EXTENSIONS OF APPROXIMATELY UNITAL OPERATOR ALGEBRAS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

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August 2013

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Abstract

Extension theory in the study of C*-algebras has been important in many aspects. For that reason, a theory of extensions of operator algebras, with as direct a link as possible to the C*-algebras theory, will be developed. In large part the theory will be developed via the natural C*-algebras generated by any operator algebra. The underlying structure used will be a generalization of Busby's theory of extensions of C*-algebras and results concerning universal completions for particular diagrams of Eilers, Loring, and Pedersen. Examples of our contributions as a result of this approach are a definition of the amalgamated free product of operator algebras, as well as a Tietze extension theorem for σ-unital operator algebras.

Contents

Chapter 1

Introduction

1.1 Description of Results and Background

Extensions of C*-algebras have been widely studied and have proven to be a valuable tool in understanding C*-algebras. However, many of the properties that are automatic for $*$-homomorphisms require additional hypotheses for general completely contractive morphisms. In Chapter 2, Lemma 2.1.2 will give a sufficient condition for the restriction of a complete quotient morphism to be a complete quotient morphism is given. Lemma 2.1.2 will be used in relating completely contractive morphisms to $*$-homomorphisms between C*-covers, which are automatically complete quotient morphisms onto their ranges. Since every two-sided ideal is the kernel of a complete quotient morphism, Lemma 2.1.2 has many consequences. Most importantly it will be used to relate an ideal of an operator algebra to the corresponding C*-covers formed in a containing C*-algebra.

The four diagrams of [17] are shown below. Diagram I is also used in [10].

Diagram I	Diagram II
$\begin{array}{ccccccccc} 0 & \longrightarrow & \circ & \dashrightarrow & \circ & \dashrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$	$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \dashrightarrow & \circ & \dashrightarrow & \circ & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$
Diagram III	**Diagram IV**
$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \dashrightarrow & \circ & \dashrightarrow & \circ & \longrightarrow & 0. \end{array}$	$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & \circ & \dashrightarrow & \circ & \dashrightarrow & C_1 & \longrightarrow & 0 \end{array}$

In 1968 Busby published his work on extensions of C*-algebras [10]. In this dissertation, results concerning non-selfadjoint operator algebras will be obtained using classic Busby extension theory as it applies in the non-selfadjoint situation. This generalization of Busby theory to general operator algebras will be developed in Chapter 3. As with C*-algebras, Busby theory relates to what are called Diagram I completions in [17]. An additional hypothesis will be that the first algebra of an extension have a cai, or contractive approximate identity. Many of the necessary applications of the multiplier algebras critical in C*-algebra extension theory require that the related algebra have a cai. The results of Chapter 4 for sub-extensions will be used to develop the theory of covering extensions in Chapter 5. A covering extension will be defined to be a C*-algebra extension which contains an extension of generating operator algebras as a sub-extension. Most of the results in Chapters 2-5 have appeared in a joint paper [6] with my advisor.

In Chapter 6 the remaining diagrams of [17] are discussed and will be generalized to operator algebras with modest additional hypotheses. Primary among the hypotheses is that the kernel of a complete quotient morphism has a cai which coincides with the same hypothesis in the extension theory of Chapter 2. The first vertical $*$-homomorphisms of a completion of a diagram of [17] is often required to be proper. This gives a natural requirement for a general operator algebra diagram completion, that is the first algebras have a cai. It will be necessary to investigate in detail one main tool used in [10] and [17]. That is the pullback algebra from [10]. Another tool is what will be termed covering extensions in Chapter 5.

Two of the primary results that are developed in the non-selfadjoint case are Tietze extension results in Chapter 7. The C*-algebra Tietze Extension Theorem for σ-unital C*-algebras was first proved in 1973 by Akemann, Pederson, and Tomiyama [1] viewing C*-algebras as noncommutative topology. The first Tietze result in this dissertation does not require a σ-unital hypothesis and so increases the class of $*$-homomorphisms that extend surjectively to the multiplier algebras. The second result gives a sufficient condition for the extension of a complete quotient morphism between σ-unital operator algebras to extend surjectively to the multiplier algebras.

1.2 The Non-selfadjoint Setting

An *operator algebra* will mean a possibly non-selfadjoint closed subalgebra of the bounded operators on a Hilbert space H. This category, with morphisms the completely bounded homomorphisms, will be designated by OA. A subcategory, $AUOA$

includes operator algebras that are approximately unital. A given operator algebra A will be said to be *approximately unital* if it has a two-sided *approximate identity.* This is a net designated $(e_t)_t$ with $\|e_t\| \leq 1$ for all t, and given an $a \in A$, $ae_t \to a$ and $e_t a \to a$ in the norm topology. Generally it will not be assumed that an operator algebra is in the category AUOA unless specified.

A *morphism* $\pi : A \to B$ will mean a homomorphism between operator algebras that is linear and respects the operations of addition and multiplication. A *proper morphism* will be a morphism which takes a cai of A to a cai of B. It has been shown that the appropriate morphisms for the category of operator algebras are the *completely bounded morphisms.* As in [5, 1.2.1], a morphism $\pi : X \to Y$ is completely bounded if for each nth amplification of π, $\|\pi_n\| \leq r$ for some real number r. In this case

$$\|\pi\|_{\text{cb}} = \sup\{\|[\pi(x_{ij})]\|_n : \|[x_{ij}]\| \leq 1, \text{ for all n } \in \mathbb{N}\}.$$

A completely contractive morphism is one that is norm decreasing at all matrix levels. Two types of completely bounded morphisms of particular interest are the complete quotient morphisms and complete isometries. A complete quotient morphism is a surjective completely contractive morphism which takes ball(A) onto ball(B) at all matrix levels. Here $\text{ball}(A) = \{a \in A : \|a\| < 1\}$. The closed ball is denoted $\text{Ball}(A) = \{a \in A : \|a\| \leq 1\}$. A complete isometry is an isometric morphism which is also an isometry at all matrix levels. See [5, 1.2.1]. Two objects, A and B, in the category of operator algebras will be considered isomorphic if there exists a completely isometric isomorphism between A and B, and in this case we write $A \cong B$.

The multiplier algebra of an operator algebra A is well defined when A has a cai [5, Theorem 2.6.2 and Section 2.6.7]. Additionally, there are important morphisms between multiplier algebras that can be formed in certain cases. A completely contractive morphism $\pi : A \to \mathcal{M}(B)$, where A is approximately unital, will be called *multiplier-nondegenerate* if for every element $b \in B$ and any cai $(e_t)_t$ of A, $b\pi(e_t) \to b$ and $\pi(e_t)b \to b$. Here A is approximately unital and, of course, $\mathcal{M}(B)$ is unital. An important result [5, Theorem 2.6.12] says that if B is also approximately unital, then π extends to a unital completely contractive morphism $\hat{\pi} : \mathcal{M}(A) \to \mathcal{M}(B)$. Even in the case that A is not approximately unital, if $A \subset B$ as a two-sided ideal, then there is a canonical completely contractive morphism, which will be designated $\sigma : B \to \mathcal{M}(A)$, that takes an element $b \in B$ to $m \in \mathcal{M}(A)$ where $\sigma(b)a = ma$ and $a\sigma(b) = am$. This is the unique such morphism which will be the identity morphism on A.

A C*-*cover* of an operator algebra A consists of a pair $(\mathcal{D}, i)$ where $\mathcal{D}$ is a C*-algebra and $i : A \to \mathcal{D}$ is a completely isometric map with the property that $i(A)$ generates $\mathcal{D}$ as a C*-algebra. There are two C*-covers of particular interest. The maximal C*-cover of A, denoted $\mathrm{C}^*_{\max}(A)$, has the universal property that any completely contractive morphism $\pi : A \to B$, where B is a C*-algebra, extends to a morphism $\varphi : \mathrm{C}^*_{\max}(A) \to B$. The C*-envelope of A, designated $\mathrm{C}^*_{\mathrm{e}}(A)$, can be defined if A is unital as the C*-algebra generated by A in its injective envelope

$(I(A), j)$. For unital operator algebras, the injective envelope is the smallest injective C*-algebra containing A completely isometrically isomorphically with j a unital homomorphism [5, Corollary 4.2.8]. If A is not unital, but contains a cai, then any injective envelope for A^1 is also an injective envelope for A [5, Corollary 4.2.8]. In the approximately unital case, the C^*-envelope is defined as the C*-algebra generated by A in $\mathrm{C}^*_{\mathrm{e}}(A^1)$ where A^1 is a unitization of A [5, §4.3.4]. The universal property of $\mathrm{C}^*_{\mathrm{e}}(A)$ is that given any other C*-cover $(\mathcal{D}, j)$ of A, there exists a necessarily unique surjective $*$-homomorphism $\pi : \mathcal{D} \to \mathrm{C}^*_{\mathrm{e}}(A)$ such that $\pi \circ j = i$ where $i : A \to \mathrm{C}^*_{\mathrm{e}}(A)$. The C*-covers will be important in Chapter 5.

We will discuss other operator algebraic constructions throughout the paper as we need them.

Chapter 2

Approximately Unital Ideals and C^*-covers

In this chapter A, B, and C will represent operator algebras with A containing a cai. Algebras represented by a scripted letter may represent C^*-algebras, but will be clearly defined as such in those cases. The morphism θ will always be a complete quotient morphism, which of course is a homomorphism. All other morphisms are completely contractive homomorphisms unless otherwise specified. The morphism σ will always map into the multiplier algebra of an operator algebra which has a cai.

2.1 The Restriction of a Complete Quotient Morphism

Among the most useful properties of C*-algebras is that $*$-homomorphisms are open maps, or equivalently that they have closed range, and when factoring by the kernel necessarily induce complete isometries. For the more general case of possibly non-selfadjoint operator algebras neither property is necessarily true. To see this, the following example is constructed using [5, Proposition 2.2.11].

Example. A completely contractive morphism in AUOA that is not a complete quotient morphism.

If E is any Banach space, then an operator space structure can be imposed on E by viewing E as a subspace of the commutative C*-algebra, call it C, of the continuous functions on the ball of E^*. Taking $C \subset B(H)$ for a Hilbert space H, E can be viewed as an operator subalgebra of $M_2(B(H))$ as follows:

$$\mathcal{U}(E) = \left\{ \begin{bmatrix} \lambda_1 I & x \\ 0 & \lambda_2 I \end{bmatrix} : \, x \in E, \; \lambda_1, \; \lambda_2 \in \mathbb{C}, \; I \text{ the identity operator on } H \right\}.$$

From the matrix multiplication it is clear that any two elements of E inside $\mathcal{U}(E)$ have zero product under this embedding. If $\alpha : E \to E_1$ is any contractive linear map from E into another Banach space E_1 with nonclosed range, then E_1 can be given a similar structure, $\mathcal{U}(E_1)$, as the subspace of a C*-algebra C_1. From [5, Proposition 2.2.11] it can be see that α can be made into a unital completely contractive operator algebra homomorphism with nonclosed range. This can then be extended to a completely contractive homomorphism $\alpha' : \mathcal{U}(E) \oplus^{\infty} \mathbb{C} \to \mathcal{U}(E_1)$ as follows.

$$\alpha' \begin{pmatrix} \lambda_1 I & x & 0 \\ 0 & \lambda_2 I & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 I & \alpha(x) \\ 0 & \lambda_2 I \end{pmatrix}.$$

Here α' is unital with $\mathrm{Ker}(\alpha')$ also unital, but not having closed range.

The next lemma is well known, but the result is used several times in the paper and so is given now with a proof.

Lemma 2.1.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be two operator algebras. Suppose that θ is a complete quotient morphism from $\mathcal{E}$ onto $\mathcal{F}$ such that $\mathcal{D} = Ker(\theta)$. Then $Ker(\theta^{**}) = \mathcal{D}^{\perp\perp} \cong \mathcal{D}^{**}$.*

Proof. Since θ is a complete quotient morphism, $\mathcal{F}$ can be taken to be $\mathcal{E}/\mathcal{D}$. The second dual morphism $\theta^{**} : \mathcal{E}^{**} \to (\mathcal{E}/\mathcal{D})^{**}$ is the unique weak* continuous extension of θ and is also a complete quotient morphism [5, Section 1.4.3]. By Banach space theory, $(\mathcal{E}/\mathcal{D})^{**} \cong \mathcal{E}^{**}/\mathcal{D}^{\perp\perp}$. With $\mathrm{ran}(\theta^{**}) \cong \mathcal{E}^{**}/\mathcal{D}^{\perp\perp}$, and unraveling these identifications yields that $\mathrm{Ker}(\theta^{**}) = \mathcal{D}^{\perp\perp}$. □

For this section it will be assumed the operator algebras are in the category of general operator algebras not necessarily containing an approximate identity unless otherwise stated. The following lemma gives a condition that will be sufficient for the restriction of a complete quotient morphism to also be a complete quotient morphism.

Lemma 2.1.2. *Let $\mathcal{E}$ and $\mathcal{F}$ be two operator algebras. Suppose that θ is a complete quotient morphism from $\mathcal{E}$ onto $\mathcal{F}$. Furthermore suppose that B is a closed subalgebra of $\mathcal{E}$ and either (1) B contains a cai for the kernel of θ, or (2) the kernel of θ contains a cai whose weak* limit is contained in $B^{\perp\perp}$. Then $\theta(B)$ is closed and $\theta|_B$ is a complete quotient morphism onto its range. Indeed $\theta(\mathrm{Ball}(B)) = \mathrm{Ball}(\theta(B))$ and at all matrix levels. Additionally $\theta(B) \cong B/\mathrm{Ker}(\theta|_B) \cong B/\mathrm{Ker}(\theta) \subset \mathcal{E}/\mathrm{Ker}(\theta)$ completely isometrically isomorphically.*

Proof. Let $\mathrm{Ker}(\theta) = \mathcal{D}$ and $A = \mathrm{Ker}(\theta|_B)$, making each an ideal in $\mathcal{E}$ and B respectively. First it will be shown that condition 1 is equivalent to condition 2. It is obvious that if B contains a cai for $\mathcal{D}$, then $\mathcal{D}$ contains a cai whose weak* limit is contained in $B^{\perp\perp}$. Now assume the weak* limit of a cai of $\mathcal{D}$ is contained in $B^{\perp\perp}$, call it p. Let $\theta^{**} : \mathcal{E}^{**} \to \mathcal{F}^{**}$ be the unique weak* continuous extension of θ. Let $(e_t)_t$ be a cai for $\mathcal{D}$. Since $\mathcal{D} = \mathrm{Ker}(\theta)$, $\theta(e_t) = 0$ for all t. This implies that $\theta^{**}(p) = 0$. Using the definition from [4], that is p is open in $\mathcal{E}$ if $p \in (p\mathcal{E}^{**}p \cap \mathcal{E})^{\perp\perp}$, it will be shown that p is open in $\mathcal{E}^{**}$. Since p is the weak* limit of a net in $\mathcal{D}$, $p \in \mathcal{D}^{\perp\perp}$. This gives

$$\mathcal{D} = \mathrm{Ker}(\theta) = p\mathcal{E}^{**}p \cap \mathcal{E},\ p \in (p\mathcal{E}^{**}p \cap \mathcal{E})^{\perp\perp}$$

and p is open in $\mathcal{E}^{**}$. Also with $A \subset \mathcal{D}$, for all $a \in A$, $ae_t \to a$ and $e_t a \to a$ so that $A \subseteq (pB^{\perp\perp}p \cap B) \subseteq \mathrm{Ker}(\theta|_B) = A$. If $\mathcal{E}$ is contained in a unital C*-algebra, call it C, with unit 1. That p is open with respect to C can be see by a similar argument to the one above. By [4, Theorem 2.3], with B a closed subalgebra of C, p is open with respect to B. The orthogonal projection to p, namely $1-p$, is closed so that by [22, Theorem 4.2] $A = pB^{\perp\perp}p \cap B$ has a cai. Call this cai $(f_s)_s$. Let $I = \{x \in \mathcal{E} : f_s x \to x \text{ and } x f_s \to x\}$, which is a hereditary subalgebra of $\mathcal{E}$ by [4, Corollary 2.2]. As a hereditary subalgebra, by [4, Proposition 5.2] I is an inner ideal. Since $f_s \to p$ in the weak* topology in $\mathcal{E}^{**}$, and since $\mathcal{D} = p\mathcal{E}^{**}p \cap \mathcal{E}$, $I \subset \mathcal{D}$. For the other inclusion, since $f_s \to p$, for all $d \in \mathcal{D}$, $f_s d \to d$ weak*. With I an ideal, for all s, $f_s d \in I$ implying $d \in I^{\perp\perp}$. With $\mathcal{D} \subset I^{\perp\perp}$ and in $\mathcal{E}$, $\mathcal{D} \subset I^{\perp\perp} \cap \mathcal{E} = I$ and $\mathcal{D} = I$ giving $(f_s)_s$ is a cai for both A and $\mathcal{D}$.

From the above, A and $\mathcal{D}$ share a common cai and are ideals in B and $\mathcal{E}$ respectively. With p the weak* limit of this common cai in the second dual, clearly $p \in B^{\perp\perp} \cong B^{**}$ as well as $p \in \mathcal{E}^{**}$. As an open projection, p is the (central) support projection of both A and $\mathcal{D}$ in $B^{\perp\perp}$ and $\mathcal{E}^{**}$ respectively. Suppose $\mathcal{E}$ is contained in a unital operator algebra with unit 1, then $\mathcal{E}^{**}(1-p) \subset \mathcal{E}^{**}$ as well as $B^{\perp\perp}(1-p) \subset B^{\perp\perp}$. Indeed the map $q : \mathcal{E}^{**} \to \mathcal{E}^{**}(1-p)$ defined by $\eta \to \eta(1-p)$ is a completely contractive projection on $\mathcal{E}^{**}$ and as well as when the above map is restricted to $B^{\perp\perp}$. This projection is also a weak* continuous homomorphism and its kernel is $\mathcal{D}^{\perp\perp}$ since $\mathcal{E}^{**} \cong \mathcal{D}^{\perp\perp} \oplus \mathcal{E}^{**}(1-p)$. It can be deduced further from Lemma 2.1.1 that

$$\mathcal{D}^{**} \cong \mathcal{D}^{\perp\perp} = \mathcal{E}^{**}p, \ \mathcal{F}^{**} \cong \mathcal{E}^{**}/\mathcal{D}^{\perp\perp} \cong \mathcal{E}^{**}(1-p),$$

with

$$\mathcal{E}^{**} = \mathcal{E}^{**}p \oplus^{\infty} \mathcal{E}^{**}(1-p) \cong \mathcal{D}^{**} \oplus^{\infty} \mathcal{F}^{**} = \mathrm{Ker}(\theta^{**}) \oplus^{\infty} \theta^{**}(\mathcal{E}^{**}).$$

Similarly,

$$A^{**} \cong A^{\perp\perp} = B^{**}(1-p), \;\; (B/A)^{**} \cong B^{\perp\perp}/B^{\perp\perp}p \cong B^{\perp\perp}(1-p).$$

Noting that $B/\mathcal{D} \subset \mathcal{E}/\mathcal{D}$, the composition of the completely contractive morphism taking $B/A \to B/\mathcal{D}$ with the canonical complete isometry taking $\mathcal{E}/\mathcal{D} \to \mathcal{E}^{**}(1-p)$ agrees with the canonical complete isometries taking $B/A \to B^{\perp\perp}(1-p) \subset \mathcal{E}^{**}(1-p)$. This implies that $B/A \cong B/\mathcal{D}$ and further that $B/\mathcal{D}$ is closed. Let $\widetilde{\theta}$ be the induced complete isometry from $\mathcal{E}/\mathcal{D} \to \mathcal{F}$. Forming the composition taking $B/A \to B/\mathcal{D} \to \mathcal{F}$ with the last arrow representing $\widetilde{\theta}$ restricted to $B/\mathcal{D}$, it is easy to see this composition takes A/B onto $X = \theta(B)$ and is a complete isometry. Hence X is a closed subalgebra of $\mathcal{F}$. It follows that this composition can be considered to be $\widetilde{\theta_{|_B}}$, a complete isometry induced by $\theta_{|_B}$ making $\theta_{|_B}$ a complete quotient morphism.

That $\theta(\mathrm{Ball}(B)) = \mathrm{Ball}(\theta(B))$ follows from both A and $\mathcal{D}$ being proximinal ([5] and [21]) as M-ideals in B and $\mathcal{E}$ respectively. As a complete quotient morphism it is the case that $\theta(\mathrm{ball}(B)) = \mathrm{ball}(\theta(B))$. For the closed ball, let $z \in \mathrm{Ball}(\theta(B))$ such that $\|z\| = 1$. With A proximinal in B, there exists $b \in B$ with $\|b\| = 1$ and $\theta(b) = z$ indicating that each such $z \in \mathrm{Ball}(\theta(B))$ has in its inverse image under θ an element in B of norm one, giving $\theta(\mathrm{Ball}(B)) = \mathrm{Ball}(\theta(B))$. $\square$

The next two corollaries will have implications for extensions of operator algebras, in particular the second one dealing with corona algebras.

Corollary 2.1.3. *Let $\mathcal{E}$ be an operator algebra with $\mathcal{D}$, B, and A closed subalgebras such that $\mathcal{D}$ is an ideal of $\mathcal{E}$ and A is an ideal in B. If $\mathcal{D}$ and A share a common cai, then $B/A \subset \mathcal{E}/\mathcal{D}$ completely isometrically isomorphically.*

Proof. Let θ be the canonical quotient morphism taking $\mathcal{E} \to \mathcal{E}/\mathcal{D}$. With $\mathcal{D}$ and A sharing a common cai, the first of the two equivalent conditions of Lemma 2.1.2 is satisfied giving that the restriction of θ to B is a complete quotient morphism. The rest follows as in the proof of Lemma 2.1.2. □

Corollary 2.1.4. *If A is a closed subalgebra of an operator algebra B and suppose that they share a common cai, then $\mathcal{Q}(A) \subset \mathcal{Q}(B)$ completely isometrically isomorphically.*

Proof. Let ι be the inclusion map of A into B which extends to a unital completely isometric isomorphism $\hat{\iota} : \mathcal{M}(A) \to \mathcal{M}(B)$ as in the introduction. This then induces a map $\tilde{\iota} : \mathcal{Q}(A) \to \mathcal{Q}(B)$ which have A and B respectively as kernels. By Corollary 2.1.3 $\mathcal{M}(A)/A \subset \mathcal{M}(B)/B$, or $\mathcal{Q}(A) \subset \mathcal{Q}(B)$ completely isometrically isomorphically. □

2.2 Approximately Unital Ideals and C*-covers

Studying ideals of an operator algebra is integral to the theory of extensions. The results from the C*-algebra theory are well known and as a way to connect the general case to the C*-algebra case, it will be helpful to compare ideals of operator algebras with ideals in the C*-algebras they generate. This will be accomplished in the next several results.

