that θ is not a complete isometry.

Consider $x = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$. Then $\|x\| = \sqrt{2}$. Since $\theta_2(x) \in M_2(\mathbb{C}^*) \cong CB(\mathbb{C}, M_2(\mathbb{R}))$, we have that

$$\|\theta_2(x)\| = \sup\{\|\theta_2(x)([z_{kl}])\| \ : [z_{kl} = x_{kl} + iy_{kl}] \in M_n(\mathbb{C})\}.$$

Consider

$$\begin{aligned} \|\theta_2(x)[x_{kl} + iy_{kl}]\| &= \left\| \begin{bmatrix} \mathrm{Re}[x_{kl} + iy_{kl}] & \mathrm{Re}[ix_{kl} - y_{kl}] \\ 0 & 0 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} [x_{kl}] & [-y_{kl}] \\ 0 & 0 \end{bmatrix} \right\|. \end{aligned}$$

Let $\vec{v}$ be a row vector of length $2n$, whose first n entries are α_i and the last n entries are β_i. Then the norm of the square of $\vec{v}$ produced by the action of $\begin{bmatrix} [x_{kl}] & [-y_{kl}] \\ 0 & 0 \end{bmatrix}$ is given by

$$\begin{aligned} \sum_{k=1}^{n} \left| \sum_{l=1}^{n} (x_{kl}\alpha_l - y_{kl}\beta_l) \right|^2 &\leq \sum_{k=1}^{n} \left| \sum_{l=1}^{n} (x_{kl} + iy_{kl})(\alpha_l + i\beta_l) \right|^2 \\ &\leq \|[x_{kl} + iy_{kl}]\|^2 . \end{aligned}$$

So $\left\| \begin{bmatrix} [x_{kl}] & [-y_{kl}] \\ 0 & 0 \end{bmatrix} \right\| \leq \|[x_{kl} + iy_{kl}]\|$, and hence $\|\theta_2(x)\| \leq 1$. In fact $\|\theta_2(x)\|$ is equal to 1, for instance if $[z_{kl}] = I_{M_2(\mathbb{R})}$, then $\|\theta_2(x)\| \geq 1$. Thus $\|\theta_2(x)\| = 1 \leq \sqrt{2} = \|x\|$. Hence θ cannot be a complete isometry. $\square$

Remark. We end this section with a list of several results from the operator space theory which can be generalized for the real operator spaces using the exact same proof as in the complex setting. Various constructions using real operator spaces like taking the quotient, infinite direct sums, c_0-direct sums, mapping spaces $CB(X,Y)$ and matrix spaces $\mathbb{M}_{I,J}(X)$ can be defined analogously, and are real operator spaces. All the results and properties of matrix spaces hold true for the real operator spaces (see e.g. [14, 1.2.26]). Further, we can define Hilbert row and Hilbert column operator space structure on a real Hilbert space by replacing $\mathbb{C}$ with $\mathbb{R}$, in the usual definition. Then $B(H,K) \cong CB(H^c,K^c)$ and $B(H,K) \cong CB(K^r,H^r)$ completely isometrically, for real Hilbert spaces H,K. Also $(H^c)^* \cong H^r$ and $(H^r)^* \cong H^c$. We can show that if $u : X \longrightarrow Z$ is completely bounded between real operator spaces X and Z, and Y is any subspace of $\mathrm{Ker}(u)$, then the canonical map $\tilde{u} : X/Y \longrightarrow Z$ induced by u is completely bounded. If $Y = \mathrm{Ker}(u)$ then u is a complete quotient if and only if $\tilde{u}$ is a completely isometric isomorphism. The duality of subspaces and quotients hold in the real case, i.e., $X^* \cong Y^*/X^\perp$ and $(Y/X)^* \cong X^\perp$ completely isometrically, where Y is a subspace of the real operator space X. It is also true that the trace class operator $S^1(H)$ is the predual of $B(H)$ for every real Hilbert space H. If X is a real operator space then $M_{m,n}(X)^{**} \cong M_{m,n}(X^{**})$ completely isometrically for all $m,n \in \mathbb{N}$. If X and Y are real operator spaces and if $u : X \longrightarrow Y^*$ is completely bounded, then its (unique) w^*-extension $\tilde{u} : X^{**} \longrightarrow Y^*$ is completely bounded with $\|\tilde{u}\| = \|u\|$. Hence $CB(X,Y^*) = w^*CB(X^{**},Y^*)$ completely isometrically.

5.2 Real Operator Algebras

Definition 5.2.1. An (abstract) real operator algebra A is an algebra which is also an operator space, such that A is completely isometrically isomorphic to a subalgebra of $B(H)$

for some (real) Hilbert space H, i.e., there exists a (real) completely isometric homomorphism $\pi : A \longrightarrow B(H)$. For any n, $M_n(A) \subset M_n(B(H)) = B(H^n)$ is a real operator algebra with product of two elements, $[a_{ij}]$ and $[b_{ij}]$ of $M_n(A)$, given by

$$[a_{ij}][b_{ij}] = [\sum_{k=1}^{n} a_{ik}b_{kj}].$$

Every real operator algebra can be embedded (uniquely up to a complete isometry) into a complex operator algebra via 'Ruan's reasonable' complexification. Let A be a real operator algebra and $A_c = A + iA$ be the operator space complexification of A. Then A_c is an algebra with a natural product

$$(x + iy)(v + iw) = (xv - yw) + i(xw + yv).$$

Suppose that $\pi : A \longrightarrow B(H)$ is a complete isometric homomorphism, for some real Hilbert space H. Then $\pi_c : A_c \longrightarrow B(H)_c$ is a (complex) complete isometry, and it is easy to see that π_c is also a homomorphism. Thus A_c is a complex operator algebra if A is a real operator algebra. As in [54], $B(H_c) = B(H) + iB(H)$ has a reasonable norm extension $\{\|.\|_n\}_{n\in\mathbb{N}}$, these norms are inherited by A_c via the complete isometric homomorphism $\pi_c : A_c \longrightarrow B(H)_c$. Thus the matrix norms on A_c satisfy $\|x + i0\|_n = \|x\|_n$ and $\|x + iy\| = \|x - iy\|$, for all $x + iy \in M_n(X_c) = M_n(X) + iM_n(X)$ and $n \in \mathbb{N}$. The conjugation "-" on A_c satisfies $\overline{xy} = \bar{x}\bar{y}$, for all $x, y \in A_c$.