Lemma 2.2.1. *Suppose that A and B are closed subalgebras of a* C*-*algebra $\mathcal{E}$ with A approximately unital and an ideal in B. Then the* C*-*algebra generated by A is an ideal in the* C*-*algebra generated by B.*

Proof. Designate the C*-algebra generated by A as $\mathcal{A}$ and the C*-algebra generated by B as $\mathcal{B}$. It is enough by linearity of multiplication to show that for all $a \in A$ and $b \in B$, all mixed products of the form $a^*b^*,\ ab^*,\ b^*a,$ and ba^* are contained in $\mathcal{A}$. By hypothesis A contains a cai, call it e_t. By [5, Lemma 2.1.6] $a^*e_t \to a^*$ and $e_ta^* \to a^*$ giving that $a^*b = \lim_t a^*e_tb \in A^*A$. Similarly for the other mixed terms. □

Lemma 2.2.2. *If A is a closed approximately unital subalgebra of an operator algebra B such that A is an ideal in B, then $C^*_e(A)$ is an ideal in $C^*_e(B)$, in the sense that the* C*-*algebra generated by A in $C^*_e(B)$ is a* C*-*envelope of A.*

Proof. It can be assumed that B is unital since, if not, $C^*_e(B) \subset C^*_e(B^1)$ is the C*-algebra generated by B in $C^*_e(B^1)$ by [5, Section 4.3]. Suppose first that B contains a central projection p such that $A = Bp$. As a central projection $pp = p$ so $p \in A$ and A is unital. It will be the case that B splits as $A \oplus^\infty B(1-p)$ where 1 is the unit of B. This gives that the C*-algebra generated by A in $C^*_e(B)$ is $C^*_e(B)p$. Let $(I(B), j)$ be an injective envelope of B which is a unital C*-algebra by [5, Corollary 4.2.8]. By [5, Theorem 4.6.3], $I(B)$ contains an injective envelope of A which again is a unital C*-algebra. Let e and e' be the units of $I(B)$ and $I(A)$

respectively. Evidently $I(B) = I(A) \oplus^{\infty} I(B)(e - e')$. With A contained in $I(A)$, then $A = I(A) \cap B = Be \cap B = Bp$. This then means $\mathrm{C}^*_{\mathrm{e}}(B)p$ is the C^*-algebra generated by A in $I(A)$ and so is $\mathrm{C}^*_{\mathrm{e}}(A)$. By Lemma 2.2.1, $\mathrm{C}^*_{\mathrm{e}}(A)$ is an ideal in $\mathrm{C}^*_{\mathrm{e}}(B)$.

More generally it will be necessary to go to the second dual algebras. Given a C^*-envelope of B^{**}, the C^*-algebra generated by B inside $\mathrm{C}^*_{\mathrm{e}}(B^{**})$ is a C^*-envelope of B by [4, Lemma 5.3]. Let p be the support projection of A inside B^{**}. With $A^{\perp\perp}$ a direct summand of B^{**} and thus an ideal, by the first paragraph the C^*-algebra generated by $A^{\perp\perp}$ is a C^*-envelope of $A^{\perp\perp}$. Again, using [4, Lemma 5.3], the C^*-algebra generated by A in $\mathrm{C}^*_{\mathrm{e}}(A^{\perp\perp}) \cong \mathrm{C}^*_{\mathrm{e}}(A^{**})$ is a C^*-envelope of A. That $\mathrm{C}^*_{\mathrm{e}}(A)$ is contained in $\mathrm{C}^*_{\mathrm{e}}(B)$ is clear since the C^*-algebra generated by A is contained in the C^*-algebra generated by B. Apply Lemma 2.2.1 to see that $\mathrm{C}^*_{\mathrm{e}}(A)$ is an ideal in $\mathrm{C}^*_{\mathrm{e}}(B)$. □

Remark. Although it will not be used, there is an interesting fact that follows from the last result. If A and B are as in the above result with A and B both approximately unital, then the injective envelope of A, $I(A)$ (whose theory may be found in e.g. [10, 24]), may be viewed as a subalgebra of $I(B)$. In fact there is a projection designated above by $e \in I(B)$ with $I(A) = eI(B)e$. This projection is the unit of $I(A)$. To see this, apply the last result with [20, Theorem 6.5] and the fact $I(\cdot) = I(C^*_e(\cdot))$.

When A is an ideal in B there is a canonical completely contractive morphism $\sigma : B \to \mathcal{M}(A)$ which takes $b \in B$ to $m \in \mathcal{M}(A)$ such that $\sigma(b)a = a' = ma$ and $a\sigma(b) = a'' = am$. It is clear that σ is the identity on A if A is approximately unital. If σ is a complete isometry, the ideal will be said to be *completely essential*. As in the

C*-algebra theory this property can be characterized in terms of the Busby invariant, which will be discussed later. The next lemma gives equivalent characterizations indicating when an ideal is completely essential.

Proposition 2.2.3. *Let A be an approximately unital closed two sided ideal in an operator algebra B. The following are equivalent:*

(i) *A is a completely essential ideal in B.*

(ii) *Any complete contraction with domain B is completely isometric if and only if its restriction to A is completely isometric.*

(iii) *There is a C*-cover $\mathcal{E}$ of B such that the C*-subalgebra J of $\mathcal{E}$ generated by A is an essential ideal in $\mathcal{E}$.*

(iv) *Same as* (iii)*, but with $\mathcal{E} = \mathrm{C}^*_{\mathrm{e}}(B)$.*

(v) *If $j : B \to I(B)$ is the canonical map into the injective envelope of B, then $(I(B), j_{|A})$ is an injective envelope of A.*

If B is nonunital, these are equivalent to

(vi) *A is a completely essential ideal in the unitization B^1.*

Proof. First it will be shown that (i) is equivalent to (vi). That (vi)$\Rightarrow$ (i) is trivial and (i) $\Rightarrow$ (vi) follows from Meyer's unitization [5, Theorem 2.1.15], i.e. since $\sigma : B \to \mathcal{M}(A)$ is a complete isometry it extends to a completely isometric unital morphism from B^1 into $\mathcal{M}(A)$.

For the last few implications below it will be helpful to assume that B is approximately unital. For this, note that if B is not unital then one may appeal to (vi) and use the fact $C_e^*(B)$ is the C*-algebra generated by B in $C_e^*(B^1)$ (see [5, 4.3.4]).

(ii) $\Rightarrow$ (i) Since the restriction of canonical morphism σ to A is the identity on A as noted above, by assuming (ii), σ is a complete isometry on B.

(iii) $\Rightarrow$ (i) The hypothesis that J is a completely essential ideal in $\mathcal{E}$ means the canonical $*$-homomorphism taking $\mathcal{E} \to \mathcal{M}(J)$ is one-to-one and a complete isometry. If ρ is the restriction to B, ρ is a complete isometry. This implies σ is a complete isometry as seen by letting (e_t) be a common cai for A and J and following the proof in [5, 2.23]

$$\|[b_{ij}]\| = \|[\rho(b_{ij})]\| = \sup_t \|[b_{ij}e_t]\| = \|[\sigma(b_{ij})]\|, \quad [b_{ij}] \in \mathrm{M}_n(B).$$

(i) $\Rightarrow$ (iv) It is supposed that $\sigma : B \to \mathcal{M}(A)$ is a complete isometry. View $\mathrm{C}_\mathrm{e}^*(A) \subset \mathrm{C}_\mathrm{e}^*(B)$ as in Lemma 2.2.2 and consider the canonical $*$-homomorphism $\sigma' : \mathrm{C}_\mathrm{e}^*(B) \to \mathcal{M}(\mathrm{C}_\mathrm{e}^*(A))$. Since $\mathrm{C}_\mathrm{e}^*(A)$ and A share a common cai, the setting is similar to the last displayed equation and the restriction ρ of σ' to B is a complete isometry. This indicates that $\mathcal{M}(\mathrm{C}_\mathrm{e}^*(A))$ contains a C*-cover of B. By the essential property of $\mathrm{C}_\mathrm{e}^*(B)$ (see e.g. 4.3.6 in [5]), σ' is a complete isometry.

(iv) $\Rightarrow$ (v) Follows from the C^*-algebraic case from [23] together with the fact that the injective envelope of an operator algebra is the same as the injective envelope of its C*-envelope.

(v) $\Rightarrow$ (ii) Suppose that θ is a completely contractive homomorphism from B into

$B(H)$ such that the restriction to A is a complete isometry. Extend θ to a complete contraction $\hat{\theta} : I(B) = I(A) \to B(H)$ by the injectivity of $B(H)$. By the essential property of $I(A)$ (see e. g. [5, Section 4.2]), $\hat{\theta}$ is a complete isometry indicating $\theta = \hat{\theta}|_B$ is a complete isometry. □

Lemma 2.2.4. *Let A be an approximately unital ideal in an operator algebra B, and if $\mathcal{D}$ is the* C^**-algebra generated by A inside of* $\mathrm{C}^*_{\max}(B)$*, then $\mathcal{D}$ is a maximal* C^**-cover of A. Additionally the quotient algebra* $\mathrm{C}^*_{\max}(B)/\mathcal{D}$ *is a maximal* C^**-cover for B/A.*

Proof. Let $\mathcal{D}$ be the C^*-algebra generated by A inside $\mathrm{C}^*_{\max}(B)$. It will be shown that $\mathcal{D}$ has the necessary universal property. Let $\theta : A \to B(H)$, for some Hilbert space H, be a completely contractive nondegenerate homomorphism. By [5, Theorem 2.6.2] there is a completely isometrically isomorphic copy of $\mathcal{M}(\theta(A))$ inside $B(H)$. With θ completely isometric and taking a cai of A to a cai of $\theta(A)$, there is a unique extension of θ by [5, Proposition 2.6.12] $\hat{\theta} : \mathcal{M}(A) \to \mathcal{M}(\theta(A))$. Also with A an ideal in B, there is a canonical morphism $\sigma : B \to \mathcal{M}(A)$. Form the composition $\hat{\theta} \circ \sigma : B \to B(H)$, which extends to a $*$-homomorphism $\mathrm{C}^*_{\max}(B) \to B(H)$. Restricting this to $\mathcal{D}$ gives a $*$-homomorphism extending θ.

Turning to the second assertion, first note by the above paragraph $\mathrm{C}^*_{\max}(B)/\mathcal{D} \cong \mathrm{C}^*_{\max}(B)/\mathrm{C}^*_{\max}(A)$. Now appealing to Corollary 2.1.4, there exists a subalgebra C of $\mathrm{C}^*_{\max}(B)/\mathcal{D}$ with $B/A \cong C$ completely isometrically isomorphically and so B/A will be identified with C. It is easy to see that B/A generates $\mathrm{C}^*_{\max}(B)/\mathrm{C}^*_{\max}(A)$ and so is a C^*-cover. Let $\varphi : \mathrm{C}^*_{\max}(B) \to \mathrm{C}^*_{\max}(B)/\mathrm{C}^*_{\max}(A)$ be the canonical surjective

$*$-homomorphism due to $\mathrm{C}^*_{\max}(A)$ being an ideal in $\mathrm{C}^*_{\max}(B)$. Since φ is surjective, if $\eta \in \mathrm{C}^*_{\max}(B)/\,\mathrm{C}^*_{\max}(A)$, then there is a pre-image under φ for η, call it ζ. With $\mathrm{C}^*_{\max}(B)$ a C^*-cover of B, ζ can be approximated by a finite sum of finite products in B and $B^* = \{b^* : b \in B\}$, the adjoint algebra of B. Let $\epsilon > 0$ and ξ a finite sum of finite products of elements from B and B^* such that $\|\zeta - \xi\| < \epsilon$. Then

$$\|\varphi(\zeta - \xi)\| = \|\varphi(\zeta) - \varphi(\xi)\| = \|\eta - \varphi(\xi)\| < \epsilon.$$

With φ a $*$-homomorphism, $\varphi(\xi)$ is a finite sum of finite products in $\varphi(B)$ and $\varphi(B^*)$. This makes $\mathrm{C}^*_{\max}(B)/\,\mathrm{C}^*_{\max}(A)$ a C^*-cover for $B/\,\mathrm{C}^*_{\max}(A)$. Since $B/\,\mathrm{C}^*_{\max}(A) \cong B/A$, $\mathrm{C}^*_{\max}(B)/\,\mathrm{C}^*_{\max}(A)$ is a C^*-cover for B/A.

It will be shown that $\mathrm{C}^*_{\max}(B)/\,\mathrm{C}^*_{\max}(A)$ has the universal property of $\mathrm{C}^*_{\max}(B/A)$. Let $\omega : B/A \to \mathcal{E}$ be a completely contractive morphism. Note that if θ is the canonical complete quotient morphism taking B to B/A, then $\omega \circ \theta : B \to \mathcal{E}$. By the universal property of $\mathrm{C}^*_{\max}(B)$, $\omega \circ \theta$ extends to a $*$-homomorphism $\widetilde{\omega \circ \theta}$ with domain $\mathrm{C}^*_{\max}(B)$. By construction it is clear that $\mathrm{C}^*_{\max}(A)$ is a subset of the kernel of this extension implying a $*$-homomorphism γ can be uniquely constructed from $\mathrm{C}^*_{\max}(B)/\,\mathrm{C}^*_{\max}(A) \to \mathcal{E}$ by the Factor Theorem. Let $q : \mathrm{C}^*_{\max}(B) \to \mathrm{C}^*_{\max}(B)/\,\mathrm{C}^*_{\max}(A)$ be the canonical morphism and by the Factor Theorem, $\gamma \circ q = \widetilde{\omega \circ \theta}$. Let $\bar{\eta} \in \mathrm{C}^*_{\max}(B)/\,\mathrm{C}^*_{\max}(A)$ and by definition $\gamma(\bar{\eta}) = \widetilde{\omega \circ \theta}(\eta)$ which agrees with ω on B/A. $\square$

2.3 Tensor Products and C^*-covers

The following results use the language of operator algebra tensor products. (See [5, §6.1])

Lemma 2.3.1. *If B is any C^*-algebra, and if A is any approximately unital operator algebra, then $C^*_{\max}(B \otimes_{\max} A) = B \otimes_{\max} C^*_{\max}(A)$. If B is in addition a nuclear C^*-algebra, then $C^*_{\max}(B \otimes_{\min} A) = B \otimes_{\min} C^*_{\max}(A)$.*

Proof. By [5, 6.1.9] it is the case that $B \otimes_{\max} A \subset B \otimes_{\max} C^*_{\max}(A)$ completely isometrically isomorphically. Clearly $B \otimes A$ generates the latter C^*-algebra. It will be shown that $B \otimes_{\max} A \subset B \otimes_{\max} C^*_{\max}(A)$ has the necessary universal property. Let $\theta : B \otimes_{\max} A \to B(H)$ be a completely contractive homomorphism. By [5, Corollary 6.1.7] there are two completely contractive homomorphisms $\pi : B \to B(H)$ and $\rho : A \to B(H)$ with commuting ranges such that $\theta(b \otimes a) = \pi(b)\rho(a)$. Now π is forced to be a $*$-homomorphism by [5, 1.2.4] and $\pi(B)$ commutes with the extension $\tilde{\rho}$ to $C^*_{\max}(A)$. To see this, first note that $\pi(B)$ has commuting range with the adjoint morphism on the adjoint algebra of A, namely $\rho^* : A^* \to B(H)$. For all $b \in B$ and $a \in A$:

$$\pi(b)\rho^*(a^*) = ((\pi(b)\rho^*(a^*))^*)^* = (\rho(a)\pi(b^*))^* = (\pi(b^*)\rho(a))^* = \rho^*(a^*)\pi(b).$$

The second to last equality is due to the commuting ranges of π and ρ. This leads to $\pi(B)$ having commuting range with a dense subset of $\tilde{\rho}(C^*_{\max}(A))$ and by continuity of multiplication by a single element of $\pi(B)$, extends to $\tilde{\rho}(\mathrm{C}^*_{\max}(A))$. This then gives a $*$-homomorphism $\tilde{\theta} : B \otimes_{\max} \mathrm{C}^*_{\max}(A) \to B(H)$ with

$$\tilde{\theta}(b \otimes a) = \pi(b)\tilde{\rho}(a) = \pi(b)\rho(a) = \theta(b \otimes a),\ a \in A,\ b \in B,$$

proving the first assertion.

Now suppose B is nuclear. By [5, 6.1.15], $B \otimes_{\max} A = B \otimes_{\min} A$. By above $\mathrm{C}^*_{\max}(B \otimes_{\min} A) = B \otimes_{\max} \mathrm{C}^*_{\max}(A) = B \otimes_{\min} \mathrm{C}^*_{\max}(A)$ by definition of nuclearity. □

Lemma 2.3.2. *If A, B are approximately unital operator algebras then $B \otimes_{\min} A$ is a completely essential ideal in $B^1 \otimes_{\min} A^1$. Here A^1 is the unitization, set equal to A if A is already unital and similarly for B^1.*

Proof. Let $\sigma : B^1 \otimes_{\min} A^1 \to \mathcal{M}(B \otimes_{\min} A)$ be the canonical morphism which is a complete contraction on $B^1 \otimes_{\min} A^1$ and a complete isometry on $B \otimes_{\min} A$. Assume A and B are nondegenerately represented on Hilbert spaces K and H respectively. Then $B^1 \otimes_{\min} A^1$ can be regarded as a unital subalgebra of $B(H \otimes^2 K)$. Let $u \in B^1 \otimes_{\min} A^1$, $\eta \in \mathrm{Ball}(H \otimes^2 K)$, and $\{f_s\}_S,\ \{e_t\}_T$ be approximate identities for B and A respectively, the following relationships exist:

$$\|\sigma(u)\| \geq \|\sigma(u)\sigma(f_s \otimes e_t)\| = \|(\sigma(u(f_s \otimes e_t))\| = \|u(f_s \otimes e_t)\| \geq \|u(f_s \otimes e_s)\eta\|.$$

Taking the limit of the contractive approximate identity for $B \otimes_{\min} A$ it follows that for all $\eta \in \mathrm{Ball}(H \otimes K)$ that $\|\sigma(u)\| \geq \|u\eta\|$, so that $\|\sigma(u)\| \geq \|u\|$ and σ is an isometry. The calculation follows at all matrix levels making σ a complete isometry as seen below:

$$\|[\sigma(u_{ij})]\| \geq \|[u_{ij}(f_s \otimes e_t)]\| \geq \|[u_{ij}(f_s \otimes e_t)]\eta_n\|,$$

$$[u_{ij}] \in \mathrm{M}_n(B^1 \otimes_{\min} A^1),\ \eta_n \in \mathrm{Ball}((H \otimes^2 K)^n).$$

Again taking the limit of the contractive approximate identity it follows that for all $\eta_n \in \mathrm{Ball}((H \otimes^2 K)^n)$ that $\|[\sigma(u_{ij})]\eta_n\| \geq \|[u_{ij}]\eta_n\|$. As in the above calculation $\|[\sigma(u_{ij})]\| \geq \|[u_{ij}]\|$ and σ is a complete isometry. $\square$

Theorem 2.3.3. *If A and B are two unital operator spaces, or two approximately unital operator algebras, then* $\mathrm{C}^*_{\mathrm{e}}(B \otimes_{\min} A) = \mathrm{C}^*_{\mathrm{e}}(B) \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)$.

Proof. First assume that A and B are unital. The result is proved in [20, Theorem 6.8] for operator systems. With A and B unital it can be assumed that A and B are operator spaces by replacing A with $A + A^*$, and similarly for B. The result follows since the C^*-envelope is the same for A and $A + A^*$. A more modern proof is given in [6, Theorem 2.10].

Next, suppose that A and B have cais. Let J be a boundary ideal (see [5, p.99]) for $B \otimes A$ in $\mathrm{C}^*_{\mathrm{e}}(B) \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)$. Again appealing to [5, p.99], $(\mathrm{C}^*_{\mathrm{e}}(B) \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A))/J \cong \mathrm{C}^*_{\mathrm{e}}(B \otimes_{\min} A)$ so that $(B \otimes_{\min} A)/J \cong B \otimes_{\min} A$. By Lemma 2.3.2, $\mathrm{C}^*_{\mathrm{e}}(B) \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)$ is a completely essential ideal in $\mathrm{C}^*_{\mathrm{e}}(B)^1 \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)^1$ and so let $\iota : \mathrm{C}^*_{\mathrm{e}}(B) \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A) \to \mathrm{C}^*_{\mathrm{e}}(B)^1 \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)^1$ be the inclusion morphism. This indicates that J is also an ideal in $\mathrm{C}^*_{\mathrm{e}}(B)^1 \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)^1$. By Lemma 2.3.2, $(B \otimes_{\min} A)$ is a completely essential ideal in

$B^1 \otimes_{\min} A^1$. Let $q : \mathrm{C}^*_{\mathrm{e}}(B^1) \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)^1 \to (\mathrm{C}^*_{\mathrm{e}}(B)^1 \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)^1)/J$ be the canonical morphism. This is a complete isometry on $B \otimes_{\min} A$ so that by Proposition 2.2.3 $(B^1 \otimes_{\min} A^1)/J \cong B^1 \otimes_{\min} A^1$. It is easy to see that for any approximately unital operator algebra C, $\mathrm{C}^*_{\mathrm{e}}(C)^1 \cong \mathrm{C}^*_{\mathrm{e}}(C^1)$ since $\mathrm{C}^*_{\mathrm{e}}(C)$ is the C^*-algebra generated by C in $\mathrm{C}^*_{\mathrm{e}}(C^1)$. With $B^1 \otimes_{\min} A^1$ a unitization of $B \otimes_{\min} A$, by the previous paragraph $\mathrm{C}^*_{\mathrm{e}}(B)^1 \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)^1 \cong \mathrm{C}^*_{\mathrm{e}}(B^1 \otimes_{\min} A^1)$. With $(\mathrm{C}^*_{\mathrm{e}}(B)^1 \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)^1)/J$ a C^*-cover for $B^1 \otimes_{\min} A^1$, there is a canonical surjective morphism $\pi : (\mathrm{C}^*_{\mathrm{e}}(B)^1 \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)^1)/J \to \mathrm{C}^*_{\mathrm{e}}(B)^1 \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(A)^1$, indicating $J = 0$. □

In the above proof it was shown that $\mathrm{C}^*_{\mathrm{e}}(A)^1 = \mathrm{C}^*_{\mathrm{e}}(A^1)$. The next result shows it is also the case for $\mathrm{C}^*_{\max}(A)$.