Remarks. 1) The complexification of a real operator algebra is unique, up to complete isometry by [54, Theorem 3.1].

2) If A is approximately unital then so is A_c. Indeed if e_t is an approximate unit for A, then for any $x + iy \in A_c$, $\|e_t(x + iy) - (x + iy)\| \leq \|e_t x - x\| + \|e_t y - y\|$. Thus e_t is an approximate identity for A_c.

Now we show that there is a real version of the BRS theorem which characterizes the real operator algebras.

Theorem 5.2.2. *(BRS Real Version) Let A be a real operator space which is also an approximately unital Banach space. Then the following are equivalent:*

(i) *The multiplication map $m : A \otimes_h A \longrightarrow A$ is completely contractive.*

(ii) *For any n, $M_n(A)$ is a Banach algebra. That is,*

$$\left\| \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right] \right\|_{M_n(A)} \leq \| [a_{ij}] \|_{M_n(A)} \| [b_{ij}] \|_{M_n(A)} ,$$

for any $[a_{ij}]$ and $[b_{ij}]$ in $M_n(A)$.

(iii) *A is a real operator algebra, that is, there exist a real Hilbert space H and a completely isometric homomorphism $\pi : A \longrightarrow B(H)$.*

Proof. The equivalence between (i) and (ii), and that (iii) implies these, follows from the property that the Haagerup tensor product of real operator spaces linearizes completely bounded bilinear maps, and the fact that each $M_n(A)$ is an operator algebra.

(iii) $\Rightarrow$ (ii) Let $\pi : A \longrightarrow B(H)$. Then by [54, Theorem 2.1], $\pi_c : A_c \longrightarrow B(H)_c$ is a complete isometric homomorphism. Let $[a_{ij}], [b_{ij}] \in M_n(A) \subset M_n(A_c)$. Then by the BRS theorem for complex operator algebras,

$$\left\| \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right] \right\|_{M_n(A)} \leq \| [a_{ij}] \|_{M_n(A)} \| [b_{ij}] \|_{M_n(A)} .$$

(ii) $\Rightarrow$ (iii) Since A is approximately unital, by the above remark, A_c is also approximately unital. Let $\theta : A_c \longrightarrow M_2(A)$ be $\theta(x + iy) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$. Then θ is a complete isometric homomorphism and each amplification, θ_n is an isometric homomorphism. Let $a = [a_{ij}],\ b = [b_{ij}] \in M_n(A_c)$. Then

$$\|ab\| = \|\theta_n(ab)\| = \|\theta_n(a)\theta_n(b)\| \leq \|\theta_n(a)\| \, \|\theta_n(b)\| = \|a\| \, \|b\| .$$

Thus by the BRS theorem for complex operator algebras, there exists a completely isometric homomorphism $\pi : A_c \longrightarrow B(K)$, for some complex Hilbert space K. Let $K = H_c$. Define $\pi_1 = \frac{\pi + \overline{\pi}}{2}$ and $\pi_2 = \frac{\pi - \overline{\pi}}{2i}$. Then $\pi_1,\ \pi_2$ are (complex) linear maps such that $\pi_1 = \overline{\pi_1}$, $\pi_2 = \overline{\pi_2}$, and $\pi(x+iy) = (\pi_1(x) - \pi_2(y)) + i(\pi_1(y) + \pi_2(x))$. Let $\tilde{\pi}$ be the composition of π with the canonical identification $B(K) \hookrightarrow M_2(B(H))$ (see e.g. (5.1.1)), so

$$\tilde{\pi}(x+iy) = \begin{bmatrix} \pi_1(x) - \pi_2(y) & -\pi_1(y) - \pi_2(x) \\ \pi_1(y) + \pi_2(x) & \pi_1(x) - \pi_2(y) \end{bmatrix} \in M_2(B(H)).$$

The restriction of $\tilde{\pi}$ to A, say $\pi_\circ$, is a complete isometric inclusion from A into $M_2(B(H))$. Also, for $x, v \in A$

$$\begin{aligned} \pi_\circ(x)\pi_\circ(v) &= \begin{bmatrix} \pi_1(x) & -\pi_2(x) \\ \pi_2(x) & \pi_1(x) \end{bmatrix} \begin{bmatrix} \pi_1(v) & -\pi_2(v) \\ \pi_2(v) & \pi_1(v) \end{bmatrix} \\ &= \begin{bmatrix} \pi_1(x)\pi_1(v) - \pi_2(x)\pi_2(v) & -\pi_1(x)\pi_2(v) - \pi_2(x)\pi_1(v) \\ \pi_1(x)\pi_2(v) + \pi_2(x)\pi_1(v) & \pi_1(x)\pi_1(v) - \pi_2(x)\pi_2(v) \end{bmatrix} \\ &= \begin{bmatrix} \pi_1(xv) & -\pi_2(xv) \\ \pi_2(xv) & \pi_1(xv) \end{bmatrix} = \pi_\circ(xv) \end{aligned}$$

Thus $\pi_\circ$ is a completely isometric homomorphism from A into $M_2(B(H)) \cong B(H^2)$. □

Theorem 5.2.3. *Let A be a complex operator algebra. Then A is a complexification of a real operator algebra B, i.e., $A = B_c$ completely isometrically if and only if there exists a complex conjugation "$-$" on A such that*

(i) *"$-$" is a complete isometry, i.e., $\|[x_{ij}]\|_n = \|[\overline{x_{ij}}]\|_n$ for all $[x_{ij}] \in M_n(A)$ and $n \in \mathbb{N}$,*

(ii) $\overline{xy} = \bar{x}\bar{y}$ *for all* $x, y \in A$.