Lemma 2.3.4. *Let A be an operator algebra. Then $\mathrm{C}^*_{\max}(A^1) \cong \mathrm{C}^*_{\max}(A)^1$ for any unitization of A.*

Proof. Let A^1 be a unitization of A. If 1 is the unit adjoined to A, then 1 can be taken to be the identity element of $B(H)$ for some Hilbert space H and is independent of H. The unitization is $A^1 = \mathrm{span}\{A, I_H\}$. Similarly, if $\mathrm{C}^*_{\max}(A)^1$ is a unitization of $\mathrm{C}^*_{\max}(A)$, then the unit is the identity of $B(K)$ for some Hilbert space K. By [5, 2.1.16], a unitization of A can be taken to be $A^1 = \mathrm{span}\{A, I_K\}$. Since the two unitizations are completely isometrically isomorphic, $\mathrm{C}^*_{\max}(A)^1 \cong \mathrm{C}^*_{\max}(A^1)$. □

A review of cones and suspensions for an approximately unital operator algebra will be given. Let $\mathcal{C}(A)$ denote the cone and $\mathcal{S}(A)$ denote the suspension of A, defined as follows:

$$\mathcal{C}(A) = C_0((0,1], A) \cong C_0((0,1]) \otimes_{\min} A, \text{ where}$$

$$C_0((0,1], A) = \{f : [0,1] \to A : f \text{ is continuous and } f(0) = 0\}.$$

$$\mathcal{C}(A)^1 = \{f : [0,1] \to A, f \text{ is continuous and } f(0) \in \mathbb{C}\, 1_A \text{ if } A \text{ is unital}\}.$$

$$\mathcal{S}(A) = C_0((0,1), A) \cong C_0((0,1)) \otimes_{\min} A, \text{ where}$$

$$C_0((0,1), A) = \{f : [0,1] \to A : f(0) = f(1) = 0\}.$$

$$\mathcal{S}(A)^1 = \{f : [0,1] \to A^1,\ f \text{ is continuous and } f(0) = f(1) \in \mathbb{C}\, 1_A, f(t) \in A + \lambda 1_A\}.$$

Corollary 2.3.5. *The cone and suspension operations both commute with* $\mathrm{C}^*_{\mathrm{e}}$ *and* $\mathrm{C}^*_{\max}$ *for approximately unital operator algebras.*

Proof. First, the above result will be proved for $\mathcal{C}(\mathrm{C}^*_{\mathrm{e}}(C))$ for an approximately unital algebra C. By the identification of $\mathcal{C}(A) \cong C_0((0,1]) \otimes_{\min} A$, $\mathcal{C}(\mathrm{C}^*_{\mathrm{e}}(C)) \cong C_0((0,1]) \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(C)$. Applying Theorem 2.3.3, then

$$C_0((0,1]) \otimes_{\min} \mathrm{C}^*_{\mathrm{e}}(C) \cong \mathrm{C}^*_{\mathrm{e}}(C_0((0,1]) \otimes_{\min} C) \cong \mathrm{C}^*_{\mathrm{e}}(\mathcal{C}(C)).$$

A similar calculation shows that $\mathcal{S}(\mathrm{C}^*_{\mathrm{e}}(C)) \cong \mathrm{C}^*_{\mathrm{e}}(\mathcal{S}(C))$.

To prove the results for both $\mathcal{C}(\mathrm{C}^*_{\max}(C)) \cong \mathrm{C}^*_{\max}(\mathcal{C}(C))$ and $\mathcal{S}(\mathrm{C}^*_{\max}(C)) \cong \mathrm{C}^*_{\max}(\mathcal{S}(C))$, begin by perform calculations similar to those above. Then, combine the calculations with the fact that both $C((0,1])$ and $C_0((0,1))$ are nuclear and use Lemma 2.3.1.

With the understanding that $\mathrm{C}^*_{\mathrm{e}}(A)^1 \cong \mathrm{C}^*_{\mathrm{e}}(A^1)$ and $\mathrm{C}^*_{\max}(A^1) \cong \mathrm{C}^*_{\max}(A)^1$, the unitization of the cone and suspension commutes with $\mathrm{C}^*_{\mathrm{e}}$ and $\mathrm{C}^*_{\max}$. $\square$

Chapter 3

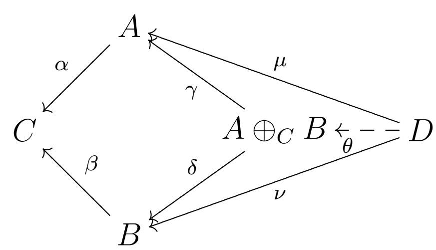
Theory of Extensions of Operator Algebras

In this chapter there are several important algebras and morphism designations that carry through the chapter. The homomorphism $\alpha : A \to B$ will be a completely isometric morphism that is the embedding of an approximately unital operator algebra A as an ideal in B. The homomorphism $\beta : B \to C$ will be a complete quotient morphism. As in the previous chapter, σ will be a completely contractive homomorphism which takes an operator algebra, usually B, into the multiplier algebra of an approximately unital operator algebra, usually A. It will usually be the case in this chapter that σ will extend to a completely contractive homomorphism $\hat{\sigma} : \mathcal{M}(B) \to \mathcal{M}(A)$. As a convention, anytime a completely contractive morphism extends to the multiplier of the domain, the hat notation will be used. For instance, often the completely contractive morphism $\mu : A_1 \to A$ will extend to $\hat{\mu} : \mathcal{M}(A) \to \mathcal{M}(A_1)$. The Greek letter τ will be reserved for a specific completely contractive homomorphism from an operator algebra into the corona algebra of an approximately unital operator algebra which will be shown to have properties similar to the Busby invariant from C^*-algebra theory. The completely contractive morphism $\pi : \mathcal{M}(A) \to \mathcal{Q}(A)$ will be the canonical one taking the multiplier algebra of an approximately unital multiplier algebra onto its corona algebra.

3.1 The Pullback Construction

In developing the theory of extensions for C^*-algebras, the pullback construction has been crucial. Given three objects in a category, and morphisms $\alpha : A \to C$ and $\beta : B \to C$, which are called the linking morphisms, the pullback of A and B along α and β, denoted $A \oplus_C B$ is the sub-object of the direct sum $A \oplus^\infty B$ defined as the ordered pairs (a, b) such that $\alpha(a) = \beta(b)$. This will be denoted PB. There are canonical morphisms, $\gamma : PB \to A$ and $\delta : PB \to B$, which are the projections of each coordinate into the associated object.

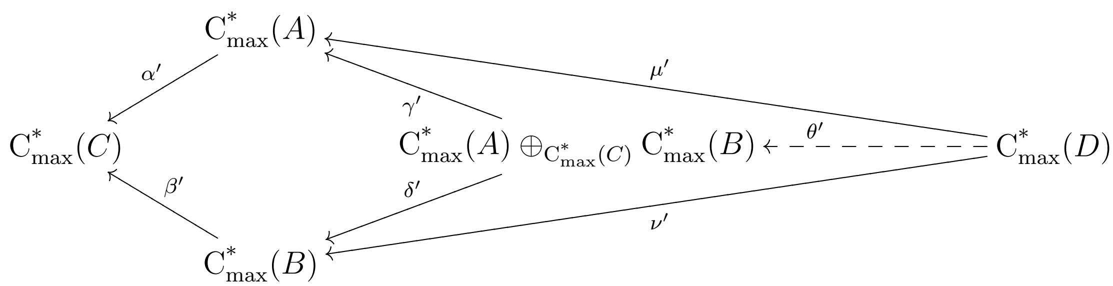
Given another object in the category with morphisms μ and ν from D into A and B respectively, these morphisms will be called *coherent* if when composed with α and β they agree in C, or more precisely, $\alpha \circ \mu = \beta \circ \nu$. The pullback has the universal property that given such an object D and coherent morphisms μ and ν, there exists a unique morphism from D into the pullback satisfying the following universal commutative diagram.



The morphism θ can be defined for $d \in D$ as $\theta(d) = (\mu(d), \nu(d))$, which by the coherency of μ and ν, maps d into PB. For given linking morphisms α and β the pullback is unique up to completely isometric isomorphism as can be seen using the above diagram. Suppose there is another pullback, call it E with morphisms γ' and δ' that make the above diagram commute. Then γ' and δ' would necessarily be coherent giving $\theta : E \to PB$ defined for all $\eta \in E$ by $\theta(\eta) = (\gamma'(\eta), \delta'(\eta))$. Alternately with E also a pullback, by the universal property of the pullback there is a morphism, $\theta' : PB \to E$ that makes the above diagram commute. This requires that $\theta((a, b)) = d$ with $a = \gamma(d)$ and $b = \delta'(d)$ by commutativity. This shows that $\theta' = \theta^{-1}$ and $\theta'^{-1} = \theta$ giving that θ is an isomorphism. Once it is shown below that both θ and θ' are complete contractions, then θ would be a completely isometric isomorphism.

In the category of operator algebras the pullback is closed by the continuity of the linking morphisms. Suppose (a_t, b_t) is net in PB converging to $(a, b) \in A \oplus^\infty B$. By definition $\alpha(a_t) = \beta(b_t)$ for all t. By continuity of α and β, $\alpha(a) = \beta(b)$ so that $(a, b) \in PB$. The morphisms α, β, μ and ν are required to be completely contractive homomorphisms of operator algebras. This then makes θ a completely contractive morphism. One way to prove this, which also shows the interaction with

C*-algebra theory, is to use the universal property of the maximal C*-cover for each algebra. Given a commutative diagram as at the beginning of this chapter, then each morphism extends to the maximal C*-cover as below:



Here the morphisms are the canonical extensions of each to the respective maximal C*-covers, except perhaps θ'. It is easy to see that in fact it is by the manner in which θ' is defined. First let $i : A \to \mathrm{C}^*_{\max}(A)$, $j : B \to \mathrm{C}^*_{\max}(B)$, and $k : D \to \mathrm{C}^*_{\max}(D)$. By definition for all $\eta \in \mathrm{C}^*_{\max}(D)$, $\theta'(\eta) = (\mu'(\eta), \nu'(\eta))$. With μ' and ν' the extensions of ν and μ, then $\mu' \circ k = i \circ \mu$ and $\nu' \circ k = j \circ \nu$ so that $(\theta' \circ k)(d) = ((i \circ \mu)(d), (j \circ \nu)(d))$. With α' and β' the extensions of α and β to the respective maximal C*-covers, $(a, b) \in A \oplus_C B$ if and only if $(i(a), j(b)) \in \mathrm{C}^*_{\max}(A) \oplus_{\mathrm{C}^*_{\max}(C)} \mathrm{C}^*_{\max}(B)$. Let $\iota : A \oplus_C B \to \mathrm{C}^*_{\max}(A) \oplus_{\mathrm{C}^*_{\max}(C)} \mathrm{C}^*_{\max}(B)$ by $\iota((a, b)) = (i(a), j(b))$ which is clearly a complete isometry since it is a complete isometry in each component. Then $\theta' \circ k = \iota \circ \pi$ and θ is a completely contractive morphism.

Unfortunately it is not the case that the pullback constructed from three objects in the category AUOA stays in that category, as in the following example.

Example 3.1.1. A Pullback that is not approximately unital.

Consider the following unital 3×3 matrices.

$$A = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}, \; B = \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let α and β be the obvious maps from A and B respectively into C. The kernel of α contains the matrices with zero entries except in the last column. The linking morphisms only agree on the 1-1 entry and the last column, so that the pullback is of the form:

$$\left(\begin{bmatrix} x & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

which is clearly not unital. Conditions on the linking morphisms will be given later in Lemma 3.4.1 so that the pullback will have an approximate unit if A, B, and C have one.

3.2 Extensions

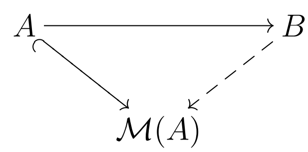
If A and C are nontrivial operator algebras with A approximately unital, then (following the C*-algebra theory) an extension of C by A will be defined as a short exact sequence, denoted for convenience by E,

$$E:\ 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

where α is a completely isometric homomorphism, so that B contains a completely isometrically isomorphic copy of A, and $\alpha(A)$ is the kernel of β. This makes $\alpha(A)$ an ideal in B. Often for simplicity it will be assumed that $A \subset B$ with α the inclusion morphism. Additionally the morphism β will be required to be a complete quotient morphism, which is of course a homomorphism, so that $C \cong B/A$ completely isometrically isomorphically in the general case. In the C^*-algebra case these last few conditions are automatic by the properties of $*$-homomorphisms. In extending the other elements of the C^*-algebra theory to general operator algebras will typically require additional hypothesis. The requirement that A has a cai is necessary for several reasons including using the theory developed concerning morphisms between multiplier algebras for approximately unital operator algebras as well as the results concerning ideals in the previous chapter.

A *unital extension* is an extension in which the middle algebra B is unital. A *split extension* or *split exact extension* is one in which there exists a morphism γ from the third algebra into the second algebra such that $\beta \circ \gamma = I_C$, the identity morphism on C. A *trivial extension* is one in which the middle algebra is an infinity direct sum of the first and third algebras and where α is a the inclusion morphism into one factor and β is the projection morphism onto the other factor. As discussed in Chapter 1, an ideal A in B is completely essential if the canonical morphism taking $B \to \mathcal{M}(A)$

is a complete isometry. An ideal is essential if the canonical morphism is one-to-one. An *essential (respectively completely essential) extension* is one in which $\alpha(A)$ is an essential (respectively completely essential) ideal in B as in the following diagram.



We shall show the largest completely essential extension with first term A is the corona extension

$$0 \to A \to \mathcal{M}(A) \to \mathcal{Q}(A) \to 0.$$

Define $E' \leq E$ if there exist completely contractive morphisms from each successive algebra of E into each successive algebra of E' as below:

$$\begin{array}{ccccccccc} E : 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \nu} & & \\ E' : 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 0. \end{array}$$

To see this applies to the corona algebra, note that the multiplier algebra of A is the largest unital algebra containing A as an essential ideal. If E' is another completely essential extension of C by A, the middle algebra, call it X, contains A as a completely essential ideal. The injection morphism $\iota : A \to \mathcal{M}(A)$ extends to X canonically, which by Proposition 2.2.3 is a complete isometry. For the first two algebras of the two extensions we have $A = A$ and $X \subset \mathcal{M}(A)$ completely isometrically isomorphically. That $C \subset \mathcal{Q}(A)$ completely isometrically isomorphically, an appeal will be made to Lemma 2.1.2. Since A has a cai, the restriction to X of the canonical quotient morphism from $\mathcal{M}(A) \to \mathcal{Q}(A)$ is a complete quotient morphism and $X/A \cong C$ by definition of exact sequence giving the third complete isometry.

If the third algebra is $\mathbb{C}$, then the middle algebra is A^1, the canonical unitization of A [5, Section 2.1.11]. The exact sequence can be given as below:

$$0 \to A \to A^1 \xrightarrow{\beta} \mathbb{C} \to 0.$$

The unitization is defined as $\mathrm{Span}\{A, 1_H\}$ for $A \subset B(H)$ for some H and contains A as an ideal. Then $A^1/A \cong \mathbb{C}$ showing the above line fits the definition of exact sequence.

Working in the category AUOA, it is desirable to know that if an exact sequence extension exists of C by A of two given operator algebras A and C, then the middle term necessarily stays in the same category. The following proposition shows this is in fact the case. It also shows that an extension of C*-algebras stays in that category in a similar fashion.

Proposition 3.2.1. *Given an extension of operator algebras in which A has a cai*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

the algebra B is approximately unital if and only if C is approximately unital. Also, A and C are C*-*algebras if and only if B is a* C*-*algebra.*

Proof. By definition of an exact sequence of operator algebras above, A is assumed to have a cai. Suppose that C is approximately unital. By definition of an exact sequence $C \cong B/A$. Let $B \subset D$, where D is a unital operator algebra with unit 1. Also, view $A \subset B$ completely isometrically isomorphically. If p is the support projection of A in B^{**}, as in the proof of Lemma 2.1.2, $B^{**} = B^{**}p \oplus^{\infty} B^{**}(1-p) \cong A^{**} \oplus^{\infty} C^{**}$, which is unital since both A and C are approximately unital. Appealing to [5, Proposition 2.5.8], B has a cai. For the other direction, it is obvious that a complete quotient morphism takes a cai to a cai, so C will be approximately unital assuming B is approximately unital.

Suppose B is a C*-algebra. With A an ideal in B, A is self-adjoint by [11, Theorem 4.3 p.245], and is a C*-algebra by [5, A.5.1]. By [5, Theorem A.5.9] the range of $\beta = C$, C is a C*-algebra. Suppose A and C are C*-algebras. Looking at the second dual $B^{**} \cong A^{**} \oplus^{\infty} C^{**}$ as in the proof of Lemma 2.1.2, the right side is a C*-algebra. By [5, Lemma 7.1.6] B is selfadjoint; that is, B is a C*-algebra. □

3.3 Morphisms Between Extensions

If a commutative diagram between two extensions can be formed as below,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \nu} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{\alpha_2} & B_2 & \xrightarrow{\beta_2} & C_2 & \longrightarrow & 0
\end{array} \tag{3.3.1}$$

with μ, δ, and ν completely contractive morphisms, it will be termed a *morphism between extensions.* The existence of the two outer morphisms will not guarantee the existence of the middle morphism. This may be the case even with C*-algebras. The middle morphism will be shown to exist uniquely under certain hypotheses including that the left vertical arrow is proper.

If the first and third algebras of each extension are the same, namely A and C, with μ and ν the identity morphisms, the existence of the middle morphism can be regarded as giving a partial ordering on the set of extensions of C by A in the following sense. If the middle morphism is also a surjective complete isometry such that the above diagram commutes, the two extensions will be said to be *strongly isomorphic* as in the C*-algebra case. The set of equivalence classes of strongly isomorphic extensions of C by A will be denoted $\mathbf{Ext}(C, A)$.

It will be the case that Diagrams I, II, III, and IV of [17] work in the setting of this thesis similar to the way they work in [17], although stronger hypotheses will sometimes be required. For example the pullback algebras of the universal Diagram II completion may not stay in the category AUOA even if all algebras of the diagram are in that category.

Looking at more general morphisms between extensions, it is of interest when all the vertical arrows are also complete quotient morphisms or completely isometric isomorphisms. A variant of the "five lemma" from algebra will be given below, and it is necessary since these characteristics of $*$-homomorphisms cannot be automatically assumed in the general operator algebra case. Two lemmas will be required first.

Lemma 3.3.1. *Let E be an extension of C by A with middle algebra B. If A is unital, then $B \cong A \oplus^{\infty} C$ completely isometrically isomorphically. Indeed, E is strongly isomorphic to a trivial extension.*

Proof. Let $e = 1_{\alpha(A)}$ which is easy to see is a central projection in B. Indeed, if $b \in B$, then $be \in \alpha(A)$ and $eb \in \alpha(A)$. As the identity of $\alpha(A)$, then $e(be) = be, (be)e = be$, and $e(eb) = eb$. As a central projection, and hence an orthogonal projection, by standard operator theory, $B = Be \oplus^{\infty} B(1-e)$ where 1 represents a unitization of B. This gives $\alpha(A) = Be$ and $C \cong B/\alpha(A) \cong B/Be \cong B(1-p)$. Now define a morphism $\delta : B \to B/\alpha(A)$ in the canonical way. With B a complete quotient morphism, define $\xi : C \to B/A$ by $\xi(c) = b + \alpha(A)$ such that $\beta(b) = c$, which, by the Factor Theorem, is a complete isometry. The composition $\delta^{-1} \circ \xi : C \to B$ such that $\beta \circ (\delta^{-1} \circ \gamma)$ is the identity on C. Let $\gamma = \delta^{-1} \circ \xi$. From the above, it is clear that $B = Be \oplus^{\infty} B(1-p) = \alpha(A) \oplus^{\infty} \gamma(C)$. The obvious map $\rho : \alpha(A) \oplus^{\infty} \gamma(C) \to B$, for $(a, c) \in \alpha(A) \oplus^{\infty} \gamma(C)$ given by $\rho(a, c) = \alpha(a) + \gamma(c)$, is a completely isometric isomorphism, as shown above. □

Lemma 3.3.2. *If extension E of C by A with middle algebra B exists, there is an induced extension, denoted E^{**} of C^{**} by A^{**} with middle algebra B^{**}. Additionally this extension of C^{**} by A^{**} is strongly isomorphic to a trivial extension.*

Proof. Suppose an extension of C by A exists with middle algebra B. Let $\alpha : A \to B$

be the complete isometry and $\beta : B \to C$ a complete quotient morphism with $\mathrm{Ker}(\beta) = A$ giving the exact sequence. Then α^{**} is a complete isometry taking $A^{**} \to A^{\perp\perp} \subset B^{**}$ and β^{**} is a complete quotient morphism onto C^{**}. By Lemma 2.1.1, $\mathrm{Ker}(\beta^{**}) = A^{\perp\perp}$ and there exists an extension of C^{**} by A^{**} with middle algebra B^{**}. To see the extension E^{**} is strongly isomorphic to a trivial extension, apply Lemma 3.3.1. $\square$

Lemma 3.3.3. *Given a morphism between extensions as in Diagram 3.3.1 with all algebras approximately unital, the middle morphism will be a completely isometric isomorphism (respectively, complete quotient morphism) if both the outer two morphisms are completely isometric isomorphisms (respectively, complete quotient morphisms).*

Proof. First, it can be assumed that each algebra is unital by going to the second dual extension. This means the extensions are trivial by Lemma 3.3.1. The morphism between extensions with this assumption is given below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_1 \oplus^\infty C_1 & \xrightarrow{\beta_1} & C_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & A_2 & \xrightarrow{\alpha_2} & A_2 \oplus^\infty C_2 & \xrightarrow{\beta_2} & C_2 & \longrightarrow & 0 \end{array}$$

The morphisms α_i can be taken to be the injection map into $A_i \oplus^\infty 0$ and the complete quotient morphism β can be taken to be $\beta_i(a, c) = c \in C_i, i = 1, 2$. Any such middle morphism, in order to be linear, would be of the form $\delta(a, c) = (\mu(a) + \lambda(c), \nu(c) + \rho(a))$ where λ and ρ are completely contractive morphisms. To see that $\lambda = 0 = \rho$, first note that both μ and ν must be unital since they are surjective morphisms if they are either completely isometric isomorphism or complete quotient morphisms. It is easy to see that δ will also be surjective since it would be surjective in each component. Since δ, as a morphism, must be unital:

$$(1_{A_2}, 1_{C_2}) = \delta(1_{A_1}, 1_{C_1}) = (1_{A_2} + \lambda(1_{C_1}), 1_{C_2} + \rho(1_{A_1})).$$

Evidently $\lambda(1_{C_1}) = 0$ and $\rho(1_{A_1}) = 0$ giving that λ and ρ are zero on C_1 and A_1 respectively.