Proof. If $A = B_c$, for a real operator algebra B, then clearly A satisfies the conditions in (i) and (ii) above. Suppose that A is a complex operator algebra such that (i) and (ii) hold. Since A is a complex operator space such that the matrix norms satisfy (i), by [54, Theorem 3.2] there exists a real operator space B such that $A = B + iB$ completely isometrically. Now the conjugation on A is $\overline{x+iy} = x - iy$, and $B = \text{Re}(A) = \{x \in A : x = \bar{x}\}$. So if $x, y \in B$, then $xy = \bar{x}\bar{y} = \overline{xy}$. Thus B is a subalgebra. Since A is a complex operator algebra, it is also a real operator algebra, and B is a (real) closed subalgebra of A. Thus B is a (real) operator algebra. □

Let $A \subset B(H)$ be a real operator algebra, for some real Hilbert space H. Define the unitization of A as $A^1 = \text{Span}_{\mathbb{R}}\{A, I_H\} \subset B(H)$. Then $A \subset A^1 \subset B(H)$ is a closed subalgebra, and A^1 is a unital real operator algebra.

Lemma 5.2.4. *Let* $A \subset B(H)$ *be a real operator algebra. Then* $(A_c)^1 = (A^1)_c \subset B(H)_c$, *completely isometrically.*

Proof. Clearly both $(A_c)^1$ and $(A^1)_c$ are subsets of $B(H)_c$. Since $A \subset A^1$, $A_c \subset (A^1)_c$ and $I_H \in (A^1)_c$. So $(A_c)^1 = \text{Span}\{A_c, I_H\} \subset (A^1)_c$. If $x \in (A^1)_c$, then

$$\begin{aligned} x &= (\alpha a + \alpha' I_H) + i(\beta b + \beta' I_H) \\ &= (\alpha a + i\beta b) + (\alpha' + i\beta')I_H \in \text{Span}\{A_c + I_H\} \subset (A_c)^1. \end{aligned}$$

Thus $(A_c)^1 = (A^1)_c$. □

The following result shows that the unitization of real operator algebras is independent of the choice of the Hilbert space H.

Theorem 5.2.5. *(Real Version of Meyer's Theorem) Let* $A \subseteq B(H)$ *be a real operator algebra, and suppose that* $I_H \notin A$. *Let* $\pi : A \longrightarrow B(K)$ *be a completely contractive homomorphism, where* K *is a real Hilbert space. We extend* π *to* $\pi^\circ : A^1 \longrightarrow B(K)$ *by* $\pi^\circ(a + \lambda I_H) = \pi(a) + \lambda I_K$, $a \in A$, $\lambda \in \mathbb{C}$. *Then* π° *is a completely contractive homomorphism.*

Proof. Consider $\pi_c : A_c \longrightarrow B(K)_c \cong B(K_c)$, which is a completely contractive homomorphism. Now extend π_c to $(\pi_c)^\circ : (A_c)^1 \longrightarrow B(K_c)$ by $(\pi_c)^\circ(a + \lambda I_{H_c}) = \pi_c(a) + \lambda I_{K_c}$, $a \in A_c$, $\lambda \in \mathbb{C}$. Then by the Meyer's Theorem for complex operator algebras ([14, Corollary 2.1.15]), $(\pi_c)^\circ$ is a completely contractive homomorphism. Let $a + \lambda I_H \in A^1$, then $(\pi_c)^\circ(a+\lambda I_H) = \pi_c(a)+\lambda I_K = \pi(a)+\lambda I_K = \pi^\circ(a+\lambda I_H)$. Thus $(\pi_c)^\circ|_{A^1} = \pi^\circ$ and hence π° is a completely contractive homomorphism. □

5.3 Real Injective Envelope

In this section we study in more detail the real injective envelopes of real operator spaces, which is mentioned by Ruan in [53].

Definition 5.3.1. Let X be a real operator space and let Y be a real operator space, such that there is a complete isometry $i : X \longrightarrow Y$. Then the pair (Y, i) is called an extension of X. An injective extension (Y, i) is a *real injective envelope* of X if there is no real injective space Z such that $i(X) \subset Z \subset Y$. We denote a real injective envelope by $(I(X), i)$ or simply by $I(X)$.

By the Arveson-Wittstock-Hahn-Banach theorem for real operator spaces, [53, Theorem 3.1], $B(H)$ is an injective real operator space for any real Hilbert space H. Thus a real operator space $X \subset B(H)$ is injective if and only if it is the range of a completely contractive idempotent map from $B(H)$ onto X.

Definition 5.3.2. If (Y, i) is an extension of X, then Y is a *rigid* extension if I_Y is the only completely contractive map which restricts to an identity map on X. We say that (Y, i) is an *essential* extension of X, if whenever $u : X \longrightarrow Z$ is a completely contractive map, for some real operator space Z, such that $u \circ i$ is a complete isometry, then u is a complete isometry.

Theorem 5.3.3. *If a real operator X is contained in a real injective operator space W, then there is an injective envelope Y of X such that $X \subset Y \subset W$.*

To prove this theorem we need to define some more terminology, and we also need the following two lemmas, which are the real analogies of [14, Lemma 4.2.2] and [14, Lemma 4.2.4], respectively. The proof of Lemma 5.3.5 uses the fact that, if X is a real operator spaces and H is any real Hilbert space, then a bounded net (u_t) in $CB(X, B(H))$ converges in weak*-topology to a $u \in CB(X, B(H))$ if and only if

$$\langle u_t(x)\zeta, \eta\rangle \to \langle u(x)\zeta, \eta\rangle \text{ for all } x \in X,\ \zeta, \eta \in H.$$

Definition 5.3.4. Let X is a subspace of a real operator space W. An *X-projection* on W is a completely contractive (real) idempotent map $\phi : W \to W$ which restricts to the identity map on X. An *X-seminorm* on W is a seminorm of the form $p(\cdot) = \|u(\cdot)\|$, for a completely contractive (real) linear map $u : W \to W$ which restricts to the identity map on X. Define a partial order $\leq$ on the sets of all X-projections, by setting $\phi \leq \psi$ if $\phi \circ \psi = \psi \circ \phi = \phi$. This is also equivalent to $\mathrm{Ran}(\phi) \subset \mathrm{Ran}(\psi)$ and $\mathrm{Ker}(\psi) \subset \mathrm{Ker}(\phi)$.

Lemma 5.3.5. *Let X be a subspace of a real injective operator space W.*

(i) *Any decreasing net of X-seminorms on W has a lower bound. Hence there exists a minimal X-seminorm on W, by Zorn's lemma. Each X-seminorm majorizes a minimal X-seminorm.*

(ii) *If p is a minimal X-seminorm on W, and if $p(\cdot) = \|u(\cdot)\|$, for a completely contractive linear map on W which restricts to the identity map on X, then u is a minimal X-projection.*

Lemma 5.3.6. *Let (Y, i) be an extension of real operator space X such that Y is injective. Then the following are equivalent:*

(i) *Y is an injective envelope of X,*

(ii) *Y is a rigid extension of X,*

(iii) *Y is an essential extension of X.*

Using the rigidity property of injective envelopes and a standard diagram chase, we can show that if (Y_1, i_1) and (Y_2, i_2) are two injective envelopes of a real operator space X then Y_1 and Y_2 are completely isometrically isomorphic via some map u such that $u \circ i_1 = i_2$. Hence the real injective envelope, if exists, is unique. The argument in [14, Theorem 4.2.6], and Lemma 5.3.5 and Lemma 5.3.6, prove Theorem 5.3.3. Thus the real injective envelope exists.

Lemma 5.3.7. *Let X be a real operator space with complexification X_c. Then X is real injective iff X_c is (complex) injective.*

Proof. First suppose that X is real injective, then there exists a completely contractive idempotent P, from $B(H)$ onto Z, for some real Hilbert space H. The complexification of P, $P_c : B(H)_c \longrightarrow X_c$ is clearly a (complex) completely contractive idempotent onto X_c. Since $B(H)_c \cong B(H_c)$, completely isometrically, Z_c is a (complex) injective operator space. Conversely, let X_c be a (complex) injective space and $Q : B(K) \longrightarrow X_c$ be a completely contractive (complex) linear surjective idempotent. Let $K = H_c$ where H is a real Hilbert space, so $Q : B(H_c) \cong B(H)_c \longrightarrow X_c$. Consider $\mathrm{Re}(Q) = \frac{Q+\bar{Q}}{2}$, where $\bar{Q}(T+iS) = \overline{Q(T-iS)}$. For any $T+iS \in B(H)_c$,

$$\overline{Q}^2(T+iS) = \overline{Q}(\overline{Q(T-iS)}) = \overline{Q^2(T-iS)} = \overline{Q(T-iS)} = \overline{Q}(T+iS).$$

Let $x+iy \in X_c$ and suppose that $Q(T+iS) = x-iy$ for some $T, S \in B(H)$. Then $\overline{Q}(T-iS) = x+iy$. Thus $\overline{Q}$ is an idempotent onto X_c. So $\overline{Q}Q(T+iS) = Q(T+iS)$ and $Q\overline{Q}(T+iS) = \overline{Q}(T+iS)$, for all $T+iS \in B(H)_c$. Thus for $T \in B(H)$, $(\mathrm{Re}(Q))^2(T) = \frac{Q^2(T)+Q\overline{Q}(T)+\overline{Q}Q(T)+\overline{Q}^2(T)}{4} = \frac{2Q(T)+2\overline{Q}(T)}{4} = \mathrm{Re}(Q)(T)$. If $x \in X \subset X_c$, then $Q(x) = x$ and $\overline{Q}(x) = x$, so $\mathrm{Re}(Q)(x) = x$. This shows that $\mathrm{Re}(Q) : B(H) \longrightarrow X$ is a (real) linear completely contractive idempotent onto X. Hence X is real injective. $\square$

The next result is a real analogy of a Choi-Effros theorem (see e.g., [14, Theorem 1.3.13]). It is shown in the last paragraph of [53, pg. 492]) that the argument in the complex version of the theorem can be reproduced to prove (i) of the following result.

Theorem 5.3.8 (Choi-Effros)**.** *Let A be a unital real C^*-algebra and let $\phi : A \longrightarrow A$ be a selfadjoint, completely positive, unital, idempotent map. Then*

(i) $R = \mathrm{Ran}(\phi)$ *is a unital real C^*-algebra with respect to the original norm, involution, and vector space structure, but new product $r_1 \circ_\phi r_2 = \phi(r_1 r_2)$,*

(ii) $\phi(ar) = \phi(\phi(a)r)$ *and* $\phi(ra) = \phi(r\phi(a))$, *for $r \in R$ and $a \in A$,*

(iii) *If B is the C^*-algebra generated by the set R, and if R is given the product $\circ_\phi$, then $\phi\,|_B$ is a $*$-homomorphism from B onto R.*

Proof. Let $\phi : A \longrightarrow A$ be a selfadjoint, completely positive, unital idempotent map. Then ϕ is completely contractive, by [53, Proposition 4.1], and hence $\phi_c : A_c \longrightarrow A_c$ is a completely contractive, unital idempotent onto $\mathrm{Ran}(\phi)_c$. By the Choi-Effros Lemma for complex operator systems, [14, Theorem 1.3.13], $\mathrm{Ran}(\phi)_c$ is a C^*-algebra with a new product given by $(r_1 + ir_2) \circ (s_1 + is_2) = \phi_c((r_1 + ir_2)(s_1 + is_2))$, $r_1, r_2, s_1, s_2 \in R$. For $r, s \in R$, $r \circ s = \phi_c(rs) = \phi(rs) \in R$. By [43, Proposition 5.1.3], R is a real C^*-algebra with this product. Further, $\phi(ar) = \phi_c(ar) = \phi_c(\phi_c(a)r) = \phi(\phi(a)r)$, and similarly $\phi(ra) = \phi(r\phi(a))$, for all $a \in A, r \in R$. Let $C = C^*(R_c)$ be the (complex) C^*-algebra generated by R_c in A_c, then by [14, Theorem 1.3.13 (iii)], $(\phi_c)|_C$ is a $*$-homomorphism from C onto R_c. Let $B = C^*(R)$ be the real C^*-subalgebra of A generated by R. It is easy to see that $C^*(R_c) = C^*(R)_c$. Clearly, since $C^*(R) \subset C^*(R_c)$, $C^*(R)_c \subset C^*(R_c)$. Also,