Suppose that both μ and ν are completely isometric isomorphisms (respectively, complete quotient morphisms). Then as shown above, $\delta((a,c)) = (\mu(a), \nu(c))$ is a completely isometric isomorphism (respectively, complete quotient morphism) in each component and so is a completely isometric isomorphism (respectively, complete quotient morphism). $\square$

The following is a case where the middle morphism between two extensions does not exist. Let $A = \mathbb{K}$ the compact operators on ℓ^2, $B = \mathbb{B} = B(\ell^2)$ and C be the Calkin Algebra $\mathbb{B}/\mathbb{K}$. The morphism $\beta : B \to C$ is the canonical complete quotient morphism onto the Calkin Algebra.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\gamma} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & A \oplus^{\infty} C & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

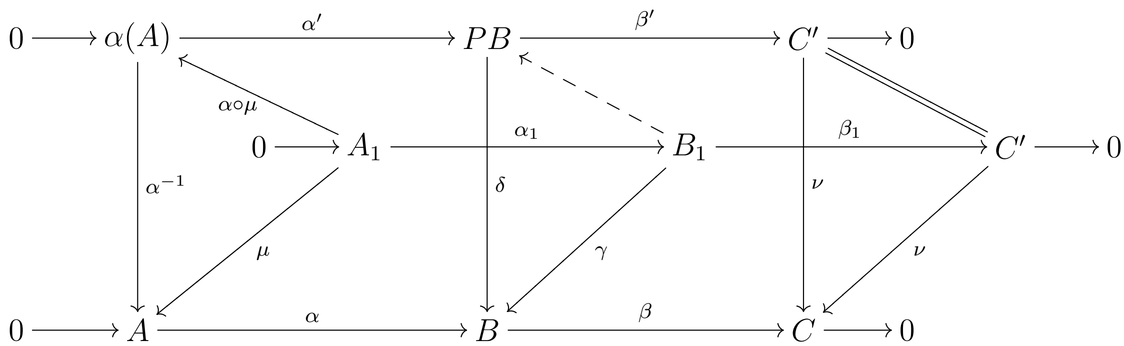
It is well known that the inverse image of the Calkin algebra under β is not an ideal in $\mathbb{B}$. If γ existed, by Lemma 3.3.3 it would be a $*$-isomorphism. In that case γ would have an inverse which would necessarily take an ideal to an ideal. By the second line $0 \oplus^\infty C$ is an ideal in $\gamma(B)$, but its image under γ^{-1}, that is the Calkin algebra, would not be an ideal.

3.4 Diagram I Completions and the Busby Invariant

The next tool to consider is referred to as Diagram I in [17] and is also a key construction in Busby's original paper. Given the following commutative diagram of a morphism between extensions with the ν a complete contraction,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \circ & \dashrightarrow & \circ & \dashrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \nu & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

the completion of the diagram consists of algebras and morphisms A', B', α', β', μ and δ forming an extension of operator algebras on the top line such that the diagram commutes. In the universal completion the first top algebra is A and the middle top algebra is constructed as a pullback of B and C' along β and ν as discussed in the beginning of this chapter. To see this is the universal completion, suppose there is another completion consisting of an extension of C' by A_1 which makes the following diagram commutative:



A morphism needs to be defined for the dotted arrow from B_1 into PB. From the definition of a Diagram I completion, the bottom face commutes showing that γ and β_1 are coherent morphisms with β and ν. Using the universal property of the pullback, the middle morphism exists as a completely contractive morphism making the second half of the rectangular diagram commute. Call this morphism λ. As discussed earlier, for all $b \in B_1$, $\lambda(b) = (\gamma(b), \beta_1(b))$. The only part that is not obviously commutative is the left half of the top face. It needs to be shown that $\alpha' \circ (\alpha \circ \mu) = \lambda \circ \alpha_1$. Let $a \in A_1$. Then $(\alpha' \circ (\alpha \circ \mu))(a) = \alpha'((\alpha \circ \mu)(a)) = ((\alpha \circ \mu)(a), 0)$. For the other direction, $(\lambda \circ \alpha_1)(a) = \lambda(\alpha_1(a)) = ((\gamma \circ \alpha_1)(a)), 0)$. By commutativity of the bottom face, $\alpha \circ \mu = \gamma \circ \alpha_1$ showing the desired commutativity.

Surprisingly, as will be shown in the next lemma, the universal completion will necessarily be in the category AUOA if A, B, C and C' are in this category.

Lemma 3.4.1. *Let B, C' and C be approximately unital operator algebras with $\beta : B \to C$ a complete quotient morphism such that* $\mathrm{Ker}(\beta)$ *has a cai and $\nu : C' \to C$ a completely contractive morphism. Then $B \oplus_C C'$ along β and ν will be approximately unital and there will exist an extension of C' by* $\mathrm{Ker}(\beta)$ *with the pullback algebra,*

$B \oplus_C C'$, being the middle algebra.

Proof. Let $A = \text{Ker}(\beta)$ and form a Diagram I completion shown below with $B \oplus_C C' = \{(b, c) : b \in B, c \in C' \text{ and } \beta(b) = \nu(c)\}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & B \oplus_C C' & \xrightarrow{q} & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

The top row is an extension as defined above with A the kernel of β giving that $\iota(A) = A \oplus 0 \subset B \oplus_C C'$ is an ideal in $B \oplus_C C'$ since $(a, 0)(b, c) = (a', 0) \in \iota(A)$. The morphism q is the projection of the second coordinate onto C' making $\iota(A) = \text{Ker}(q)$. That q is surjective is due to the subjectivity of β. To see q takes the $\text{ball}(B \oplus_C C')$ onto $\text{ball}(C')$, first it is observed that if $c' \in \text{ball}(C')$, with β a complete quotient morphism, there exists $b \in \text{ball}(B)$ such that $\beta(b) = \nu(c')$. In fact, $(b, c') \in \text{ball}(B \oplus_C C')$. Similarly, at all matrix levels this would be the case, with q being a complete quotient morphism and ν being a completely contractive morphism. This gives a pre-image under q for c' in the $\text{ball}(B \oplus_C C')$, indicating $q(\text{ball}(B \oplus_C C')) = \text{ball}(C')$. By Proposition 3.2.1 the pullback will be approximately unital. $\square$

If the bottom row of a Diagram I completion is the corona extension, then the universal diagram I completion is of particular interest and has the following form:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & \mathcal{M}(A) \oplus_{\mathcal{Q}(A)} C & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} & & \\
0 & \longrightarrow & A & \longrightarrow & \mathcal{M}(A) & \xrightarrow{\pi} & \mathcal{Q}(A) & \longrightarrow & 0.
\end{array}
$$

The middle term is shown as a pullback, which exists by the above discussion. Here τ is what will be shown to be the Busby invariant for the category AUOA and will be assumed to be a completely contractive morphism. Traditionally the term Busby invariant is associated with C*-algebras, but in the chapter on covering extensions the use of the term in this setting will be justified. As shown above, σ is the projection of the first coordinate of the pullback algebra.

In a more general setting, for instance an extension of C by A with middle algebra B, a morphism between extensions can be constructed with the corona extension. This is the purpose of the next lemma.

Lemma 3.4.2. *Given an extension of C by A there exists a morphism between extensions as follows:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} & & \\
0 & \longrightarrow & A & \longrightarrow & \mathcal{M}(A) & \xrightarrow{\pi} & \mathcal{Q}(A) & \longrightarrow & 0.
\end{array}
\tag{3.4.1}
$$

Proof. The morphism σ will be taken to be the canonical morphism defined as $\sigma(b)a = \sigma(b(\alpha(a)))$ and $a\sigma(b) = \alpha(a)b$. With A an ideal in B this defines a multiplier $\sigma(b)$ on $A \subset \mathcal{M}(A)$. This is a completely contractive morphism since $\|ba\| \leq \|b\|\|a\|$ for all $a \in A$ and at all matrix levels.

The necessary definition of τ which makes the diagram commutative is $\tau(\beta(b)) = \sigma(b)+A$. Besides making the diagram commutative, it can be shown that τ as defined in this way is a complete contraction. To makes the calculations more transparent, define $\widetilde{\beta} : B/A \to C$ to be the canonical completely isometric isomorphism induced by virtue of β being a complete quotient morphism and $\widetilde{\tau} : B/A \to \mathcal{Q}(A)$ by $\widetilde{\tau}(b+A) = \sigma(b) + A$. To see $\widetilde{\tau}$ is well defined, first note that for $b, b' \in B$, $b,\ b' \in b + A$ if and only if $b - b' \in A$ and if and only if $\sigma(b) - \sigma(b') \in A$ (by linearity of σ) giving $\sigma(b) + A = \sigma(b') + A$. With A containing a cai, it is proximinal and for every coset $x + A \in B/A$, there exists a $b \in B$ such that $\|x + A\| = \|b\|$ and $b + A = x + A$. For that b,

$$\|b + A\| = \|b\| \geq \|\sigma(b)\| \geq \|\sigma(b) + A\| \in \mathcal{M}(A)/A = \mathcal{Q}(A).$$

With σ being a complete contraction and $M_n(A)$ also in AUOA, the above relationship can be redone at all matrix levels indicating $\widetilde{\tau}$ is a completely contractive morphism. As defined above, $\tau = \widetilde{\tau} \circ \widetilde{\beta}^{-1}$ and is a completely contractive morphism. □

Suppose that $\tau : C \to \mathcal{Q}(A)$, is a completely contractive morphism. Then the middle algebra of an extension of C by A can be constructed as a pullback $PB = \mathcal{M}(A) \oplus_{\mathcal{Q}(A)} C$ along π and τ as above. This is called the *pullback extension constructed from* τ, and will be denoted by E_τ. The first concern is that this completion stays in the category AUOA when A and C are in this category. However, as above by Proposition 3.2.1 this will be the case. The next two theorems show that as in the C*-algebra case there is a bijective relationship between completely contractive morphisms from C into $\mathcal{Q}(A)$ and strongly isometric extensions of C by A.

Theorem 3.4.3. *Let τ be a completely contractive morphism from $C \to \mathcal{Q}(A)$ making the following diagram commute.*

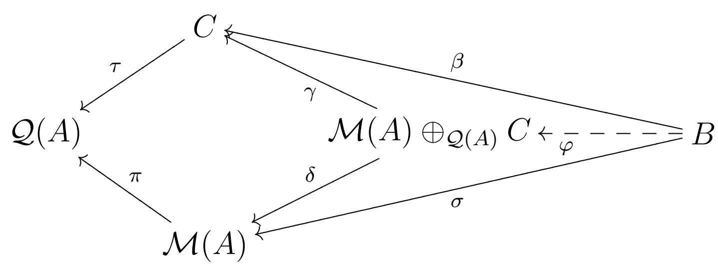
$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} & & \\
0 & \longrightarrow & A & \longrightarrow & \mathcal{M}(A) & \xrightarrow{\pi} & \mathcal{Q}(A) & \longrightarrow & 0
\end{array}$$

Then there exists a completely isometric isomorphism φ from B onto PB, the pullback construction discussed above, making this extension strongly isomorphic to E_τ.

Proof. A morphism between extensions with middle term B on the top line and pullback extensions constructed from τ on the bottom line will be constructed as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \varphi} & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & PB & \xrightarrow{\gamma} & C & \longrightarrow & 0.
\end{array}$$

Evidently A and C will be the first and third algebras in each extension, and it will be enough by Lemma 3.3.3 that a morphism from $B \to PB$ exists. Using the universal property of the pullback, it only needs to be seen that in the following diagram $\tau \circ \beta = \pi \circ \sigma$.



By assumption the top line is an exact sequence, and by the commutative Diagram 3.4.1 from Lemma 3.4.2, $\tau \circ \beta(b) = \pi \circ \sigma(b)$. This gives the existence of φ as defined in the discussion of the pullback and its universal property. By Lemma 3.3.3, φ is a completely isometric isomorphism. □

The next theorem follows closely with [25, Theorem 1.2.11]. In the beginning of Section 2.2 a definition of split extension was given. That is, if there exists a completely contractive morphism $\gamma : C \to B$ such that $\beta \circ \gamma = I_C$, the identity morphism on C, the extension is defined to be split. The map τ in the next theorem is called the Busby invariant of the extension.

Theorem 3.4.4. *There is a bijection between* $\mathbf{Ext}(C, A)$ *and* $\mathrm{Mor}(C, \mathcal{Q}(A))$*, the set of completely contractive morphisms* $\tau : C \to \mathcal{Q}(A)$*. When* C *is unital this restricts to a bijection between unital extensions, in which the middle algebra is unital, and unital morphisms, those taking 1 to 1, from* $C \to \mathcal{Q}(A)$*. There is also a bijective correspondence between the morphisms* $\gamma : C \to B$ *associated with classes of split extensions and elements* $\eta \in \mathrm{Mor}(C, \mathcal{M}(A))$ *for which* $\pi \circ \eta = \tau$ *where* $\pi : \mathcal{M}(A) \to \mathcal{Q}(A)$*. In addition an extension is trivial if and only if it is split with* $\tau = 0$*.*

Proof. Define $\theta_1 : \mathrm{Mor}(C, \mathcal{Q}(A)) \to \mathbf{Ext}(C, A)$ by $\theta(\tau) = [\mathrm{E}_\tau]$, the equivalence class of extensions strongly isomorphic to E_τ where E_τ is the pullback extension. By Theorem 3.4.3, this mapping is well defined since all extensions of C by A with τ as the Busby invariant are strongly isomorphic to the pullback extension. To see θ_1 is surjective, given an extension of C by A a morphism from $C \to \mathcal{Q}(A)$ will be constructed using the following diagram.

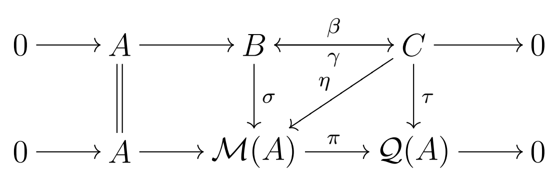
$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & \mathcal{M}(A) & \xrightarrow{\pi} & \mathcal{Q}(A) & \longrightarrow & 0
\end{array}$$

The obvious way to define τ is by $\tau(\beta(B)) = \sigma(b) + A$, as discussed at the beginning of this section. It was pointed out that this is a completely contractive morphism, showing that θ_1 is surjective. To see θ_1 is injective, note that if $\theta_1(\tau_1) = \theta_1(\tau_2)$, then $[\mathrm{E}_{\tau_1}] = [\mathrm{E}_{\tau_2}]$. Note that if $(m, c) \in \mathrm{PB}_i$, then $\tau_i(c) = \pi(m), i = 1, 2$, where $\pi : \mathcal{M}(A) \to \mathcal{Q}(A)$. This gives that $\tau_1(c) = \pi(m) = \tau_2(c)$ and $\tau_1 = \tau_2$.

For the unital case, if B is unital, then $C \cong B/A$ is unital since $1_B + A \in B/A$. The completely contractive morphism τ is defined to take $1_C \to \sigma(1_B) + A$ which is the unit of $\mathcal{Q}(A)$. This gives that if B is unital, τ is unital. Conversely, if τ is unital then let $1_C \in C$ such that $\tau(1_C) = 1_A + A = \pi(1_A)$, the unit of $\mathcal{M}(A)$. This means $(1_A, 1_C) \in PB$ and there is an element $x \in B$ such that $\sigma(x) = 1_A$ and $\beta(x) = 1_C$. Let $J = \mathrm{Ker}(\sigma)$ and, by the Factor Theorem, let $\tilde{\sigma} : B/J \to \mathcal{M}(A)$ be the canonical morphism which is one-to-one and a complete contraction. Since $\tilde{\sigma}$ is multiplicative, it is clear $x + J$ is a unit for B/J. If $b \in B$, then if $b + J \in B/J$, $x(b + J) = b + J$ so

$xb - b \in J$. Alternately taking $C = B/A$ for clarity, $\beta(x)(b + A) = b + A$ implies that $xb - b \in A$. Since σ is the identity morphism on A, $A \cap J = 0$. With $xb - b \in A$ and $xb - b \in J$ $xb - b \in A \cap J = 0$ so that $xb = b$ and similarly for right multiplication giving x is a unit for B.

Working with the second assertion, define $\theta_2 : \mathrm{Mor}(C, B) \to \mathrm{Mor}(C, \mathcal{M}(A))$ for $\gamma \in \mathrm{Mor}(C, B)$ by $\theta_2(\gamma) = \sigma \circ \gamma = \eta$. It needs to be shown that this definition of θ_2 yields a morphism from $C \to \mathcal{M}(A)$ such that $\tau = \pi \circ \eta$. Let E be a split extension and γ a right inverse of β which is not necessarily unique. For a given γ, $\theta_2(\gamma) = \sigma \circ \gamma = \eta$ using the diagram below.



With $\eta = \sigma \circ \gamma$, then $\tau(c) = \pi \circ (\sigma \circ \gamma)(c) = (\pi \circ \eta)(c)$ for all $c \in C$ so that η satisfies the requirement in the hypothesis. This definition of η is unique for a given γ by the uniqueness of σ giving that θ_2 is well defined. To see it is injective, suppose that $\theta_2(\gamma_1) = \theta_2(\gamma_2) = \sigma \circ \gamma_i, i = 1, 2$. Let $b_i = \gamma_i(c)$ for $c \in C$. This means $c = \beta(b_1) = \beta(b_2)$ so that $b_1 - b_2 \in \alpha(A)$, call this difference $\alpha(a)$. However, with $\eta_1 = \eta_2$, $\sigma(b_1) = m = \sigma(b_2)$ and

$$(\sigma \circ \alpha)(a) = \sigma(b_1 - b_2) = \sigma(b_1) - \sigma(b_2) = 0.$$

With σ a complete isometry on $\alpha(A)$, the above indicates $b_1 = b_2$ giving $\gamma_1 = \gamma_2$.

Next it will be shown that θ_2 is onto. Suppose that there exists a $\eta : C \to \mathcal{M}(A)$ such that $\pi \circ \eta = \tau$. Notice that $(\eta(c), c) \in PB$, the pullback construction from above. Let $\delta : C \to PB$ be the completely contractive morphism defined as $\delta(c) = (\eta(c), c)$. If $\xi : PB \to B$ is the inverse of the canonical completely isometric isomorphism from $B \to PB$ discussed above, then $\xi((\sigma(b), \beta(b))) = b$. Now define $\gamma : C \to B$ for all $c \in C$ by $\gamma(c) = (\xi \circ \delta)(c)$. As the composition of a completely contractive morphism and a complete isometry, γ is a completely contractive morphism from C to B. To see $\beta \circ \gamma = I_C$, it needs to be shown that if $b = \gamma(c)$, then $\beta(b) = c$. Let $c \in C$ with $b = \gamma(c) = \xi \circ \delta(c)$. Then $\delta(c) = (\eta(c), c) = (\sigma(b), \tau(b))$ for some $b \in B$ since $(\eta(c), c) \in PB$. Looking at the second coordinate, $\tau(b) = c$, completing the proof of this assertion.

Now suppose $\tau = 0$. As above, $(A, 0) \in PB$ since $A = \mathrm{Ker}(\pi)$. Similarly $(0, C) \in PB$ since $C = \mathrm{Ker}(\tau)$. This gives a splitting of the extension with $\gamma : C \to (0, C)$. The pullback $PB = A \oplus^\infty C$ since for all $a \in A$ and all $c \in C$, $\pi(a) = 0 = \tau(c)$. $\square$

Remarks. 1) By the above bijection, elements of $\mathrm{Mor}(C, \mathcal{Q}(A))$ will be referred to as *extensions of* C *by* A.

2) If an extension has Busby invariant the trivial morphism, i.e. $\tau = 0$, then the middle algebra has the form $A \oplus^\infty C$. If both A and C are unital, this is the only extension of C by A since $\mathcal{Q}(A) = 0$. This agrees with Lemma 3.3.1, which was proved using the Busby invariant of an extension of maximal C*-covers.

3) A split extension will be called *strongly unital* if γ can be chosen to be unital. A unital extension results when the middle algebra has a unit. From the above

diagram it is clear that γ is unital if and only if η is unital. However, even if τ is unital ζ could take the unit of C to an element of $\mathcal{M}(A)$ equivalent via π to the unit in $\mathcal{Q}(A)$.

3.5 Proper Morphisms Between Extensions

A morphism between two algebras A and A_1 in AUOA will be called a *proper morphism* if it takes a cai from A to a cai in A_1. It will be necessary to connect this concept with that of *multiplier nondegenerate morphisms*, [5, §2.6.11], which are defined as $\beta : A \to \mathcal{M}(A_1)$ such that A_1 is a nondegenerate module with respect to the natural module action of A on A_1 via β. This will be done in the next lemma.

Lemma 3.5.1. *Let A and A_1 be in the category AUOA. A morphism $\mu : A \to A_1$ is proper morphism if and only if it is multiplier nondegenerate.*

Proof. First suppose μ is proper so that if (e_t) is a cai for A, then $(\mu(e_t)) = (f_t)$ is a cai for A_1.Then for any $b \in A_1$, $bf_t \to b$ implying b is nondegenerately expressed with respect to the right module action of $\mu(A)$ on A_1. A similar calculation shows nondegeneracy with respect to the left module action. Viewing $A_1 \subset \mathcal{M}(A_1)$, μ can be considered to map into $\mathcal{M}(A_1)$ and by definition, μ is multiplier nondegenerate.

Now suppose that μ is multiplier nondegenerate. Then A_1 can be considered to be a nondegenerate $\mu(A)$-module. By Cohen's Factorization Theorem, every element of $a_1 \in A_1$ can be written as $a_1 = \mu(a)y$ for some $\mu(a) \in \mu(A)$ and $y \in A_1$. With μ continuous, then $\mu(e_t)\mu(a)y = \mu(e_t a)y \to \mu(a)y$. This means $\mu(e_t)a_1 \to a_1$ and

$(\mu(e_t))_t$ is a cai for A_1. □

A theorem from [5] was used in the previous chapter, but will be used frequently throughout the rest of this thesis. It is stated below and will be referenced in this form.

Theorem 3.5.2. *[5, Theorem 2.6.12] If A, B are approximately unital operator algebras, and if $\sigma : A \to \mathcal{M}(B)$ is a multiplier-nondegenerate morphism then σ extends uniquely to a unital completely contractive homomorphism $\hat{\sigma} : \mathcal{M}(A) \to \mathcal{M}(B)$. Moreover $\hat{\sigma}$ is completely isometric if and only if σ is completely isometric.*

Returning to the notion of morphisms between extensions, if μ in the diagram below is proper,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha_1} & B & \xrightarrow{\beta_1} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_2} & B_1 & \xrightarrow{\beta_2} & C_1 & \longrightarrow & 0 \end{array}$$

then μ extends to a unital morphism $\hat{\mu} : \mathcal{M}(A) \to \mathcal{M}(A_1)$ by Theorem 3.5.2. To further take this extension of μ and apply it to the two corona algebras of A and A_1 requires using the Factor Theorem.

Lemma 3.5.3. *Let μ be a proper completely contractive morphism from $A \to A_1$. Then there exists a unital completely contractive morphism $\tilde{\mu} : \mathcal{Q}(A) \to \mathcal{Q}(A_1)$ induced by the extension of μ to the multiplier algebras. Additionally if $\hat{\mu} : \mathcal{M}(A) \to$*

$\mathcal{M}(A_1)$ is surjective, then $\tilde{\mu}$ will be also. If μ is a complete isometry, then $\tilde{\mu}$ will be also. Lastly, if $\hat{\mu} : \mathcal{M}(A) \to A_1$ is a complete quotient morphism, then $\tilde{\mu}$ will be also.

Proof. The natural way to define $\tilde{\mu}$ for $m + A \in \mathcal{Q}(A)$ is $\tilde{\mu}(m + A) = \hat{\mu}(m) + A_1$. To show this is a completely contractive morphism, first define $\varphi : \mathcal{M}(A) \to \mathcal{Q}(A_1)$ by $\varphi = \pi_1 \circ \hat{\mu}$. Here π_1 is the canonical morphism taking $\mathcal{M}(A_1) \to \mathcal{Q}(A_1)$ and $\hat{\mu}$ as in the above discussion. As the composition of two complete contractions, φ is a complete contraction. Using the Factor Theorem and noting $A \subset \mathrm{Ker}(\varphi)$, φ descends to a morphism $\tilde{\varphi} : \mathcal{M}(A)/A = \mathcal{Q}(A) \to \mathcal{Q}(A_1)$ with completely bounded norm no larger than that of φ. That is $\|\tilde{\varphi}\| \leq 1$. It is clear that $\tilde{\varphi}(m + A) = \hat{\mu}(m) + A_1$. This agrees with the above described morphism $\tilde{\mu}$, so that $\|\tilde{\mu}\| \leq 1$.

For the last assertions, let $\pi_1 : \mathcal{M}(A) \to \mathcal{Q}(A)$ and $\pi_2 : \mathcal{M}(A_1) \to \mathcal{Q}(A_1)$ be the canonical morphisms. If $\hat{\mu}$ is surjective, then it is clear $\tilde{\mu}$ will be (since each $m_1 \in \mathcal{M}(A_1)$ has a pre-image in $\mathcal{M}(A)$ under $\hat{\mu}$, so that $m_1 + A_1$ will have a pre-image under $\tilde{\mu}$). Now suppose μ is a complete isometry which would make $\hat{\mu}$ a complete isometry Theorem 3.5.2. With μ proper, $\mu(A)$ shares a cai with A_1 so that the restriction of π to $\hat{\mu}(\mathcal{M}(A))$ is a complete quotient morphism by Lemma 2.1.2 making $\hat{\mu}(\mathcal{M}(A))/\hat{\mu}(A) \cong \hat{\mu}(\mathcal{M}(A))/A_1$. With $\hat{\mu}$ a complete isometry, it has an inverse. This shown that $\pi_1 \circ \hat{\mu}^{-1}$ is a complete quotient morphism with $\mathrm{Ker}(\pi_1 \circ \hat{\mu}^{-1}) = \hat{\mu}(A)$. By the factor theorem there is a completely isometric isomorphism from $\hat{\mu}(\mathcal{M}(A))/\hat{\mu}(A) \to \mathcal{Q}(A)$. Putting this together, $\mathcal{Q}(A) \cong \hat{\mu}(\mathcal{M}(A))/\hat{\mu}(A) \cong \hat{\mu}(\mathcal{M}(A))/A$ and $\tilde{\mu}$ is a complete isometry.

Suppose $\hat{\mu} : \mathcal{M}(A_1) \to \mathcal{M}(A)$ is a complete quotient morphism. There is the following commutative diagram.

$$\begin{array}{ccc} \mathcal{M}(A_1) & \xrightarrow{\pi_1} & \mathcal{Q}(A_1) \\ \downarrow{\scriptstyle \hat{\mu}} & & \downarrow{\scriptstyle \tilde{\mu}} \\ \mathcal{M}(A) & \xrightarrow{\pi_2} & \mathcal{Q}(A) \end{array}$$

With $\pi_2 \circ \hat{\mu}$ a complete quotient morphism, then $\tilde{\mu} \circ \pi_1$ must be also, implying $\tilde{\mu}$ is a complete quotient morphism since π_1 is a complete quotient morphism. □

Continuing with determining a criterion for the existence of the middle morphism in the general diagram of a morphism between extensions, the development follows from [17] with proof the same as in the C*-algebra case with details added.

Lemma 3.5.4. *Given a commutative diagram with the horizontal morphisms the embedding of ideals and μ proper,*

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & X_1 \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \delta} \\ A & \xrightarrow{\alpha} & X \end{array}$$

and letting

$$\sigma : X \to \mathcal{M}(A),\ \sigma_1 : X_1 \to \mathcal{M}(A_1),\ \textit{and}\ \hat{\mu} : \mathcal{M}(A_1) \to \mathcal{M}(A),$$

then

$$\sigma \circ \delta = \hat{\mu} \circ \sigma_1.$$

Proof. The proof will show the following expanded diagram commutes.

$$\begin{array}{ccccc} A_1 & \xrightarrow{\alpha_1} & X_1 & \xrightarrow{\sigma_1} & \mathcal{M}(A_1) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \hat{\mu}} \\ A & \xrightarrow{\alpha} & X & \xrightarrow{\sigma} & \mathcal{M}(A) \end{array}$$

By assumption μ is proper, so that for a cai of A_1, $(e_t)_t$ say, then $\mu((e_t)_t)$ is a cai for A. If $a_1 \in A_1$ and $a \in A$, equivalently the elements of the form $\mu(a_1)a$ are dense in A. The above assertion for left multiplication needs to only be proved for manner in which the multiplier algebra $\mathcal{M}(A)$ acts on elements of this form as a (possibly not closed) subalgebra of A. Right multiplication can be demonstrated in a symmetric manner.

Recall that canonical morphisms $\sigma(x)a = \sigma(x\alpha(a))$ and $\sigma_1(x_1)a_1 = \sigma_1(x_1\alpha_1(a_1))$. In other words, the composition $\sigma_i \circ \alpha_i$ is the identity on $A_i, i = 1, 2$. By commutativity of the left square of the above diagram, $\delta \circ \alpha_1 = \alpha \circ \mu$ on A_1. Looking at the manner in which elements of $\hat{\mu}(\mathcal{M}(A_1))$ act on elements of the form $\mu(a_1)a \in A$, and in particular $\sigma_1(x_1)$ for $x_1 \in X_1$:

$$(\hat{\mu} \circ \sigma_1)(x_1)\mu(a_1)a = (\hat{\mu} \circ \sigma_1)(x_1)(\hat{\mu} \circ \sigma_1 \circ \alpha_1)(a_1)a = (\hat{\mu} \circ \sigma_1)(x_1\alpha_1(a_1))a =$$

$$\mu(\sigma_1(x_1)a_1)a = (\sigma \circ \delta)(x_1\alpha_1(a_1))(a) = \sigma \circ \delta(x_1)(\sigma \circ \delta \circ \alpha)(a_1)a =$$

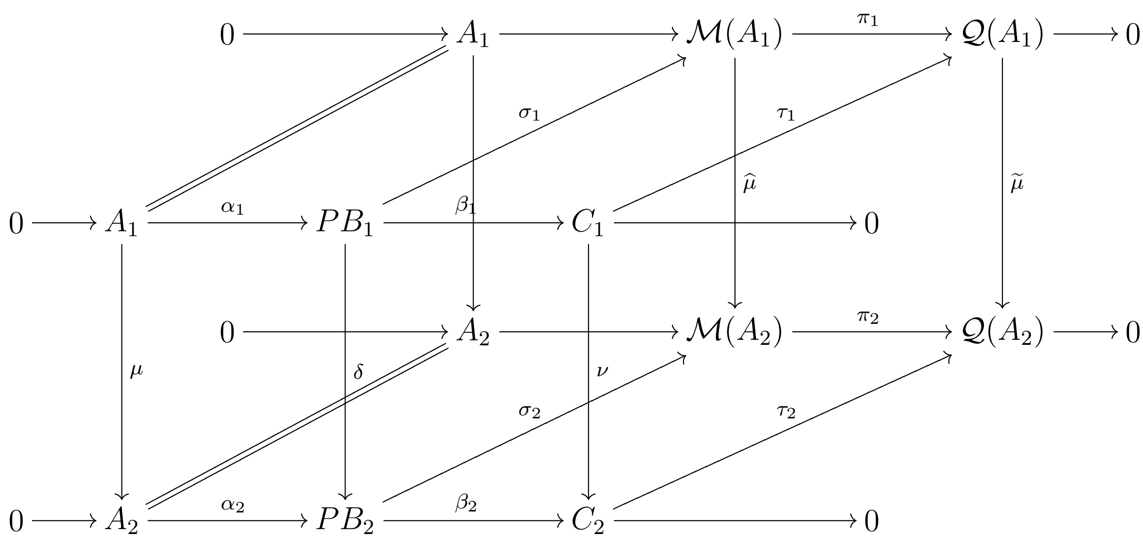
$$(\sigma \circ \delta)(x_1)(\sigma \circ \alpha \circ \mu)(a_1)a = (\sigma \circ \delta)(x_1)\mu(a_1)a.$$

Putting the first expression with the last expression gives the result. □

Again the proof of the next theorem follows the proof of [17, Theorem 2.2] using Lemma 3.5.3 above.

Theorem 3.5.5. *Given two extensions as in the diagram above and morphisms μ and ν with μ proper, then the completely contractive morphism δ exists if and only if $\tilde{\mu}\circ\tau_1 = \tau_2\circ\nu$ where τ_1 and τ_2 are the associated Busby invariants of the two extensions. Furthermore, δ is the unique such completely contractive morphism which makes the diagram commutative.*

Proof. The three-dimensional diagram below will be used. Note that by construction the leftmost sidewise square, top, bottom, and back faces commute. Also the pullback algebras will be substituted as the middle algebras by Theorem 3.4.3. This will allow an exact definition for the morphisms $\sigma_1, \sigma_2, \beta_1,$ and β_2 as the projection on the associated coordinate.



Suppose that δ exists so that the front faces commute. By Lemma 3.5.4, the middle square commutes forcing the rightmost face to commute as shown in the following equation.

$$\tilde{\mu} \circ \tau_1(\beta_1(b_1)) = \pi_2 \circ \hat{\mu}(\sigma(b_1)) = \pi_2 \circ \sigma_2(\delta(b_1)) = \tau_2 \circ \beta_2(\delta(b_1)) = \tau_2 \circ \nu(\beta_1(b_1)).$$

Now suppose the rightmost sideways face commutes and a morphism δ will be constructed. Define $\delta : PB_1 \to PB_2$ as $\delta((m_1, c_1)) = (\hat{\mu}(m_1), \nu(c_1))$. Assuming this mapping can be shown to be well defined, it is clear is a completely contractive homomorphism since the component morphisms are completely contractive homomorphisms. To see $(\hat{\mu}(m_1), \nu(c_1)) \in PB_2$, recall that $\pi_1(m_1) = \tau_1(c_1)$, which gives the following relationships:

$$\pi_2(\hat{\mu}(m_1)) = \tilde{\mu}(\pi_1(m_1)) = \tilde{\mu}(\tau_1(c_1)) = \tau_2(\nu(c_1)).$$

□

3.6 Completely Essential Extensions and Quotient Extensions

An ideal A of B was defined in Section 3.2 to be essential if the canonical morphism $\sigma : B \to \mathcal{M}(A)$ from above is one-to-one. The ideal A is completely essential is σ is a complete isometry. To begin this section, it will be shown the associated Busby invariant τ will, in these cases, also have the respective property of being one-to-one or a complete isometry.

Lemma 3.6.1. *An extension in the category of AUOA is essential (respectively, completely essential) if and only if the associated Busby invariant is one-to-one (respectively, a complete isometry.)*

Proof. The definition of the Busby invariant gives the following morphisms of extensions.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} & & \\
0 & \longrightarrow & A & \longrightarrow & \mathcal{M}(A) & \xrightarrow{\pi} & \mathcal{Q}(A) & \longrightarrow & 0
\end{array}$$

For clarity it will be assumed that $A \subset B$. For the essential case, apply the Five Lemma from module theory. If τ is a complete isometry then by Lemma 3.3.3 σ is a complete isometry. Suppose σ is a complete isometry. With $\text{Ker}(\sigma) = 0$, then $\text{Ker}(\pi \circ \sigma) = A$. By the Factor Theorem there exists a unique complete isometry, call it $\tilde{\sigma} : B/A \to \mathcal{Q}(A)$. Now let $\tilde{\beta}^{-1} : C \to B/A$ be the inverse of the canonical completely isometric isomorphism due to β being a complete quotient morpism. It is clear that $\tilde{\sigma} \circ \tilde{\beta} : C \to \mathcal{Q}(A)$ is a complete isometry. By the uniquness of τ, it needs to be shown that $\tilde{\sigma} \circ \tilde{\beta} \circ \beta = \pi \circ \sigma$ on B. Let $b \in B$. Then

$$(\tilde{\sigma} \circ \tilde{\beta} \circ \beta)(b) = (\tilde{\sigma} \circ \tilde{\beta})(c) = \tilde{\sigma}(b + A) = (\pi \circ \sigma)(b),$$

giving the result. □

The next lemma gives a method for creating an essential or a completely essential extension from a given extension. The Busby invariant τ determines whether this induced extension is either essential or completely essential. Quotient extensions will not have much application in Diagram II, III, or IV completions, or the chapter relating operator algebra extensions to C*-algebra extension. This is because, in the case of universal diagram completions, either the first or third algebras are repeated. Also a complete isometry between operator algebras does not necessarily induce an injective $*$-homomorphism between C*-covers. However, the next lemma will have an application in situations where, in particular, a completely essential extension is preferred.

Before beginning the next lemma, an important morphism will be defined. Suppose there is an extension of C by A with middle algebra B. Considering the canonical morphisms, let $J = \text{Ker}(\sigma)$ and $K = \text{Ker}(\tau)$. The morphism $\lambda : B/J \to C/K$ can be defined by $\lambda(b + J) = \beta(b) + K$. This will be proved to be a completely contractive homomorphism in the following proof.

Lemma 3.6.2. *Suppose there is an extension of C by A with middle algebra B, all with cais and let $\tau : C \to \mathcal{Q}(A)$ be the associated completely contractive morphism (respectively complete quotient morphism onto its range.) If $K = Ker(\tau)$ and $J = Ker(\sigma)$, let $\theta_1 : B \to B/J$ and $\theta_2 : C \to C/K$ be the induced complete quotient morphisms. Then there is an induced essential extension (respectively completely essential extension) of C/K by $A \cong A/J$ with middle algebra B/J.*

Proof. Let $\beta : B \to C$ be the completely contractive morphism from the extension of C by A giving the following morphism between extensions.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \mathcal{M}(A) & \xrightarrow{\pi} & \mathcal{Q}(A) & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle\tau} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\beta} & C & \longrightarrow & 0
\end{array}$$

First it will be shown that $\beta(J) = K$. Since σ is a complete isometry on A, it is clear $A \cap J = 0$. By commutativity of the right square, $\pi \circ \sigma = \tau \circ \beta$ and $\beta(J) \subset K$. Since $A = \text{Ker}(\beta)$, then $A = \beta^{-1}(0)$, so let $k \in K$ be nonzero requiring that $\tau(k) = 0$ be nontrivial. If $x \in \beta^{-1}(k)$, and since $k \neq 0$, $x \notin A$. With $(\pi \circ \sigma)(x) = 0$, then $\sigma(x) \in A$. Let $\sigma(x) = a$ giving $\sigma^{-1}(a) = \{a + j : j \in J\}$. Let $x = a + j_0$ with $j_0 \neq 0$

since $x \notin A$. By assumption $x \in \beta^{-1}(k)$, so that $\beta(a + j_0) = k$. Since $A = \text{Ker}(\pi)$, $k = \beta(a + j_0) = \pi(a) + \pi(j_0) = 0 + \pi(j_0) = \pi(j_0)$. With j_0 in the inverse image of k under π, $K \subset \beta(J)$ so that $\beta(J) = K$. Let $\theta_1 : B \to B/J$ and $\theta_2 : C \to C/K$ be the canonical morphisms. Define $\lambda : B/J \to C/K$ by $\lambda(b + J) = \beta(b) + K$ for all $b + J \in B/J$. This will obviously make the following diagram commute:

$$\begin{array}{ccc} B/J & \xrightarrow{\lambda} & C/K \\ \theta_1 \big\uparrow & & \big\uparrow \theta_2 \\ B & \xrightarrow{\beta} & C \end{array} \tag{3.6.1}$$

With $J \subset \text{Ker}(\theta_2 \circ \beta)$, by the Factor Theorem λ is a surjective completely contractive morphism since $\theta_2 \circ \beta$ is a complete contraction onto C/K. Furthermore, λ is a complete quotient morphism since all other morphisms in the diagram are complete quotient morphisms. It needs to be seen that $A/J = \text{Ker}(\lambda)$. Let $x \in \text{Ker}(\lambda)$ with $y \in (\theta_1)^{-1}(x)$. If $y \in A$, then $\beta(y) = 0 = \theta_2(0)$ giving $A/J \subset \text{Ker}(\lambda)$. If $y \notin A$, then $\beta(y) \neq 0$ so that if $\theta_2(\beta(y)) = 0$, then $\beta(y) \in K$ forcing $y = a + j$ for some $a \in A$ and some $j \in J$. Evidently $\theta_1(y) = x = a + J \in A/J$ and $\text{Ker}(\lambda) = A/J$. It remains to see that $A/J \cong A$. Let $\tilde{\sigma} : B/J \to \mathcal{M}(A)$ be the induced completely contractive morphism with $\tilde{\sigma} \circ \theta_1 = \sigma$. With σ is a completely contractive morphism by the Factor Theorem $\tilde{\sigma}$ is also giving $\|a + J\| \geq \|\tilde{\sigma}(a + J)\| = \|\sigma(a)\| = \|a\|$ for all $a \in A$ and at all matrix levels. With θ a completely contractive morphism, $\|a + J\| \leq \|a\|$ so that $\|a\| = \|a + J\|$ also at all matrix levels. This gives an extension of C/K by $A/J \cong A$ with middle algebra B/J.

To see this extension is essential (respectively, completely essential), the above induced morphism $\tilde{\sigma}$ is a complete contraction. Since $J = \text{Ker}(\sigma)$, $\tilde{\sigma}$ is one-to-one (respectively, a complete isometry) with $\tilde{\sigma} \circ \theta_1 = \sigma$. This means in particular that $\tilde{\sigma}(a + J) = \sigma(a) = a$. Let $\sigma' : B/J \to \mathcal{M}(A)$ be the induced morphism due to the extension of C/K by A. The complete isometry $\alpha : A \to A/J$ must be defined by $\alpha(a) = a + J$ and $\sigma'(a + J) = a$ for all $a + J \in A/J$ since $\sigma \circ \alpha$ must the identity morphism on A. Since $\tilde{\sigma}$ and σ' agree on A, by uniqueness $\tilde{\sigma} = \sigma'$. By definition the induced extension of C/K by A is essential (respectively, completely essential). □

Another way to approach the proof of the above lemma in the completely essential case would be to note that $\tilde{\tau} : C/K \to \mathcal{Q}(A)$ is a complete isometry and form the pullback construction as a Diagram I completion. In showing the pullback is completely isometrically isomorphic to B/J, it would need to be shown that $\beta(J) = K$, and by commutativity of the morphism between extensions of the original extension, $\tau(c + k) = \sigma(b + j) + A$ for all $j \in J$ and all $k \in K$. This leads to the definition of λ which then gives a morphism $\gamma : B/J \to PB$ defined by $\gamma(b + J) = (\tilde{\sigma}(b + J), \lambda(b + J))$. If β is a complete quotient morphism, it is straightforward that γ is a complete isometry. A little more work would be required in the essential case.

Given an extension of C by A with τ a complete quotient morphism, then a more explicit definition of the pullback due to τ which is given in the following corollary.

Corollary 3.6.3. *Let E be an extension of C by A with middle algebra B with τ a complete quotient morphism. Then $B \cong B/J \oplus_{C/K} C$ along λ and θ_2 completely isometrically isomorphically.*

Proof. By Lemma 3.6.2, $\mathrm{Ran}(\sigma) \cong B/J$ and $\mathrm{Ran}(\tau) \cong C/K$ via $\tilde{\sigma}$ and $\tilde{\tau}$ respectively. This means that all first coordinates of the pullback, PB, from the original extension come from $\tilde{\sigma}(B/J)$ and all second coordinates come from C. Define a completely isometric isomorphism from $\delta : PB \to B/J \oplus_{C/K} C$ by $\delta((m, c)) = (\hat{\sigma}^{-1}(m), c) = (b+J, c)$. This is well defined as long as $\lambda(b+J) = \theta_2(c)$ since $\hat{\sigma}$ is a complete isometry and so is one to one. With $(m, c) \in PB \cong B$, $(m, c) = (\sigma(b), \beta(b))$ for some $b \in B$. For that b, by the way λ was defined for the commutative diagram, Diagram 3.6.1, $(\lambda \circ \theta_1)(b) = \lambda(b + J) = \beta(b) + K = (\theta_2 \circ \beta)(b) = \theta_2(c)$. An inverse can be defined in the obvious way, that is $\delta^{-1}((\hat{\sigma}^{-1}(m), c)) = (m, c)$, making δ a completely isometric isomorphism. Composing δ with the inverse of the completely isometric isomorphism from B onto the pullback gives a completely isometric isomorphism from $B/J \oplus_{C/K} C$ onto B. □

3.7 Functoriality

For this section the category AUOA, denoted by $\mathcal{A}$, will contain approximately unital operator algebras with the morphisms the proper morphisms discussed above. In the case where morphisms are not required to be proper in AUOA the functoriality is much deeper. With Blecher we hope to present this elsewhere. The category OP, denoted by $\mathcal{C}$ will contain all operator algebras with the morphisms the completely contractive morphisms. For any operator algebra $C \in \mathcal{C}$ and any operator algebra in $A \in \mathcal{A}$ a category consisting of the sets $\mathbf{ext}(\mathcal{C}, \mathcal{A})$ will be regarded containing all equivalence classes of extensions of C by A. In particular **ext(C,A)** will be regarded as a subset of $\mathbf{ext}(\mathcal{C}, \mathcal{A})$. The morphisms will simply be the functions between sets.

Before the two families of functors can be defined below, a lemma will be required.