$$\mathrm{Span}\{s_1 s_2 \dots s_n \ : \ n \in \mathbb{N}\} = \mathrm{Span}_{\mathbb{R}}\{r_1 r_2 \dots r_n \ : \ n \in \mathbb{N}\} + i\mathrm{Span}_{\mathbb{R}}\{r_1^{'} r_2^{'} \dots r_n^{'} \ : \ n \in \mathbb{N}\},$$

where s_i is in R_c, and $r_i, r_i^{'}$ is an element of R. If $a \in C^*(R_c) \subset A_c$ then $a = x + iy$ is the limit of $a_t \in \mathrm{Span}\{s_1 s_2 \dots s_n \ : \ n \in \mathbb{N}\}$. Then $a_t = x_t + iy_t$, where $x_t \in \mathrm{Span}_{\mathbb{R}}\{r_1 r_2 \dots r_n \ : \ n \in \mathbb{N}\}$, $y_t \in \mathrm{Span}_{\mathbb{R}}\{r_1^{'} r_2^{'} \dots r_n^{'} \ : \ n \in \mathbb{N}\}$. Also, if we suppose that $A_c \subset B(H)_c$, for some real Hilbert space H, then it is easy to see that $x_t \longrightarrow x$, $y_t \longrightarrow y$. Hence, $(\phi_c)_B = \phi|_B$ is a $*$-homomorphism from B onto R. ☐

Remark. Let A and B be real C^*-algebras, and let $\phi : A \longrightarrow B$ be a unital completely contractive map. Then ϕ_c is a (complex) completely contractive linear map between complex C^*-algebras A_c and B_c. So ϕ_c is completely positive and hence selfadjoint. Since $\phi = \phi_c|_A$, ϕ is also selfadjoint. Thus a completely contractive unital map between real C^*-algebras is selfadjoint. As a result, we can replace the completely positive and selfadjoint condition in Theorem 5.3.8 above, with the condition that ϕ is completely contractive.

Theorem 5.3.9. *X be a unital real operator space, then there is an injective envelope*

$I(X)$ which is a unital real C^*-algebra.

Proof. Let $X \subset B(H)$ for some real Hilbert space H. Since $B(H)$ is injective, we can find an injective envelope of X such that $X \subset I(X) \subset B(H)$. As $I(X)$ is injective, so the identity map on $I(X)$ extends to $\phi : B(H) \longrightarrow B(H)$ such that ϕ is a completely contractive idempotent onto $I(X)$. By Theorem 5.3.8 and the remark above, $\mathrm{Ran}(\phi) = I(X)$ becomes a unital real C^*-algebra with the new product. $\square$

Proposition 5.3.10. *Let X be a real (or complex) Banach space, then* $\mathrm{Min}(I(X)) = I(\mathrm{Min}(X))$.

Proof. Let X be a real Banach space. Since $I(X)$ is an injective Banach space, and contractive maps into $\mathrm{Min}(X)$ are completely contractive, it clear that $\mathrm{Min}(I(X))$ is a real injective operator space. Let $i : X \longrightarrow I(X)$ be the canonical isometry, and let $j : I(X) \longrightarrow C(\Omega, \mathbb{R})$ be an isometric embedding of $I(X)$, for some compact, Hausdorff space Ω. Then $j : \mathrm{Min}(I(X)) \longrightarrow C(\Omega, \mathbb{R})$ and $j \circ i : \mathrm{Min}(X) \longrightarrow C(\Omega, \mathbb{R})$ are complete isometries. Thus $(\mathrm{Min}(I(X)), i)$ is a real injective extension of $\mathrm{Min}(X)$. Further suppose that $u : \mathrm{Min}(I(X)) \longrightarrow \mathrm{Min}(I(X))$ is a complete contraction which restricts to the identity map on $\mathrm{Min}(X)$. Then by the rigidity of $I(X)$, u is an isometry into $\mathrm{Min}(I(X))$, and hence a complete isometry. Thus $(\mathrm{Min}(I(X)), i)$ is a rigid extension of $\mathrm{Min}(X)$, and hence $I(\mathrm{Min}(X)) = \mathrm{Min}((I(X)))$. $\square$

Definition 5.3.11. Let X be a real unital operator space. Then we define a *C^*-extension* of X to be a pair (B, j) consisting of a unital real C^*-algebra B, and a complete isometry $j : X \longrightarrow B$, such that $j(X)$ generates B as a C^*-algebra. A C^*-extension (B, i) is a *C^*-envelope* of X if it has the the following universal property: Given any C^*-extension (A, j)

of X, there exists a (necessarily unique and surjective) real $*$-homomorphism $\pi : A \longrightarrow B$, such that $\pi \circ j = i$.

Using Theorem 5.3.8, Theorem 5.3.9, and the argument in [14, 4.3.3], we can show that the C^*-subalgebra of $I(X)$ generated by $i(X)$ is a C^*-envelope of X, where the pair $(I(X), i)$ is an injective envelope of X. Thus the C^*-envelope exists for every unital operator space X.

A *real operator system* is a (closed) subspace $\mathcal{S}$ of B(H), H a real Hilbert space, such that $\mathcal{S}$ contains I_H, and $\mathcal{S}$ is selfadjoint, i.e., $x^* \in \mathcal{S}$ if and only if $x \in \mathcal{S}$. Note that a positive element in $B(H)$, H a real Hilbert space, need not be selfadjoint. For instance, consider the 2×2 matrices over $\mathbb{R}$, then $x = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ is positive, i.e., $\langle x\zeta, \zeta \rangle \geq 0$ for all $\zeta \in \mathbb{R}^2$, but $x \neq x^*$. Thus, we say that an element $x \in \mathcal{S}(X) \subset B(H)$ is *positive*, if for all $\zeta, \eta \in H$, $\langle x\zeta, \eta \rangle = \langle \zeta, x\eta \rangle$ (selfadjoint), and $\langle x\zeta, \zeta \rangle \geq 0$. If $x \in B(H, K)$, H and K real Hilbert spaces, then

$$\begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} \geq 0 \iff \|x\| \leq 1. \tag{5.3.1}$$

In [53], Ruan considers real operator systems and shows that a unital selfadjoint map between two real operator systems is completely contractive if and only if it is completely positive. It is also shown that the Stinespring theorem, the Arveson's Extension Theorem, and the Kadison-Schwarz inequality hold true, with an added hypothesis that the maps be selfadjoint. We can show using the Stinespring theorem that Proposition 1.3.11 and 1.3.12 from [14] are true in the real setting.