Lemma 3.7.1. *Let $\mu_1 : A \to A_1$ and $\mu_2 : A_1 \to A_2$ each be proper. Then $\mu_2 \circ \mu_1$ is proper and $\hat{\mu}_2 \circ \hat{\mu}_1 = \widehat{\mu_2 \circ \mu_1} : \mathcal{M}(A) \to \mathcal{M}(A_2)$. Additionally $\tilde{\mu}_2 \circ \tilde{\mu}_1 = \widetilde{\mu_2 \circ \mu_1} : \mathcal{Q}(A) \to \mathcal{Q}(A_2)$.*

Proof. By Theorem 3.5.2, $\widehat{\mu_2 \circ \mu_1}$ is the unique morphism extending $\mu_2 \circ \mu_1$. Since $\hat{\mu}_2 \circ \hat{\mu}_1$ does the same thing, then must agree. For the second assertion, note that

$$\widetilde{\mu_2 \circ \mu_1}(m + A) = \widehat{\mu_2 \circ \mu_1}(m) + A_2 = (\hat{\mu}_2 \circ \hat{\mu}_1)(m) + A_2 =$$

$$\tilde{\mu}_2(\hat{\mu}_1(m) + A_1) = (\tilde{\mu}_2 \circ \tilde{\mu}_1)(m + A).$$

□

By fixing an $A \in \mathcal{A}$, a contravariant functor can be defined for $C \in \mathcal{B}$ by $\mathbb{R}_A(C) = \mathbf{ext}(C, A)$. Given any operator algebra $A \in \mathcal{A}$, there is always the trivial extension $A \oplus^\infty C$ so that $\mathbf{ext}(C, A)$ is nonempty. Let $\nu : C_1 \to C$ be a completely contractive morphism. For any $A \in \mathcal{A}$, given an equivalence class of extensions $E_\tau : 0 \to A \to B \to C \to 0$ with τ the associated Busby invariant, a completely contractive morphism $\tau' = \tau \circ \nu : C_1 \to \mathcal{Q}(A)$ can be formed. Designate the equivalence class of extensions due to τ' by $E_{\tau'}$. Define $\mathbb{R}_A(\nu)(E_\tau) = E_{\tau'}$. It is clear $\mathbb{R}_A(\nu) : \mathbb{R}_A(C) \to \mathbb{R}_A(C_1)$ and interestingly, $\mathbb{R}_A(\nu)$ takes E_τ to the equivalence class of extensions in the universal completion under Diagram I. This is also the mapping defined above as γ. Suppose $\nu_1 : C_2 \to C_1$ is a completely contractive morphism. Then $\nu \circ \nu_1 : C_2 \to C$

is a completely contractive morphism. Define $\tau'' = \tau \circ (\nu \circ \nu_1) : C_2 \to \mathcal{Q}(A)$. Note that as defined, $\mathbb{R}_A(\nu_1)(E_{\tau'}) = E_{\tau''}$. Then

$$\mathbb{R}_A(\nu \circ \nu_1)(E_\tau) = E_{\tau''} = \mathbb{R}_A(\nu_1)(E_{\tau'}) = (\mathbb{R}_A(\nu_1) \circ \mathbb{R}_A(\nu))(E_\tau),$$

making $(\mathbb{R}_A)_{A \in \mathcal{A}}$ a family of contravariant functors.

The second family of functors can be defined for all $C \in \mathcal{C}$. Let $A \in \mathcal{A}$ and defined $\mathbb{L}_C(A) = \mathbf{ext}(C, A)$. Since given any operator algebra $C \in OP$, there is always the trivial extension $A \oplus^\infty C$ as before giving $\mathbf{ext}(C, A)$ is nonempty. Suppose $\mu : A \to A_1$ is a proper morphism between approximately unital algebras in AUOA as defined above. Given $\tau : C \to \mathcal{Q}(A)$, there is a related equivalence class of extensions in $E \in \mathbf{ext}(C, A)$. Form the composition $\tilde{\mu} \circ \tau : C \to \mathcal{Q}(A_1)$ which gives an equivalence class of extensions E_1 of C by A_1. Let $\mathbb{L}_C(\mu)(E) = E_1$ showing $\mathbb{L}_C(\mu) : \mathbf{ext}(C, A) \to \mathbf{ext}(C, A_1)$ and by Lemma 3.7.1,

$$(\mathbb{L}_C(\mu) \circ \mathbb{L}_C)(A) = E_1 = \mathbb{L}_C(A_1) = \mathbb{L}_C(\mu(A)) = (\mathbb{L}_C \circ \mu)(A).$$

Now suppose that $\mu : A \to A_1$ and $\mu_1 : A_1 \to A_2$ with both morphisms proper. The composition $\mu_1 \circ \mu : A \to A_2$ is proper. It is clear

$$\mathbb{L}_C(\mu_1 \circ \mu) : \mathbb{L}_C(A) \to \mathbb{L}_C(A_2) = \mathbb{L}_C(\mu_1) \circ \mathbb{L}_C(\mu).$$

The functor $\mathbb{L}_C$ is a covariant functor. Fixing C and requiring μ to be proper gives a functor which takes an equivalence class of extensions in $\mathbf{ext}(C, A)$ to an equivalence class of the extension in $\mathbf{ext}(C, A_1)$. This definition of $\mathbb{L}_C(\mu)$ coincides with the δ mapping described above. In Chapter 6 Diagram III completions will be discussed in which the first vertical arrow is required to be proper. The functor $\mathbb{L}_C$ takes an extension which is the first line of a Diagram III form to the equivalence class of the second line of the universal completion. This is discussed in Section 6.2.

Before a bifunctor can be defined using the above families of functors, it needs to be shown that for $\mu : A \to A_1$ and $\nu : C \to C_1$ that:

$$\mathbb{L}_{C_1}(\mu) \circ \mathbb{R}_A(\nu) = \mathbb{R}_{A_1}(\nu) \circ \mathbb{L}_C(\mu).$$

Let $E_\tau \in \mathbf{ext}(C, A) = \mathbb{R}_A(C)$. By definition $\mathbb{R}_A(\nu)(E_\tau) = E_{\tau_1} \in \mathbf{ext}(C_1, A)$ with $\tau_1 = \tau \circ \nu$. With μ proper, define $\tau_2 = \tilde{\mu} \circ \tau_1 = \tilde{\mu} \circ \tau \circ \nu$. Then $\mathbb{L}_{C_1}(\mu)(E_{\tau_1}) = E_{\tau_2} \in \mathbf{ext}(C_1, A_1)$. Working on the other side of the above displayed equation, $\mathbb{L}_C(\mu)(E_\tau) = E_{\tau_3} \in \mathbf{ext}(A_1, C)$ with $\tau_3 = \tilde{\mu} \circ \tau$. As defined, $\mathbb{R}_{A_1}(\nu)(E_{\tau_3}) = E_{\tau_4}$ where $\tau_4 = \tau_3 \circ \nu = \tilde{\mu} \circ \tau \circ \nu = \tau_2$ which shows the equality of the above displayed equation.

Now a bifunctor can be defined by $\mathbb{F} : \mathcal{C} \times \mathcal{A} \to \mathbf{ext}(\mathcal{C}, \mathcal{A})$ by $\mathbb{F}(C, A) = (L)_C(A) = (R)_A(C) = \mathbf{ext}(C, A)$. Restricting $\mathbb{F}$ to $(C, \mathcal{A})$ is the functor $\mathbb{L}_C$ and restricting $\mathbb{F}$ to $(\mathcal{C}, A)$ is the functor $\mathbb{R}_A$.

Chapter 4

Sub-extensions

4.1 Definition and Conditions for Existence

A sub-extension will be defined as a short exact sequence in which the following morphism between extensions exists and all vertical arrows are inclusion morphisms into the another short exact sequence.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D} & \xrightarrow{\alpha} & \mathcal{E} & \xrightarrow{\beta} & \mathcal{F} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha|_A} & B & \xrightarrow{\beta|_B} & C & \longrightarrow & 0 \end{array}$$

According to the definition of an extension, $\alpha_{|A}$ must be a complete isometry, which is obvious. Also $\alpha(A) = \mathrm{Ker}(\beta_{|B})$ must contains a cai, but not necessarily a shared cai with $\mathcal{D}$. Then $\beta_{|B}$ must be a complete quotient morphism. The only remaining matter to check is that $A = \alpha^{-1}(B)$. Let $X = \alpha^{-1}(B) \subset \mathcal{D}$. This means that $\alpha(X) = B \cap \alpha(\mathcal{D})$. This shows that $(\beta \circ \alpha)(X) = 0$ so that $A \subseteq X$. With both β and $\beta_{|_B}$ being complete quotient morphisms, $B/\alpha(A) \cong C \cong B/\mathcal{D}$. Also, $X = B \cap \alpha(\mathcal{D})$ is an ideal in B, and there is the following ordering on ideals: $\alpha(A) \subseteq \alpha(X) \subset \alpha(\mathcal{D})$. By the Factor Theorem there are canonical morphisms $\pi_1 : B/\alpha(A) \to B/\alpha(X)$, and $\pi_2 : B/\alpha(X) \to B/\alpha(\mathcal{D})$. Forming the composition $\pi_2 \circ \pi_1 : B/\alpha(A) \to B\alpha(\mathcal{D})$, which is a completely isometric isomorphism. Clearly π_1 is a completely isometric isomorphism and $A = \alpha^{-1}(B)$.

Proposition 4.1.1. *Let* $\mathrm{E} \in \boldsymbol{ext}(\mathcal{F}, \mathcal{D})$ *with middle operator algebra* $\mathcal{E}$. *Let* $B \subset \mathcal{E}$ *be a nontrivial closed subalgebra. There exists a sub-extension of E with middle algebra* B *if and only if* $\alpha(\mathcal{D}) \cap B$ *is approximately unital and* $\beta_{|_B}$ *is a complete quotient morphism. If* B *contains a cai for* $\alpha(\mathcal{D})$, *then both conditions are automatically satisfied.*

Proof. Suppose a sub-extension exists as in the definition implying A is approximately unital and $\beta_{|_B}$ is a complete quotient morphism with $\mathrm{Ker}(\beta_{|_B}) = \alpha(A)$. It needs to be shown $\alpha(A) = \alpha(\mathcal{D}) \cap B$ which follows from the above discussion.

For the other direction, suppose $B \subset \mathcal{E}$ and $\alpha(\mathcal{D}) \cap B = \alpha(A')$ for a subalgebra A' of $\mathcal{D}$ which has a cai. Further suppose that $\beta_{|_B}$ is a complete quotient morphism. This immediately gives $C = \beta(B)$ is a closed subalgebra of $\mathcal{F}$. A sub-extension will be created. If B is taken as the middle algebra and C the last algebra, it will be demonstrated that $\alpha^{-1}(\mathrm{Ker}(\beta_{|_B})) = A'$. Let $X = \mathrm{Ker}(\beta_{|_B})$. With $\beta_{|_B}(X) = \beta(X) = 0$, so that $X \subset \alpha(\mathcal{D})$. Since α is a complete isometry, $\alpha^{-1}(X)$ is a closed subalgebra of $\mathcal{D}$. As the kernel of $\beta_{|_B}$, $X \subset B$, so by the definition of A', $X \subset \alpha(A')$. Let $\alpha(a) \in \alpha(A')$. This gives $\beta(\alpha(a)) = 0$ and $\alpha(a) \in \mathrm{Ker}(\beta_{|_B}) = X$ so that $X = \alpha(A')$.

For the final assertion, suppose $\alpha(A) = B \cap \alpha(\mathcal{D})$ and that B contains a cai for $\alpha(\mathcal{D})$. This cai would also be contained in A by definition of $\alpha(A)$ indicating that A is approximately unital. By Lemma 2.1.2 $\beta_{|_B}$ is a complete quotient morphism. Evidently both conditions are met for a sub-extension to exist. □

From Proposition 4.1.1 fixing B uniquely determines A and C. It is not true that fixing either subalgebras A or C uniquely determines the other two algebras. Given a closed subalgebra C of $\mathcal{F}$ the existence of at least one sub-extension can be guaranteed as will be seen in the next result. Determining the existence of sub-extensions beginning with a specified subalgebra of $\mathcal{D}$ will be addressed later in this section.

Lemma 4.1.2. *Given $E \in \boldsymbol{ext}(\mathcal{F}, \mathcal{D})$ with middle term $\mathcal{E}$, let C be a closed subalgebra of $\mathcal{F}$. Then there exists at least one sub-extension with last term C which is universal as a Diagram I type completion. In fact any other sub-extension ending with C is a sub-extension of the universal one.*

Proof. Let $B = \beta^{-1}(C)$, which is closed. It is claimed the universal sub-extension is as follows:

$$0 \longrightarrow \mathcal{D} \xrightarrow{\alpha} B \xrightarrow{\beta_{|_B}} C \longrightarrow 0$$

This is a sub-extension by Proposition 4.1.1 since $\alpha(\mathcal{D}) = \beta^{-1}(0) \subset B$, and B

trivially contains a cai for $\alpha(\mathcal{D})$. To see the above is the universal completion of the Diagram I form below,

$$\begin{array}{ccccccccc} & & & & & & C & & \\ & & & & & & \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{D} & \xrightarrow{\alpha} & \mathcal{E} & \xrightarrow{\beta} & \mathcal{F} & \longrightarrow & 0 \end{array}$$

one must check that any sub-extension with last algebra C has a subalgebra of B as its middle algebra and the first algebra a subalgebra of $\mathcal{D}$. This is also trivial since the middle algebra, call it B_1, is a subset of $\beta^{-1}(C) = B$. The first algebra of any sub-extension by Proposition 4.1.1 is $\alpha^{-1}(\alpha(\mathcal{D}) \cap B_1) \subset \mathcal{D}$. $\square$

For a given closed subalgebra C of $\mathcal{F}$ there will generally be many sub-extensions with last algebra C. It is also generally true that given A and C as closed subalgebras of $\mathcal{D}$ and $\mathcal{F}$ respectively with A containing a cai, the middle algebra is not uniquely determined. As in the proof of Lemma 4.1.2, if B_1 and B_2 are subalgebras of B with $\mathcal{D} \cap B_1 = \mathcal{D} \cap B_2$, then there would be two sub-extensions with middle algebras B_1 and B_2, and same first and last algebras.

Looking at the situation where $A \subset \mathcal{D}$ contains a cai, by Proposition 4.1.1 the subalgebras B of $\mathcal{E}$ such that $\mathcal{D} \cap B = A$ and $\beta_{|B}$ is a complete quotient morphism, are in bijective correspondence to sub-extensions with A as the first term. If further the sub-extensions are to be characterized by A and C rather than A and B, the situation is more complicated as seen in Lemma 4.1.2. If, however, A is required to contain a cai for $\mathcal{D}$, then such a characterization is more straightforward. This requirement is not without merit on its own since it would be true in the case $\mathcal{D}$ is a C*-algebra generated by A, which by [5, Lemma 2.1.7] implies A and $\mathcal{D}$ share a cai.

Lemma 4.1.3. *Let E be an extension as in Proposition 4.1.1 with τ is the Busby invariant of E. Let A be a closed subalgebra of $\mathcal{D}$ which contains a common cai for $\mathcal{D}$. Denote by $\hat{j} : \mathcal{Q}(A) \to \mathcal{Q}(\mathcal{D})$ the canonical complete isometry from Corollary 2.1.4 which is induced by the inclusion morphism $j : A \to \mathcal{D}$. If C is any nontrivial closed subalgebra of $\mathcal{F}$, a sub-extension can be formed beginning with A and ending with C if and only if there exists a completely contractive morphism $\tau' : C \to \mathcal{Q}(A)$ such that $\tau' = \hat{j}^{-1} \circ \tau_{|_C}$. Additionally, given C and A, the middle algebra is uniquely determined. By the requirement on τ', it will necessarily be the case that $\tau(C) \subset \hat{j}(\mathcal{Q}(A))$ for the sub-extension to exist.*

Proof. Suppose that $\tau' = \hat{j}^{-1} \circ \tau_{|_C} : C \to \mathcal{Q}(A)$ exists. This gives an exact sequence with first term A and last term C. Let PB_1 be the pullback construction due to τ', with α' and γ_1 the canonical associated morphisms to the pullback. A morphism between extensions will be constructed as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha'} & PB_1 & \xrightarrow{\gamma_1} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{D} & \xrightarrow{\alpha} & \mathcal{E} & \xrightarrow{\beta} & \mathcal{F} & \longrightarrow & 0 \end{array}$$

With $\tau' = \hat{j}^{-1} \circ \tau$ by hypothesis, Theorem 3.5.5 guarantees the existence of a unique morphism, call it $\delta : PB_1 \to \mathcal{E}$, which is a complete isometry by Lemma 3.3.3. Let B be the range of this complete isometry giving a sub-extension with B uniquely determined via the uniqueness of δ.

Suppose a sub-extension exists beginning with A and ending with C with Busby invariant τ'. Let $\delta : B \to \mathcal{E}$ be the middle morphism which makes the diagram commute. Again by Theorem 3.5.5 the existence of the middle morphism requires $\hat{j} \circ \tau' = \tau \circ k$ where k is the inclusion of C into $\mathcal{F}$ and apparently $\tau' = \hat{j}^{-1} \circ \tau_{|C}$. Using Theorem 3.5.5, the morphism between the sub-extension and original extensions which makes the diagram commute, is unique. □

The following lemma will be helpful for the next theorem and could be considered a corollary of Theorem 3.5.2 or Lemma 3.5.3.

Lemma 4.1.4. *Suppose that A and $\mathcal{D}$ are in the category UAOA and $A \subset \mathcal{D}$ such that each shares a common cai. Then $\mathcal{M}(A) \subset \mathcal{M}(\mathcal{D})$ completely isometrically isomorphically.*

Proof. Let $\iota : A \to \mathcal{D}$ be the inclusion morphism which is a complete isometry and multiplier nondegenerate since they share a common cai. By Theorem 3.5.2 this extends to a complete isometry $\hat{\iota} : \mathcal{M}(A) \to \mathcal{M}(\mathcal{D})$ giving the result. □

Theorem 4.1.5. *Given an extensions of operator algebras as follows:*

$$\text{E} : \qquad 0 \longrightarrow \mathcal{D} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \longrightarrow 0$$

and an approximately unital nontrivial closed subalgebra A of $\mathcal{D}$ such that A contains a cai for $\mathcal{D}$, then there is a bijective correspondence between the equivalence classes *of sub-extensions of E (under strong isomorphism) beginning with A and nontrivial closed subalgebras C of $\mathcal{G} \subset \mathcal{F}$ defined as:*

$$\mathcal{G} = \{c \in \mathcal{F} : \exists\, b \in \mathcal{E} \text{ with } b\alpha(A) + \alpha(A)b \subset \alpha(A) \text{ and } \beta(b) = c\}.$$

Proof. First suppose there is an sub-extension of C by A with middle algebra B. As an exact sequence, A is an ideal in B so that for all $b \in B$, $b\alpha(A) + \alpha(A)b \subset \alpha(A)$. To see $C \subset \mathcal{G}$, note that with $\beta(B) = C$, for every $c \in C$ there is a $b \in B$ with $b \in \beta^{-1}(c)$ such that $b\alpha(A) + \alpha(A)b \subset \alpha(A)$. To see that this B is the unique middle algebra in such a sub-extension, apply Lemma 4.1.3.

Now let C be a closed subalgebra of $\mathcal{G}$ and define

$$B = \{b \in \beta^{-1}(C) : b\alpha(A) + \alpha(A)b \subset \alpha(A)\}.$$

This is a closed subalgebra of $\mathcal{E}$. Also $\alpha(A) \subset B$ since $\beta(\alpha(A)) = 0$ and A is a trivial ideal of itself. To see $\alpha(A) = \mathrm{Ker}(\beta_{|_B})$, first note that $\mathrm{Ker}(\beta_{|_B}) \subset \alpha(\mathcal{D})$. Let $\alpha(x) \in \mathrm{Ker}(\beta_{|_B})$ and (e_t) be a common cai for A and $\mathcal{D}$ which exists by hypothesis. By definition of B, $xe_t \in A$ giving that $x \in A$. By Proposition 4.1.1, there is a sub-extension with algebras A, B and C. That this is the unique such sub-extension, follows from the previous paragraph. □

Remarks. 1.)There is a largest such sub-extension, that is

$$0 \longrightarrow A \xrightarrow{\alpha} \{b \in \mathcal{E} : bA + Ab \subset A\} \xrightarrow{\beta} \mathcal{G} \to 0.$$

It is easy to see that any other sub-extension with first algebra A would be a sub-extension of the above extension since the above middle algebra is the largest subalgebra of $\mathcal{E}$ containing A as an ideal.

2.) If $A = \mathcal{D}$ the situation is covered in Chapter 2 as a Diagram I completion.

4.2 Examples

The next lemma will be used to construct examples where sub-extensions can be used to prove certain types of extensions are split.

Lemma 4.2.1. *Given an extension E as in Proposition 4.1.1 with $\mathcal{D}$ also a C^*-algebra, then there exists a nontrivial sub-extension of C^*-algebras if and only if $\triangle(\mathcal{F}) := \mathcal{F} \cap \mathcal{F}^*$ is nontrivial.*

Proof. It is know that a completely contractive homomorphism from a C^*-algebra into a Banach algebra has a C^*-algebra as its range by [5, 2.1.2]. From this, if $\mathcal{E}$ is the middle algebra of E, then $\alpha(\mathcal{D}) \subset \triangle(\mathcal{E}) \neq 0$ since α is a one-to-one. As an ideal in $\mathcal{E}$, it is also an ideal in $\triangle(\mathcal{E})$. Suppose $\triangle(\mathcal{F})$ is nontrivial. From Lemma 4.1.3 a sub-extension exists with first algebra $\mathcal{D}$ and last algebra $\triangle(\mathcal{F})$ by restricting τ to $\triangle(\mathcal{F})$ which again is a $*$-homomorphism. By Proposition 3.2.1 the middle algebra is a C^*-algebra, call it X. As in the proof of Theorem 4.1.5, $X = \beta^{-1}(\triangle(\mathcal{F}))$. It is claimed $\triangle(\mathcal{E}) = X$. As a C^*-algebra, $X \subset \triangle(\mathcal{E})$. With $\beta(\triangle(\mathcal{E})) \subset \triangle(\mathcal{F})$, $\triangle(\mathcal{E}) \subset X$ so $X = \triangle(\mathcal{E})$.

Conversely, from the above discussion supposing there is a sub-extension of

C*-algebras, then $\triangle(\mathcal{F})$ is nontrivial. $\square$

Examples. 1.) Every extension of the upper triangular matrix algebra T_n by $\mathbb{K}$ is split. To see this, suppose that we have an extension

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} T_n \longrightarrow 0,$$

where A is a C*-algebra. It follows that $\alpha(A) \subset \triangle(B) = B \cap B^*$. By Lemma 4.2.1 there is a sub-extension

$$0 \longrightarrow A \xrightarrow{\alpha} \triangle(B) \xrightarrow{\beta} D_n \longrightarrow 0,$$

where the diagonal matrix algebra $D_n = \triangle(T_n)$.