If $X \subset B(H)$ is a real operator space, then we can define the *real Paulsen system* as

$$\mathcal{S}(X) = \begin{bmatrix} \mathbb{R}I_H & X \\ X^\star & \mathbb{R}I_H \end{bmatrix} = \left\{ \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} : x, y \in X \text{ and } \lambda, \mu \in \mathbb{R} \right\} \subset M_2(B(H)).$$

The next lemma is the real version of Paulsen lemma, and it can be proved using the argument in [14, Lemma 1.3.15], Equation (5.3.1), and that the map ϕ, defined below, is selfadjoint. This lemma shows that as a real operator system (i.e., up to complete order isomorphism) $\mathcal{S}(X)$ only depends on the operator space structure of X, and not on its representation on H.

Lemma 5.3.12. *For $i = 1, 2$, let H_i and K_i be real Hilbert spaces, and $X_i \subset B(K_i, H_i)$. Suppose that $u : X_1 \to X_2$ is a real linear map. Let $\mathcal{S}_i$ be the real Paulsen systems associated with X_i inside $B(H_i \oplus K_i)$. If u is contractive (resp. completely contractive, completely isometric), then*

$$\phi : \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \to \begin{bmatrix} \lambda & u(x) \\ u(y)^* & \mu \end{bmatrix}$$

is positive (resp. completely positive and completely contractive, a complete order injection) as a map from $\mathcal{S}_1$ to $\mathcal{S}_2$.

Let $X \subset B(H)$ be a real operator space and let $\mathcal{S}(X) \subset M_2(B(H))$ be the associated real Paulsen system. Then $I(\mathcal{S}(X)) \subset M_2(B(H))$ is a unital C^*-algebra, by Theorem 5.3.9, and there is a completely positive idempotent map ϕ from $M_2(B(H))$ onto $I(\mathcal{S}(X))$. Let p and q be the canonical projections $I_H \oplus 0$ and $0 \oplus I_H$, then $\phi(p) = p$ and $\phi(q) = q$. So,

$$I(\mathcal{S}(X)) = \begin{bmatrix} pI(\mathcal{S}(X))p & pI(\mathcal{S}(X))q \\ qI(\mathcal{S}(X))p & qI(\mathcal{S}(X))q \end{bmatrix}.$$

Using Lemma 5.3.6 and Lemma 5.3.12, and the argument in [14, Theorem 4.4.3], we can show that the 1-2-corner, $pI(\mathcal{S}(X))q$, of $I(\mathcal{S}(X))$ is an injective envelope of X. As a corollary, we get the following which is the real analogue of the Hamana-Ruan characterization of injective operator spaces.

Theorem 5.3.13. *A real operator space X is injective if and only if $X \cong pA(1-p)$ completely isometrically, for a projection p in an injective real C^*-algebra A.*

A real TRO is a closed linear subspace Z of $B(K, H)$, for some real Hilbert spaces K and H, satisfying $ZZ^\star Z \subset Z$. For $x, y, z \in Z$, xy^*z is called the *triple or ternary product* on Z, sometimes written as $[x, y, z]$. A *subtriple* of a TRO Z is a closed subspace Y of Z satisfying $YY^\star Y \subset Y$. A *triple morphism* between TROs is a linear map which respects the triple product: thus $T([x, y, z]) = [Tx, Ty, Tz]$. In the construction of the real injective envelope, discussed above, let $Z = pI(\mathcal{S}(X))q$, then $ZZ^\star Z \subset Z$ with the product of the C^*-algebra $I(\mathcal{S}(X))$. In terms of the product in $B(H)$, $[x, y, z] = P(xy^*z)$ for $x, y, z \in Z$. So if X is a TRO, then the triple product on X coincides with the triple product on X coming from $I(X)$. Thus $pI(\mathcal{S}(X))q = I(X)$ is a TRO. If two TROs X and Y are completely isometrically isomorphic, via say u, then by Lemma 5.3.12, we can extend u to a complete order isomorphism between the Paulsen systems. Further, this map extends to a completely isometric unital surjection $\tilde{u}$ between the the injective envelopes $I(\mathcal{S}(X))$ and $I(\mathcal{S}(Y))$, which are (real) unital C^*-algebras. By Lemma 5.1.2, $\tilde{u}$ is a $*$-isomorphism, and hence a ternary isomorphism between when restricted to X. Thus u is a triple isomorphism. Thus a real operator space can have at most one triple product (up to complete isometry).

Define $T(X)$ to be the smallest subtriple of $I(X)$ containing X. Then it is easy to see that

$$T(X) = \overline{\text{Span}}\{x_1 x_2^* x_3 x_4^* \dots x_{2n+1} \ : \ x_1, x_2, \dots x_{2n+1} \in X\}.$$

Let $B = T(X)^\star T(X)$, $T(X)$ regarded as a subtriple of $I(X)$ in $I(\mathcal{S}(X))$. Then B is a C^*-subalgebra of 2-2-corner of $I(\mathcal{S}(X))$, and hence of $I(\mathcal{S}(X))$. Define $\langle y, z\rangle = y^*z$ for $y, z \in T(X)$, a B-valued inner product. This inner product is called the *Shilov inner product* on X.

5.4 One-Sided Real M-Ideals

Let X be a real operator space. If P is a projection, i.e., $P = P^2$ and $P^* = P$ (equivalently $\|P\| \le 1$), then define linear mappings

$$\nu_P^c : X \longrightarrow C_2(X) : x \mapsto \begin{bmatrix} P(x) \\ x - P(x) \end{bmatrix},$$

$$\mu_P^c : C_2(X) \longrightarrow X : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto P(x) + (\mathrm{Id} - P)(y).$$

Then $\mu_P^c \circ \nu_P^c = I$.