If $A = \mathbb{K}$, then this is just an extension of ℓ_n^∞ by $\mathbb{K}$ and the n minimal projections in D_n can be lifted via β to n mutually orthogonal projections p_i in $\triangle(B)$ (see [9]). It is common notation to define e_{ij} as the matrix with all zeros except a 1 in the i, j position. Pick contractions $R_i \in B$ with $\beta(R_i) = e_{i,i+1} \in T_n$. By replacing R_i with $p_i R_i p_{i+1}$ it can be assumed $R_i = p_i R_i p_{i+1}$. Define $b_{ij} = R_i R_{i+1} \cdots R_{j-1}$, for $i < j$ and $b_{ii} = p_i$. A map, call it γ, can be defined which takes a matrix in T_n of the form $(i \leq j)$ $[\lambda_{ij}] \to \sum_{ij} \lambda_{ij} b_{ij} \in B$ is a completely contractive homomorphism by a result in McAsey and Muhly (see [24]). To be splitting for the extension, $\beta \circ \gamma$ must be the identity on T_n. This can be shown by the following calculation:

$$(\beta \circ \lambda)([\lambda_{i,j}]) = \beta(\sum_{ij} \lambda_{ij} b_{ij}) = \sum_{ij} \beta(\lambda_{ij} b_{ij}) = \sum_{ij} \lambda \beta(b_{ij}).$$

The calculation is done by showing $\beta(b_{ij}) = e_{ij}$. By the way b_{ij} is defined, this would be the case.

2.) Every unital extension of the disk algebra, or more generally of Popescu's noncommutative disk algebra A_n (see [27]), has a strongly unital splitting. This follows from the associated noncommutative von Neumann inequality. For example, unital morphisms on the disk algebra are in bijective correspondence with contractions in the algebra that the morphism maps into; and if the latter algebra is a quotient algebra then this contraction can be lifted and hence the morphism too. By a similar argument, every nonunital extension of A_n by $\mathbb{K}$ splits. For the bidisk algebra $A(\mathbb{D}^2)$, the obvious argument for the disk algebra can be followed to see that the splitting of the unital extension of $A(\mathbb{D}^2)$ by the compacts amounts to lifting commuting pairs of contractions in $\mathbb{B} / \mathbb{K}$ to commuting pairs of contraction in $\mathbb{B}$ and Ando's theorem for such pairs ([5, 2.4.13]). It is known that some such pairs do lift, while others do not and so there are quite nontrivial extensions in this case. However for the tridisk algebra $A(\mathbb{D}^3)$, and for algebras of analytic functions on other classical domains, the argument above based on von Neumann inequalities fails, although it is clear that one will usually get non-split unital extensions, and an *Ext* semigroup which is nontrivial.

The subject of corona extendability will be treated later, but the following proposition deals with a simple, but common, example of it.

Proposition 4.2.2. *Given an extension* $\mathrm{E} \in \boldsymbol{ext}(C, A)$ *with* C *nonunital there exists an extension of* C^1 *by* A *containing* E *as a sub-extension. Furthermore, if* $E_1 \in \boldsymbol{ext}(C^1, A)$*, then there is a sub-extension of* E_1 *contained in* $\boldsymbol{ext}(C, A)$*.*

Proof. For the first assertion, note that τ, the Busby invariant of the original extension, extends to a unital completely contractive morphism from $\tau_1 : C^1 \to \mathcal{Q}(A)$ as does $\beta_1 : B^1 \to C^1$ by Meyer's Unitization Theorem. Designate $\alpha' : A \to B^1$ with $\alpha'(a) = \alpha(a)$. It is clear $\alpha'(A)$ is an ideal in B^1 and is the kernel of β_1 thus giving a morphism of extensions as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha'} & B^1 & \xrightarrow{\beta_1} & C^1 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0. \end{array}$$

Suppose there is an extension of C^1 by A with middle algebra B. Since $\alpha(A) \subset B$, by Lemma 2.1.2, $\beta_{|_B}$ is a complete quotient morphism. It is clear $C \subset \mathcal{G}$ as defined in Theorem 4.1.5. Since $\beta(1_B) = 1_C$, and $\beta^{-1}(1_C) = \{1_B + a : a \in A\}$, it is clear $B = \beta^{-1}(C)$. This gives a sub-extension of C by A by Theorem 4.1.5. $\square$

Chapter 5

Covering Extensions

5.1 Existence of Covering Extensions

Given an extension of operator algebras that contains a sub-extension of subalgebras, the original extension will be said to be a *super-extension* of the sub-extension. The question which will be considered in this chapter is given an extension of operator algebras, can a super-extension be constructed extending the original morphisms α and β to containing operator algebras $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ with the extension of β a complete quotient morphism onto $\mathcal{F}$ and $\mathrm{Ker}(\beta) = \mathcal{D}$. In other words, can the following diagram be formed, with α' and β' the extensions of α and β:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{D} & \xrightarrow{\alpha'} & \mathcal{E} & \xrightarrow{\beta'} & \mathcal{F} & \longrightarrow & 0. \end{array}$$

One way to construct a superextension is to let $\mathcal{F} = C$ and employ a Diagram III technique which will be developed later. The universal completion will require A and $\mathcal{D}$ to contain a common cai. Of particular interest will be when the algebras in the second line are the C*-algebras generated by the algebras in line one, and this will be used in developing Diagram IV theory in particular. In this C*-cover case the superextension will be called a *covering extension*. To simplify the discussion it will sometimes be assumed $A \subset B$ and $C = B/A$ as well as considering the algebras in the top line to be subalgebras of the respective C*-covers as shown in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i(A) & \xrightarrow{\alpha} & j(B) & \xrightarrow{\beta} & k(C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathcal{D}, i) & \xrightarrow{\alpha'} & (\mathcal{E}, j) & \xrightarrow{\beta'} & (\mathcal{F}, k) & \longrightarrow & 0. \end{array} \quad (5.1.1)$$

Generally for the following results the injection morphisms, namely $i : A \to C_i^*(A)$, $j : B \to C_j^*(B)$ and $k : C \to C_k^*(C)$, will be suppressed unless needed for a specific purpose as in the next result. This theory will now be developed starting with the relationship of the respective Busby invariants for the original and covering extensions.

Lemma 5.1.1. *Given an extension of C by A and a covering extension as in Diagram 5.1.1, if τ is the Busby invariant of the original extension and τ' the Busby invariant of the covering extension, then*

$$\tau'_{|C} \circ k = \tilde{i} \circ \tau$$

where $k : C \to \mathcal{F}$ *and* $\tilde{i} : \mathcal{Q}(A) \to \mathcal{Q}(\mathcal{D})$ *is the canonical morphism induced by* i *as in Lemma 3.5.3. Additionally, if* $\hat{i} : \mathcal{M}(A) \to \mathcal{M}(\mathcal{D})$*,* $\sigma : B \to \mathcal{M}(A)$*,* $\sigma' : \mathcal{E} \to \mathcal{M}(\mathcal{D})$ *are the canonical morphisms, then* $\sigma'_{|_B} \circ j = \hat{i} \circ \sigma$*.*

Proof. Note that any cai of A is a cai of $\mathcal{D}$, giving i is proper, so the first inclusion morphism is proper. By Theorem 3.5.5, given the existence of the middle morphism, the Busby invariants are related by $\tilde{i} \circ \tau = \tau'_{|_C} \circ k$. The second assertion follows directly from Lemma 3.5.4 since i is proper and the first two morphisms are the embedding of ideals. □

Proposition 5.1.2. *Given an approximately unital ideal* A *in an operator algebra* B *and an extension* E *of* C *by* A *with middle algebra* B*, then there is a bijective correspondence between the equivalence classes of* C*-*covers* $(\mathcal{E}, j)$ *of* B *and equivalence classes (with respect to strong isomorphism) of covering extensions of* E*.*

Proof. Using notation from Diagram 5.1.1, let $(\mathcal{E}, j)$ be a C*-cover of B. Set $\mathcal{D}$ to be the C*-algebra generated by $j(A)$ inside $\mathcal{E}$ which is an ideal by Lemma 2.2.1. Now set $\mathcal{F} = \mathcal{E}/\mathcal{D}$ giving an extension of C*-algebras. It remains to see that $\mathcal{F}$ is a C*-cover of $C \cong B/A$. This follows in a similar manner to the second half of the proof of Lemma 2.2.4. To see this is unique up to strong isomorphism, by Lemma 5.1.1, $\tau'|_C = \hat{j}_{|_A} \circ \tau$ and with C generating, τ uniquely determines τ'. □

In Chapter 3 it was noted that the term "Busby invariant" typically applies to the C*-algebra theory. However, by Proposition 5.1.2 the term Busby invariant for extensions of operator algebras as defined in this thesis is appropriate since it can be considered the restriction of the associated C*-algebra Busby invariant for a covering extension. This section ends with two lemmas regarding properties of a covering extension. The second lemma could be considered a corollary of Lemma 5.1.1.

Lemma 5.1.3. *In the definition of a covering extension it is unnecessary to designate the middle* C*-*algebra as a* C*-*cover for* B. *This is automatic assuming* $\mathcal{D}$ *and* $\mathcal{F}$ *are* C*-*covers of* A *and* B *respectively. Additionally, if the first two* C*-*algebras are* C*-*covers, the third one will be also.*

Proof. It will be assumed the following morphism between extensions exists with either the outer C*-algebras, or the first two C*-algebras in the second line, being C*-covers. For simplicity it will be taken that $C = B/A$ and $\mathcal{F} = \mathcal{E}/\mathcal{D}$ as well as suppressing the horizontal arrows in the calculations.

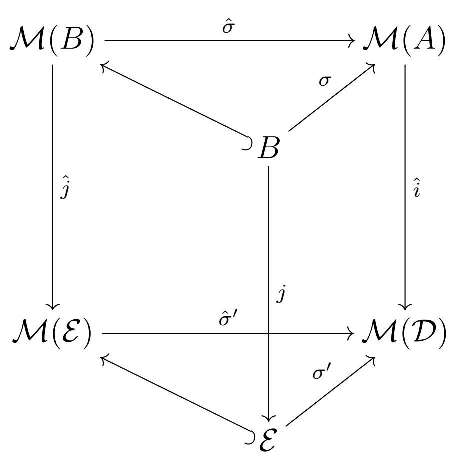
$$\begin{array}{ccccccccc} \mathrm{E}_1 : 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & B/A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\nu} & & \\ \mathrm{E}_2 : 0 & \longrightarrow & \mathcal{D} & \xrightarrow{\alpha'} & \mathcal{E} & \xrightarrow{\beta'} & \mathcal{E}/\mathcal{D} & \longrightarrow & 0. \end{array}$$

First assume that $\mathcal{D}$ and $\mathcal{E}/\mathcal{D}$ are C*-covers of A and B/A respectively and let $\mathcal{G}$ be the C*-algebra generated by B inside $\mathcal{E}$. The image of $\mathcal{G}$ under β' must be a C*-cover of B/A as in the proof of Lemma 3.5.3, so must be all of $\mathcal{E}/\mathcal{D}$. The $*$-homomorphism, $\beta'_{|\mathcal{G}}$ a complete quotient morphism, then by Proposition 4.1.1 a sub-extension of $\mathcal{F}$ by $\mathcal{D}$ exists with middle algebra $\mathcal{G}$. By Lemma 4.1.3, the middle algebra is uniquely determined by the other two. Hence, $\mathcal{G} \cong \mathcal{E}$.

Now suppose that the first two C*-algebras are C*-covers. By the proof of Proposition 5.1.2, $\mathcal{E}/\mathcal{D}$ is a C*-cover for B/A. $\square$

Lemma 5.1.4. *Let E be an extension of operator algebras of C by A with middle algebra B, and let E_* be a covering extension. Let $(\mathcal{D}, i)$ be the associated* C*-*cover of A and $(\mathcal{E}, j)$ the associated* C*-*cover of B. If $\sigma : B \to \mathcal{M}(A)$ and $\sigma' : \mathcal{E} \to \mathcal{M}(\mathcal{D})$, then $\sigma' \circ j = \hat{i} \circ \sigma$ where $\hat{i} : \mathcal{M}(A) \to \mathcal{M}(\mathcal{D})$. Also, if $\hat{\sigma} : \mathcal{M}(B) \to \mathcal{M}(A)$, $\hat{\sigma}' : \mathcal{M}(\mathcal{E}) \to \mathcal{M}(\mathcal{D})$ and $\hat{j} : \mathcal{M}(B) \to \mathcal{M}(\mathcal{E})$ are the canonical extensions, then $\hat{i} \circ \hat{\sigma} = \hat{\sigma}' \circ \hat{j}$.*

Proof. With $i : A \to \mathcal{D}$ being proper, Lemma 5.1.1 shows that $\sigma' \circ j = \hat{i} \circ \sigma$. This gives that the right face on the diagram below commutes.



The bottom and top faces commute based on standard extensions to the multiplier algebras. The left side commutes as the typical embedding of $\mathcal{M}(A) \subset \mathcal{M}(\mathcal{E})$. It is easy to see that the back face commutes if $\hat{\sigma}$ and $\hat{j}$ are restricted to B. This leaves just the right face. Since $\hat{i} \circ \hat{\sigma}_{|_B} = \hat{\sigma}' \circ \hat{j}_{|B}$, and each can extend to $\mathcal{M}(B)$, by the uniqueness of the extensions we have $\hat{i} \circ \hat{\sigma} = \hat{\sigma}' \circ \hat{j}$. $\square$

5.2 Partial Ordering of Covering Extensions

There is an ordering on the C*-covers of an algebra in the category OA. If $(j_1, \mathcal{D}_1)$ and $(j_2, \mathcal{D}_2)$ are C*-covers of A, then $\mathcal{D}_1 \leq \mathcal{D}_2$ if and only if there exists a surjective $*$-homomorphism from $\pi : \mathcal{D}_2 \to \mathcal{D}_1$ such that $j_1 = \pi \circ j_2$. The next two lemmas show this ordering on the C*-covers of the middle algebra of an extension translates to an associated ordering on the other two C*-covers of a covering extension. The ordering of C*-covers of B, the middle algebra of an extension of C by A, extends to a partial ordering on the covering extensions themselves. This ordering of covering extension will be based on the existence of a morphism between covering extensions.

Lemma 5.2.1. *Given an extension*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0,$$

consider the mapping suggested in the proof of Proposition 5.1.2 of equivalence classes C**-covers of* B *to the equivalence classes of* C**-covers of* A*, and the mapping of the equivalence classes of* C**-covers of* B *to equivalence classes of* C**-covers of* C*. These mappings preserve the natural ordering of* C**-covers, with the first mapping also surjective.*

Proof. Let $\mathcal{S}_A$ be the set of equivalence classes of C*-covers of A, $\mathcal{S}_B$ the similar set for B and $\mathcal{S}_C$ the similar set for C. First, consider the mapping suggested in the proof of Proposition 5.1.2 and call it $\gamma : \mathcal{S}_B \to \mathcal{S}_A$. It is clear $\gamma((j, \mathcal{E})) = (j_{|A}, \mathcal{D})$ where $\mathcal{D}$ is the C*-algebra generated by $j(A)$ inside $\mathcal{E}$. This is a C*-cover of A showing γ is defined for all C*-covers of B. To see γ respects the natural ordering of C*-covers, let $(j, \mathcal{E}_1) \leq (j', \mathcal{E}')$ be C*-covers of B and $(j_{|_A}, \mathcal{D})$ and $(j'_{|_A}, \mathcal{D}')$ the respective C*-covers of A inside $\mathcal{E}$ and $\mathcal{E}'$. Let $\pi : \mathcal{E}' \to \mathcal{E}$. Clearly π restricts to a $*$-homomorphism $\pi_{|_\mathcal{D}}$ with range $\mathcal{D}'$ giving $(j_{|_A}, \mathcal{D}) \leq (j'_{|_A}, \mathcal{D}')$.

To see γ is surjective, let $(\mu, \mathcal{D})$ be a C*-cover for A. By Corollary 2.1.4, $\mathcal{Q}(A) \subset \mathcal{Q}(\mathcal{D})$ completely isometrically isomorphically. Let $\tilde{\mu}$ be this complete isometry. If τ is the Busby invariant of the original extension, then $\tilde{\mu} \circ \tau : C \to \mathcal{Q}(\mathcal{D})$. This extends to a $*$-homomorphism $\tau^* : \mathrm{C}^*_{\max}(C) \to \mathcal{Q}(\mathcal{D})$ giving an extension of $\mathrm{C}^*_{\max}(C)$ by $\mathcal{D}$. Suppose X is the middle algebra of this extension. Since $\tilde{\mu} \circ \tau = \tau^*_{|_C}$, by Theorem 3.5.5 there exists a morphism from $\delta : B \to X$, which is a complete isometry by Lemma 3.3.3. Evidently X is a C*-cover of B by Lemma 5.1.3, containing $\mathcal{D}$ completely isometrically isomorphically.

Concerning the second mapping, let $\lambda : \mathcal{S}_B \to \mathcal{S}_C$. If $(j, \mathcal{E})$ is a C*-cover of B it will contain a C*-cover of A, namely $\mathcal{D}$, generated by $j(A)$. By Lemma 5.1.3, $\mathcal{F}$ is a C*-cover of B/J and hence of C. To see this preserves the natural ordering, let $(i, \mathcal{E}_1) \leq (j, \mathcal{E}_2)$ be C*-covers of B with $\pi : \mathcal{E}_2 \to \mathcal{E}_1$ the canonical $*$-homomorphism with $\pi \circ j = i$, and $(j_{|_A}, \mathcal{D}_1)$ and $(j_{|_A}, \mathcal{D}_2)$ the respective C*-covers of A inside $\mathcal{E}$ and $\mathcal{E}'$. There is a natural $*$-homomorphism $\pi' : \mathcal{E}_2/\mathcal{D}_2 \to \mathcal{E}_1/\mathcal{D}_1$. That is for $\eta \in \mathcal{E}_2$, $\pi'(\eta + \mathcal{D}_2) = \pi(\eta) + \pi'(\mathcal{D}_2) = \pi(\eta) + \mathcal{D}_1$. Let j' be the natural imbedding of B/A into $\mathcal{E}_2/\mathcal{D}_2$, which takes $b + A$ to $j(b) + \mathcal{D}_2$. Define i' as the natural imbedding of B/A into $\mathcal{E}_1/\mathcal{D}_1$ by $i'(b + A) = i(B) + \mathcal{D}_1$ for $b + A \in B/A$. With these definitions, it is clear $i'(b + A) = i(b) + \mathcal{D} = \pi'(b + \mathcal{D}_2) = (\pi \circ j')(b + A)$. $\square$

The next two examples show that the first map is not generally injective and the second map is not generally surjective.

Example 5.2.2. That γ is not generally injective will be demonstrated by the following example. Let $A = \mathbb{C}$ and

$$B = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

It is known $\mathrm{C}^*_{\mathrm{e}}(B) = \mathbb{C} \oplus^{\infty} \mathcal{M}_2$ and

$$\mathrm{C}^*_{\max}(B) = \mathbb{C} \oplus^{\infty} \{f \in \mathcal{M}_2(C([0,1]) : f(0) \text{ is a diagonal matrix}\}$$

with A embedded as the 1-1 entry in each C^*-algebra.

Example 5.2.3. To see that the second map in Lemma 5.2.1 is not generally surjective even if A is completely essential in B, let $A = \mathbb{K}$ and $C = \mathcal{T}_2$, the upper triangular matrices in $\mathcal{M}_2$. Let u be the infinite matrix with a zeroes everywhere except as follows. There will be a one in the 1,2 entry. Thereafter the nonzero entries, which equal 1, will follow the recursive formula: for $n = 0$, $i_0, j_0 = 1, 2$ and for $n \geq 1$, $i_n, j_n = i_{n-1} + 4, j_{n-1} + 4$. Let p be the projection with a one in the $i - i$ position if i is odd and zero if i is even. Then $p^{\perp}$ is the diagonal matrix with zero when i is odd and one when i is even. Note that $uu^* - p \notin \mathbb{K}$ and $u^*u - p^{\perp} \notin \mathbb{K}$ and $pp^{\perp} = u^2 = up = p^{\perp}u = 0$ and $u = pu = up^{\perp}$.. Define a homomorphism $\tau : C \to \mathbb{B} / \mathbb{K}$ by

$$\tau\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = a\dot{p} + b\dot{u} + c\dot{p}^{\perp}$$

where $\dot{p}$, $\dot{u}$ and $\dot{p}^{\perp}$ are the corresponding elements in $\mathbb{B}\,/\,\mathbb{K}$. It is easy to see τ is linear. To see it is multiplicative:

$$\tau\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)\tau\left(\begin{bmatrix} r & s \\ 0 & t \end{bmatrix}\right) = (a\dot{p} + b\dot{u} + c\dot{p}^{\perp})(r\dot{p} + s\dot{u} + t\dot{p}^{\perp}) =$$

$$ar\dot{p}^2 + as\dot{p}\dot{u} + at\dot{p}\dot{p}^{\perp} + br\dot{u}\dot{p} + bs\dot{u}\dot{u} + bt\dot{u}\dot{p}^{\perp} + cr\dot{p}\dot{p}^{\perp} + cs\dot{p}^{\perp}\dot{u} + ct(\dot{p}^{\perp})^2 =$$

$$ar\dot{p} + (as+bt)\dot{u} + ct\dot{p}^{\perp} = \tau\left(\begin{bmatrix} ar & as+bt \\ 0 & ct \end{bmatrix}\right) = \tau\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} r & s \\ 0 & t \end{bmatrix}\right).$$

Since τ is linear, it is bounded, but to see it is a complete contraction, we shall look at $\eta : C \to \mathbb{B}$ by

$$\eta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = ap + bu + cp^{\perp}.$$

The image under η is the infinite diagonal matrix with alternating copies of $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$ along the diagonal. For a diagonal matrix, the norm is the supremum of the norms of the diagonal submatrices, so η is a complete isometry and commutes with τ. This shows that τ is completely contractive.

Suppose that τ were to be extended to a $*$-homomorphism $\tau' : \mathrm{C}^*_{\mathrm{e}}(C) = \mathcal{M}_2 \to \mathbb{B}\,/\,\mathbb{K}$. As a $*$-homomorphism it would need to be the case that:

$$\tau'\left(\begin{bmatrix} 0 & 0 \\ \bar{b} & 0 \end{bmatrix}\right) = \tau\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right)^* = \bar{b}\dot{u}^*.$$

Let e_{12} and e_{21} be the canonically denoted matrices in $\mathbb{M}_2$ with the second matrix the adjoint of the first. Clearly $\tau'(e_{12}e_{21}) = \tau'(e_{11}) = \dot{p}$. By the above $\tau'(e_{12}e_{21}) = \tau'(e_{12})\tau'(e_{21}) = \dot{u}\dot{u}^*$. Since $uu^* - p \notin \mathbb{K}$, then $\dot{u}\dot{u}^* \neq \dot{p}$.

In proving τ above is a completely contractive morphism, it was also shown, with the existence of η, there is a splitting of the extension by Theorem 3.4.3. Let $\gamma : C \to PB$ be the morphism due to the splitting, where PB is the pullback due to τ. Then $\sigma : PB \to \mathbb{B}$ is a complete isometry. To see this, note that η is a complete isometry, and hence so is $\sigma \circ \gamma$. If X is the range of η, then $\gamma \circ \sigma : X \to C$ is the inverse of $\sigma \circ \gamma$ making each complete isometries. Consequently the extension of C by $\mathbb{K}$ is completely essential. With $\mathbb{K}$ a C^*-algebra, and the manner in which γ is defined, indicates that $\mathrm{C}^*_{\mathrm{e}}(PB)$ is the C^*-algebra generated by PB in $\mathbb{B}$. This example shows that given operator algebras A and B, the quotient of the C^*-envelopes $\mathrm{C}^*_{\mathrm{e}}(B)/\,\mathrm{C}^*_{\mathrm{e}}(A) \neq \mathrm{C}^*_{\mathrm{e}}(B/A)$.