Definition 5.4.1. A *complete left M-projection* on X is a linear idempotent on X such that the map $\nu_P^c : X \longrightarrow C_2(X) : x \mapsto \begin{bmatrix} P(x) \\ x - P(x) \end{bmatrix}$ is a complete isometry.

Proposition 5.4.2. *If X is a real operator space and $P : X \longrightarrow X$ is a projection, then P is a complete left M-projection if and only if μ_P^c and ν_P^c are both completely contractive.*

Proof. If ν_P^c is completely isometric, then

$$\|P(x) + y - P(y)\| = \left\| \begin{bmatrix} P(x) \\ y - P(x) \end{bmatrix} \right\| \le \left\| \begin{bmatrix} P(x) \\ x - P(x) \\ P(y) \\ y - P(y) \end{bmatrix} \right\| = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|,$$

and thus μ_P^c is contractive. These calculations work as well for matrices. The converse follows from the fact that $\mu_P^c \circ \nu_P^c = I$. $\square$

Proposition 5.4.3. *The complete left M-projections in a real operator space X are just the mappings $P(x) = ex$ for a completely isometric embedding $X \hookrightarrow B(H)$ and an orthogonal projection $e \in B(H)$.*

Proof. If $P : X \longrightarrow X$ is a complete left M-projection, then fix an embedding $X \subset B(H)$ for some real Hilbert space H. By the definition, the mapping

$$\sigma : X \hookrightarrow B(H \oplus H) : x \mapsto \begin{bmatrix} P(x) & 0 \\ (I-P)(x) & 0 \end{bmatrix}$$

is completely isometric. We have that

$$\sigma(P(x)) = \begin{bmatrix} P(x) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sigma(x),$$

and thus $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in B(H \oplus H)$ is the desired left projection relative to the embedding σ. The converse follows from the following:

$$\begin{aligned} \left\| \begin{bmatrix} P(x) \\ x - P(x) \end{bmatrix} \right\|^2 &= \left\| \begin{bmatrix} ex \\ x - ex \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} x^*e & x^* - x^*e \end{bmatrix} \begin{bmatrix} ex \\ x - ex \end{bmatrix} \right\| \\ &= \|x^*ex + x^*(1-e)x\| = \|x^*x\| = \|x\|^2 . \end{aligned}$$

$\square$

Let X be a real operator space. We say a map $u : X \longrightarrow X$ is a *left multiplier* of X if there exists a linear complete isometry $\sigma : X \longrightarrow B(H)$ for some real Hilbert space H, and an operator $S \in B(H)$ such that

$$\sigma(u(x)) = S\sigma(x),$$

for all $x \in X$. We denote the set of all left multipliers of X by $\mathcal{M}_\ell(X)$. Define the multiplier norm of u, to be the infimum of $\|S\|$ over all such possible H, S, σ. We define a *left adjointable map* of X to be a linear map $u : X \longrightarrow X$ such that there exists a linear complete isometry $\sigma : X \longrightarrow B(H)$ for some real Hilbert space H, and an operator $A \in B(H)$ such that

$$\sigma(u(x)) = A\sigma(x) \text{ for all } x \in X, \text{ and } A^*\sigma(X) \subset \sigma(X).$$

The collection of all left adjointable maps of X is denoted by $\mathcal{A}_\ell(X)$. Every left adjointable map of X is a left multiplier of X, that is, $\mathcal{A}_\ell(X) \subset \mathcal{M}_\ell(X)$.

Theorem 5.4.4. *Let X be a real operator space and let $u : X \longrightarrow X$ be a linear map. Then the following are equivalent:*

(i) *u is a left multiplier of X with norm ≤ 1.*

(ii) *The map $\tau_u : C_2(X) \longrightarrow C_2(X) : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} u(x) \\ y \end{bmatrix}$, is completely contractive.*

(iii) *There exists a unique 'a' in the 1-1-corner of $I(\mathcal{S}(X))$ such that $\|a\| \leq 1$ and $u(x) = ax$ for all $x \in X$.*

By a direct application of the argument in [14, Theorem 4.5.2], we get that (i) ⇒ (ii) and (iii) ⇒ (i). For the implication (ii) ⇒ (iii), using the machinery we developed for real operator spaces in the last section, we can replicate the elegant proof due to Paulsen mentioned in [14, Theorem 4.5.2]. Note that the map Φ' in [14, Theorem 4.5.2], is selfadjoint, therefore by the real version of the Arveson's extension theorem from [53], Φ' extends to a completely positive and selfadjoint map Φ, on the C^*-algebra M. By the real version of the Stinespring's Theorem [53, Theorem 4.3], the argument in [14, Proposition

1.3.11] can be reproduced, and hence [14, Proposition 1.3.11] holds for real C^*-algebras. Since Φ fixes the C^*-subalgebra

$$B = \begin{bmatrix} \mathbb{C} & 0 & 0 \\ 0 & I_{11} & I(X) \\ 0 & I(X)^\star & I_{22} \end{bmatrix}$$

of M, so Φ is a $*$-homomorphism on B. By [14, Proposition 1.3.11], $\Phi : M \longrightarrow M$ is a bimodule map over B. The rest of the argument follows verbatim.

Theorem 5.4.5. *Let X be a real operator space then $\mathcal{M}_\ell(X)$ is a real operator algebra. Further, $\mathcal{A}_\ell(X)$ is a real C^*-algebra.*

Proof. We use the completely isometric embeddings $X \subset I(X) \subset \mathcal{S}(X)$, and the notation from Section 5.3. Let

$$IM_l(X) = \{a \in pI(\mathcal{S}(X))p \ : \ aX \subset X\}.$$

Then $IM_l(X)$ is a subalgebra of the real C^*-algebra $pI(\mathcal{S}(X))p$, and hence is a real operator algebra. Define $\theta : IM_l(X) \longrightarrow \mathcal{M}_\ell(X)$ as $\theta(a)(x) = ax$ for any $x \in X$. Then θ is an isometric isomorphism. Using the canonical identification $M_n(\mathcal{M}_\ell(X)) \cong \mathcal{M}_\ell(C_n(X))$, define a matrix norm on $M_n(\mathcal{M}_\ell(X))$ for each n. With these matrix norms, and a matricial generalization of the argument after Theorem 5.4.4 (see e.g. [14, 4.5.4]), θ is a complete isometric isomorphism. Hence all the 'multiplier matrix norms' are norms, and $\mathcal{M}_\ell(X) \cong IM_l(X)$ is a real operator algebra. Since $\mathcal{A}_\ell(X) = \mathcal{M}_\ell(X) \cap \mathcal{M}_\ell(X)^\star$, we have that

$$\mathcal{A}_\ell(X) \cong \{a \in pI(\mathcal{S}(X))p \ : \ aX \subset X \text{and } a^*X \subset X\}.$$

Hence $\mathcal{A}_\ell(X)$ is a real C^*-algebra. □

Theorem 5.4.6. *If P is a projection on a real operator space X, then the following are equivalent:*

(i) *P is a complete left M-projection.*

(ii) *τ_P^c is completely contractive.*

(iii) *P is an orthogonal projection in the real C^*-algebra $\mathcal{A}_\ell(X)$.*

(iv) *$P \in \mathcal{M}_\ell(X)$ with the multiplier norm ≤ 1.*

(v) *The maps ν_P^c and μ_P^c are completely contractive.*

The above theorem can be easily seen from Proposition 5.4.2, Proposition 5.4.3, and Theorem 5.4.4.

Definition 5.4.7. A subspace J of a real operator space X is a *right M-ideal* if $J^{\perp\perp}$ is the range of a complete left M-projection on X^{**}.

Proposition 5.4.8. *A projection $P : X \longrightarrow X$ is a complete left M-projection if and only if P_c is a (complex) complete left M-projection on X_c.*

Proof. We first note that $C_2(X_c) \cong C_2(X)_c$, completely isometrically, via the shuffling map

$$\begin{bmatrix} \begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix} \\ \begin{bmatrix} y_1 & -y_2 \\ y_2 & y_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} & - \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} & \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \end{bmatrix}.$$

Also,

$$\begin{aligned}(\tau_P)_c\left(\begin{bmatrix} x \\ v \end{bmatrix} + i\begin{bmatrix} y \\ w \end{bmatrix}\right) &= \begin{bmatrix} P(x) \\ v \end{bmatrix} + i\begin{bmatrix} P(y) \\ w \end{bmatrix} \\ &= \begin{bmatrix} P(x)+iP(y) \\ v+iw \end{bmatrix} \\ &= \tau_{(P_c)}\left(\begin{bmatrix} x+iy \\ v+iw \end{bmatrix}\right).\end{aligned}$$

If P is a complete left M-projection, then by Theorem 5.4.6, τ_P and hence, $(\tau_P)_c$ is completely contractive. By the above $\tau_{(P_c)}$ is completely contractive and so, P_c is a complete left M-projection. Conversely, if P_c is a complete left M-projection, then $\tau_{(P_c)}$ is completely contractive. Since $\tau_{(P_c)}|_{C_2(X)} = \tau_P$, τ_P is a complete contraction and hence P is a complete left M-projection, by Theorem 5.4.6. □

Corollary 5.4.9. *A subspace J in a real operator space X is a right M-ideal if and only if J_c is a (complex) right M-ideal in X_c.*

Proof. Since $\begin{bmatrix} x_t & -y_t \\ y_t & x_t \end{bmatrix}$ converge weak* in $(X_c)^{**}$ if and only if both (x_t) and (y_t) converge weak* in X^{**}, if $J \subset X$, then $(J_c)^{\perp\perp} = \overline{J_c}^{w^*} = (\overline{J}^{w^*})_c = (J^{\perp\perp})_c$. If J is a real right M-ideal and if $P : X^{**} \longrightarrow J^{\perp\perp}$ is a (real) left M-projection, then by the above corollary $P_c : (X^{**})_c \longrightarrow (J^{\perp\perp})_c$ is a (complex) left M-projection. Let Q be the induced map from $(X_c)^{**}$ onto $(J_c)^{\perp\perp}$. So the diagram

$$\begin{tikzcd}(X^{**})_c \arrow[r, "P_c"] \arrow[d, leftrightarrow] & (J^{\perp\perp})_c \arrow[d, leftrightarrow] \\ (X_c)^{**} \arrow[r, "Q"'] & (J_c)^{\perp\perp}\end{tikzcd}$$

commutes and thus Q is an idempotent. Also, since the diagram

$$\begin{tikzcd}
C_2((X^{**})_c) \arrow[r, "\tau_{(P_c)}"] \arrow[d, leftrightarrow, "\text{c.i.}"'] & C_2((X^{**})_c) \arrow[d, leftrightarrow, "\text{c.i.}"] \\
C_2((X_c)^{**}) \arrow[r, "\tau_Q"'] & C_2((X_c)^{**})
\end{tikzcd}$$

commutes, and $\tau_{(P_c)}$ is a complete contraction, so τ_Q is a complete contraction. Hence J_c is a right M-ideal in X_c. Conversely, if P is a complete left M-projection from $(X_c)^{**} = (X^{**})_c$ onto $(J_c)^{\perp\perp} = (J^{\perp\perp})_c$, then let $Q = \text{Re}(P)$. Then a similar argument as in Lemma 5.3.7 shows that Q is an idempotent from X^{**} onto $J^{\perp\perp}$. Also since τ_Q is the restriction of τ_P to $C_2(X^{**})$, τ_Q is completely contractive. Thus J is a real right M-ideal. $\square$

Corollary 5.4.10. *The right M-ideals in a real C^*-algebra A are precisely the closed right ideals in A.*

Note that by Corollary 5.4.9, and Proposition 5.1.3, it is clear that X is right M-ideal in X^{**} if and only if X_c is right M-ideal in $(X_c)^{**}$.

We say that a real operator space X is right M-embedded if X is a right M-ideal in X^{**}. Now we have everything in place to start the real version of Chapter 3. This will be presented elsewhere.

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