The above example shows it is possible that there may be no covering extension for given C^*-covers of A and C. This motives the next proposition.

Proposition 5.2.4. *Given an extension of C by A with C^*-covers $\mathcal{F}$ and $\mathcal{D}$ of C and A respectively, a covering extension will exist with $\mathcal{D}$ and $\mathcal{F}$ as the first and last* *algebras if and only if the Busby invariant, τ extends to a $*$-homomorphism τ' from $\mathcal{F}$ into $\mathcal{Q}(\mathcal{D})$.*

Proof. Suppose that τ extends to a $*$-homomorphism from $\tau' : \mathcal{F} \to \mathcal{Q}(\mathcal{D})$. By Theorem 3.4.4 there is an extension with first algebra $\mathcal{D}$ and last algebra $\mathcal{F}$. By Lemma 5.1.3 the middle algebra is a C^*-cover of B giving a covering extension with the C^*-cover of B.

Now suppose there exists a covering extension with first and third algebras $\mathcal{D}$ and $\mathcal{F}$ respectively. By Lemma 5.1.1 the Busby invariant of the extension of C by A is associated with the Busby invariant of the covering extension by $\tau'_{|C} = \tilde{i} \circ \tau$, where $\tilde{i}$ is the complete isometry from $\mathcal{Q}(A)$ into $\mathcal{Q}(\mathcal{D})$. Since C generates $\mathcal{F}$, the restriction to C uniquely determines τ', which can be taken to extend τ. □

Lemma 5.2.5. *There exists a partial ordering on the equivalence classes of covering extensions as follows. Given two covering extensions E_1 and E_2 of E, an extension of operator algebras, then $\mathrm{E}_1 \leq \mathrm{E}_2$ if and only if there exist a morphism of extensions from E_2 onto E_1 with each vertical arrow the unique morphism from the ordering on the C^*-covers of A, B, and C as follows:*

$$\begin{array}{ccccccccc} \mathrm{E}_2 : 0 & \longrightarrow & \mathcal{D}_2 & \xrightarrow{\alpha_2} & \mathcal{E}_2 & \xrightarrow{\beta_2} & \mathcal{F}_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \nu} & & \\ \mathrm{E}_1 : 0 & \longrightarrow & \mathcal{D}_1 & \xrightarrow{\alpha_1} & \mathcal{E}_1 & \xrightarrow{\beta_1} & \mathcal{F}_1 & \longrightarrow & 0. \end{array}$$

This is equivalent to saying $\mathcal{E}_1 \leq \mathcal{E}_2$.

Proof. To show the definition of the ordering on covering extensions is equivalent to saying $\mathcal{E}_1 \leq \mathcal{E}_2$, first, given the above morphisms of extensions, then there exists $\delta : \mathcal{E}_2 \to \mathcal{E}_1$ fitting with the ordering on C*-covers of B. For the other direction, given $\delta : \mathcal{E}_2 \to \mathcal{E}_1$, by Proposition 5.1.2 there exist two covering extensions with middle C*-algebras $\mathcal{E}_2$ and $\mathcal{E}_1$. The bijection also indicates that the first C*-algebras, say $\mathcal{D}_2$ and $\mathcal{D}_1$ respectively, can be taken to be the C*-algebras generated by A in the respective C*-algebras of B. The third C*-algebras can be taken to be $\mathcal{E}_2/\mathcal{D}_2$ and $\mathcal{E}_1/\mathcal{D}_1$. Furthermore, by Lemma 5.2.1 the morphisms μ and ν exist and respect the order on C*-covers of A and C respectively. It only remains to show the above diagram would be commutative. By uniqueness of μ, that is $i_1 = \mu \circ i_2$, and the manner in which $\mathcal{D}_2$ and $\mathcal{D}_1$ were defined in Lemma 5.2.1, μ can be taken to be the restriction of δ to $\mathcal{D}_2$. This gives commutativity of the first square. For the second square, as in the proof of Lemma 5.2.1, $\mathcal{F}_1$ and $\mathcal{F}_2$ can be considered C*-covers of B/A. Let $j_1 : B \to \mathcal{E}_1$, $j_2 : B \to \mathcal{E}_2$, $k_1 : B/A \to \mathcal{F}_1$, $k_2 : B/A \to \mathcal{F}_2$, with $\beta : B \to B/A$ from the original extension. From the definition of the ordering on C*-covers, let $\delta(j_2(B)) = j_1(B)$, $\nu(k_2(B/A)) = k_1(B/A)$. The following diagram, taking $\mathcal{E}_1/\mathcal{D}_1, \mathcal{E}_2/\mathcal{D}_2$ as the third algebras if the respective covering extensions, shows the relationship of the these morphisms.

$$\begin{array}{ccc} B & \xrightarrow{\beta} & B/A \\ \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle k_2} \\ \mathcal{E}_2 & \xrightarrow{\beta_2} & \mathcal{E}_2/\mathcal{D}_2 \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \nu} \\ \mathcal{E}_1 & \xrightarrow{\beta_1} & \mathcal{E}_1\mathcal{D}_1 \\ {\scriptstyle j_1}\uparrow & & \uparrow{\scriptstyle k_1} \\ B & \xrightarrow{\beta} & B/A \end{array}$$

Commutativity of the top and bottom squares follows by noting that both $k_2 \circ \beta$ and $\beta_2 \circ j_2$ take B to $B/\mathcal{D}_2$. A similar argument works for the bottom square. For the commutativity of the middle square, using the relationship between ν and the injection morphisms,

$$\nu \circ \beta_2 \circ j_2 = \nu \circ k_2 \circ \beta = k_1 \circ \beta = \beta_1 \circ j_1 = \beta_1 \circ \delta \circ j_2.$$

With B generating $\mathcal{E}_1$ and $\mathcal{E}_2$, the same holds for the $*$-homomorphisms $\nu \circ \beta_2$ and $\beta_1 \circ \delta$, and the diagram commutes with the specified $*$-homomorphisms. $\square$

As discussed in [5, p.99], for any operator algebra B, if J is an ideal in $\mathrm{C}^*_{\max}(B)$ such that $\mathrm{C}^*_{\max}(B)/J \cong \mathrm{C}^*_{\mathrm{e}}(B)$, then there is an order-reversing bijective correspondence between $\mathcal{C}(B)$, the set of equivalence classes of C^*-covers of B and the closed ideals of J. Here the closed ideals of J relate to the universal property of the maximal C^*-cover. According to this universal property, if $\mathcal{E}$ is a C^*-cover of B, then there exists a necessarily surjective $*$-homomorphism $\pi : \mathrm{C}^*_{\max}(B) \to \mathcal{E}$. If $I = \mathrm{Ker}(\pi)$, then $I \subset J$. To see this, note that $\mathcal{E} \cong \mathrm{C}^*_{\max}(B)/I$ and by the universal property of the enveloping C*-cover, there exists a necessarily surjective $*$-homomorphism $\pi' : \mathrm{C}^*_{\max}(B)/I \to \mathrm{C}^*_{\mathrm{e}}(B)$. The composition of π and π' takes $\mathrm{C}^*_{\max}(B) \to \mathrm{C}^*_{\mathrm{e}}(B)$, so $\mathrm{Ker}(\pi' \circ \pi) = J$ showing $I \subseteq J$. Conversely, any closed ideal I of J gives a C*-cover. Note that a succession of surjective $*$-homomorphisms can be formed. That is $\pi_1 : \mathrm{C}^*_{\max}(B) \to \mathrm{C}^*_{\max}(B)/I,$ and $\pi_2 : \mathrm{C}^*_{\max}(B)/I \to \mathrm{C}^*_{\max}(J)$. The last arrow exists by the Factor Theorem. Since $\pi_2 \circ \pi_1$ is a complete isometry on B, π_1 must be a complete isometry on B. With B/I generating on $\mathrm{C}^*_{\max}(B)/I$, then $\mathrm{C}^*_{\max}(B)/I$ a C*-cover of B. This bijective correspondence then takes a C*-cover $\mathcal{E} \to I_{\mathcal{E}}$. From the above discussion it can be deduced that $\mathcal{E}_1 \leq \mathcal{E}_2$ if and only if $I_{\mathcal{E}_2} \subseteq I_{\mathcal{E}_1}$.

There is a complete lattice on the closed ideals of J with the ordering $I_1 \geq I_2$ if $I_2 \subseteq I_1$. This is a reverse ordering on the C*-covers of B by the last statement in the previous paragraph. The set $\hat{J}$, called the spectrum of J, is the equivalence classes of irreducible representations of J. It can be given a topology, called the Jacobson topology [13, 3.1]. With this topology, by [13, Theorem 3.2.2], there is a lattice isomorphism from the closed two-sided ideals of J and the open sets in the Jacobson topology on $\hat{J}$. Relating this isomorphism to the bijection described above, there is a lattice anti-isomorphism between the open sets in $\hat{J}$ and $\mathcal{C}(B)$. From Lemma 5.2.5, there is an ordering on the covering extensions of an extension E of C by A relating to the ordering on the C*-covers of the middle algebra B. This order can be related to the closed ideals of $J \subset \mathrm{C}^*_{\max}(B)$. This gives a order reversing bijection from the covering extensions and the open sets in $\hat{J}$.

Another proof of Lemma 2.2.2 can now be offered. Without assuming $\mathrm{C}^*_{\mathrm{e}}(B)$ contains an enveloping C*-algebra for A as an ideal, as in the proof of Lemma 5.2.1

a covering extension for any C*-cover of A can be found using the universal property of the maximal C*-cover. This can be done for $\mathrm{C}^*_{\mathrm{e}}(A)$ as follows:

$$0 \longrightarrow \mathrm{C}^*_{\mathrm{e}}(A) \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathrm{C}^*_{\max}(C) \longrightarrow 0.$$

By Proposition 5.1.2 there is a covering extension with middle C*-algebra $\mathrm{C}^*_{\mathrm{e}}(B)$. With the ordering on covering extensions from Lemma 5.2.5, the first C*-algebra of this covering extension dominates $\mathrm{C}^*_{\mathrm{e}}(A)$ in the ordering of C*-covers of A. Hence it must be $\mathrm{C}^*_{\mathrm{e}}(A)$ indicating $\mathrm{C}^*_{\mathrm{e}}(A)$ is an ideal in $\mathrm{C}^*_{\mathrm{e}}(B)$.

5.3 Maximal and Minimal Covering Extensions

In Proposition 2.2.3, several equivalent conditions for an ideal A in B to be completely essential was given. Using covering extensions, an additional equivalent expression can be given. Recall that for C*-algebras an extension is essential if and only if the first C*-algebra is an essential ideal in the second C*-algebra, or the morphism we have called σ is one-to-one. A covering extension will be called an *essential covering extensions of* E if it is a covering extension of E and it is essential in the C*-algebra sense.

Proposition 5.3.1. *Given operator algebras* A, B *and suppose that* A *contains a cai and is an ideal in* B*. Then there is an extension of* B/A *by* A *and the following are equivalent:*

(i) *A is a completely essential ideal in B.*

(ii) *There exists an essential covering extension of the extension of B/A by A.*

(iii) *There exists an essential covering extension as in (ii) where the first two terms are* C^**-envelopes.*

Proof. (i) $\Rightarrow$ (ii) By Proposition 2.2.3 to say A is completely essential in B is equivalent to saying there exists a C^*-cover of B, call it $\mathcal{E}$, containing a C^*-cover of A, call it $\mathcal{D}$ as an essential ideal. Form the covering extension of $\mathcal{E}/\mathcal{D}$ by $\mathcal{D}$ which is an essential extension which is a covering extension by Proposition 5.1.2.

(i) $\Rightarrow$ (iii) By Proposition 5.1.2 there is a covering extension with $\mathrm{C}^*_{\mathrm{e}}(A)$ and $\mathrm{C}^*_{\mathrm{e}}(B)$ as the first two terms. By 2.2.3 $\mathrm{C}^*_{\mathrm{e}}(B)$ contains $\mathrm{C}^*_{\mathrm{e}}(A)$ as an essential ideal giving that the referenced covering extension is essential.

(iii) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) Let $C^*_j(B)$ be a C^*-cover of B such that the C^*-algebra generated by A in $C^*_j(B)$ is an essential ideal. The canonical morphism $\sigma' : C^*_j(B) \to \mathcal{M}(C^*_i(A))$ is an injective $*$-homomorphisms and so is a complete isometry. The restriction to B will also be a complete isometry. Suppressing the inclusion morphisms into the C^*-covers, as a covering extension, $\sigma'_{|_B}$ must agree with σ from the original extension into the completely isometrically isomorphic copy of $\mathcal{M}(A)$ in $\mathcal{M}(C^*_i(A))$ by Lemma 5.1.1. This gives the restriction of B into $\mathcal{M}(A)$ is a complete isometry and the original extension is completely essential. $\square$

In the statement of the following corollary the term quotient extension is used. Since all $*$-homomorphism between C^*-covers, when they exist, are surjective, these surjective $*$-homomorphisms are, in operator algebraic terms, complete quotient morphisms. The terminology in Corollary 5.3.2 is the same as that used in Lemma 3.6.2.

Corollary 5.3.2. *Given a completely essential extension*

$$E: \qquad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

and let $(\mathcal{D}, i)$ *be a* C^**-cover of* A*, then there exists a 'smallest' and 'largest' covering extension with first term* $\mathcal{D}$*:*

$$E_{\min}: \qquad 0 \longrightarrow \mathcal{D} \xrightarrow{\alpha} \mathcal{E}_1 \xrightarrow{\beta} \mathcal{F} \longrightarrow 0,$$

$$E_{\max}: \qquad 0 \longrightarrow \mathcal{D} \xrightarrow{\alpha} \mathcal{E}_2 \xrightarrow{\beta} \mathrm{C}^*_{\max}(C) \longrightarrow 0.$$

Each covering extension has an associated universal property. The minimal covering extension E_{min} *is a quotient extension of any other covering extension with first term* $\mathcal{D}$*. Also,* E_{min} *is essential. Any other covering extension with first term* $\mathcal{D}$ *is a quotient extension of the maximal covering extension.*

Proof. With the original extension being completely essential the Busby invariant τ is a complete isometry by Lemma 3.6.1. Composing τ with the complete isometry $\tilde{\mu} : \mathcal{Q}(A) \rightarrow \mathcal{Q}(\mathcal{D})$ is a complete isometry taking C into $\mathcal{Q}(\mathcal{D})$. Let $\mathcal{F}$ be the C^*-algebra generated by $(\tilde{\mu} \circ \tau)(C)$ in the C^*-algebra $\mathcal{Q}(\mathcal{D})$. The inclusion morphism taking $\mathcal{F}$ into $\mathcal{F} \subset \mathcal{Q}(\mathcal{D})$, call it τ', is a complete isometry giving an essential extension of $\mathcal{F}$ by $\mathcal{D}$. Let $\mathcal{E}_1$ be the pullback C*-algebra for this extension.

Given any other covering extension with first term $\mathcal{D}$ and last term $\mathcal{F}'$, say, let $\mathcal{E}_1'$ be the middle algebra. Let τ_1 be the Busby invariant for this extension and it is clear $\tau_1(C) \subset \mathcal{F}$ with τ_1 mapping $\mathcal{F}'$ into $\mathcal{F}$. With this understanding, $\tau_1 = \tau' \circ \tau_1$ trivially, and by Theorem 3.5.5 there is a $*$-homomorphism $\gamma : \mathcal{E}_1' \to \mathcal{E}_1$. As both τ' and γ are the canonical such morphisms, by Lemma 5.2.5 there exists a commutative diagram with arrows being the canonical such surjective $*$-homomorphisms making $E_{\min}$ a quotient extension of this (or any other) covering extension with first term $\mathcal{D}$.

For the second assertion, let $(\mathcal{F}', k)$ be a C*-cover of C such that $\mathcal{F}'$ is the third algebra in a covering extension with first algebra $\mathcal{D}$. Let $\tau' : \mathcal{F}' \to \mathcal{Q}(\mathcal{D})$ be the Busby invariant for that extension. Additionally suppose that $k' : C \to \mathrm{C}^*_{\max}(C)$. Let $\pi : \mathrm{C}^*_{\max}(C) \to \mathcal{F}'$ and $\tau_{\max} : \mathrm{C}^*_{\max}(C) \to \mathcal{Q}(\mathcal{D})$ be the canonical $*$-homomorphisms by the universal property of $\mathrm{C}^*_{\max}(C)$. The second $*$-homomorphism is then the Busby invariant for the maximal covering extension above. To use Theorem 3.5.5 it needs to be shown that $\tau_{\max} = \tau' \circ \pi$. Viewing $C \subset \mathcal{Q}(\mathcal{D})$ by virtue of the original extension being completely essential, $(\tau_{\max} \circ k)(c) = c$ and $(\tau' \circ k')(c) = c$ for all $c \in C$. Combining this with the universal property of $\mathrm{C}^*_{\max}(C)$, namely $\pi \circ k = k'$, gives $\tau_{\max} \circ k = \tau' \circ \pi \circ k$. Since this is true for C which generates the two C*-covers, is true on the C*-algebras giving the existence of a surjective $*$-homomorphism between $\mathcal{E}$ and the middle algebra in the second covering extension. $\square$

For the next corollary, the C^*-algebra generated by C in $\mathcal{Q}(\mathrm{C}^*_{\mathrm{e}}(A))$ from the proof of Corollary 5.3.2 will be denoted by $\mathcal{F}_{\min}$.

Corollary 5.3.3. *When A is completely essential in B the surjective mapping γ indicated in Lemma 5.2.1 which maps the C^*-covers of B onto the C^*-covers of A is injective if and only if and $\mathrm{C}^*_{\max}(C) \cong \mathcal{F}_{\min}$.*

Proof. Suppose $\mathrm{C}^*_{\max}(C) = \mathcal{F}_{\min}$. Then by Corollary 5.3.2 and Lemma 3.3.3, the minimal and maximal covering extension with first term $\mathcal{D}$, where $\mathcal{D}$ is a C^*-cover of A, are the $*$-isomorphic. Let $\mathcal{E}$ be the middle algebra of this extension. Suppose there is another C^*-cover, call it $\mathcal{E}'$ of B which contains $\mathcal{D}$ as an ideal. Using the universal property of the minimal extension, there is a surjective $*$-homomorphism from $\mathcal{E}' \to \mathcal{E}$. Now using the universal property of the maximal extension, there is a surjective $*$-homomorphism from $\mathcal{E} \to \mathcal{E}'$ showing $\mathcal{E} \cong \mathcal{E}'$.

Now suppose A is completely essential in B and the mapping is injective, meaning there is only one extension with first term $\mathcal{D}$ for each C^*-cover $\mathcal{D}$ of A. By Corollary 5.3.2, the minimal and maximal covering extensions exist with third C^*-algebras $\mathcal{F}_{\min}$ and $\mathrm{C}^*_{\max}(C)$ respectively. Since there is only one such covering extension with first term $\mathcal{D}$, these are $*$-isomorphic and $\mathrm{C}^*_{\max}(C) \cong \mathcal{F}_{\min}$. □

Proposition 5.3.4. *With notation as in Proposition 5.2.4, suppose $\mathcal{D}$ and $\mathcal{F}$ are C^*-envelopes (respectively, maximal C^*-covers) of A and C, and if a covering extension exists with these as the first and last algebras, then the middle term will be a C^*-envelope of B (respectively, maximal C^*-cover of B).*

Proof. Suppose $\mathcal{D}$ and $\mathcal{F}$ are C^*-envelopes of A and C respectively and that a covering extension exists. Call this covering extension E_1. By Lemma 5.1.3, the middle algebra $\mathcal{E}$ is a C^*-cover of B. By Proposition 5.1.2, there exists a covering extension with middle algebra $\mathrm{C}^*_{\mathrm{e}}(B)$, call it E_2. By Lemma 5.2.1 there is a morphism of extensions from E_1 onto E_2, which implies that the first and last C^*-algebras in E_2 are the C^*-envelopes of A and B respectively.

Let E_1 be a covering extension of $\mathrm{C}^*_{\max}(C)$ by $\mathrm{C}^*_{\max}(A)$. By Lemma 5.1.3 there exists a covering extension with middle algebra $\mathrm{C}^*_{\max}(B)$, call it E_2. By Lemma 2.2.4, $\mathrm{C}^*_{\max}(A)$ is an ideal in $\mathrm{C}^*_{\max}(B)$ and $\mathrm{C}^*_{\max}(B)/\mathrm{C}^*_{\max}(A)$ is a maximal C^*-cover of B/A. With $B/A \cong C$, $\mathrm{C}^*_{\max}(B)/\mathrm{C}^*_{\max}(A)$ is also a C^*-cover of C. So E_2 can be considered to have $\mathrm{C}^*_{\max}(A)$ as the first algebra and $\mathrm{C}^*_{\max}(B)/\mathrm{C}^*_{\max}(A)$ as the last algebra. If X is the C^*-cover of B and the middle algebra in E_1, then $X \leq \mathrm{C}^*_{\max}(B)$ as C^*-covers of B. By Lemma 5.2.1 there is a morphism of extensions from E_2 onto E_1. Since $\mathrm{C}^*_{\max}(B)/\mathrm{C}^*_{\max}(A)$ is also a maximal C^*-cover of C, the rightmost vertical arrow of the morphism between extensions is a complete isometry. By Lemma 3.3.3, the middle arrow of the morphism between extensions is also a complete isometry, making X a maximal C^*-cover of B. $\square$

As in the last proposition, if a covering extension exists such that all C^*-algebras are actually C^*-envelopes, then it will be call a *C^*-enveloping extension.* The original extension itself will be said to be *C^*-enveloped.* If the Busby invariant τ of an extension of C by a C^*-algebra A extends to a $*$-representation $\mathrm{C}^*_{\mathrm{e}}(C) \to \mathcal{Q}(A)$, then the extension E is C^*-enveloped. For an extension to be C^*-enveloped it would need to be the case that $\mathrm{C}^*_{\mathrm{e}}(C) \cong \mathrm{C}^*_{\mathrm{e}}(B)/\mathrm{C}^*_{\mathrm{e}}(A)$ which is not always the case. It is trivially true that any extension of a C*-algebra by a non-selfadjoint operator algebra is C^*-enveloped.

It will always be true a covering extension of maximal C*-covers exists by the universal properties of the maximal C*-covers. The next result shows this is driven by the middle C*-algebra of a covering extension.

Lemma 5.3.5. *If the middle* C**-algebra of a given covering extension is a maximal* C**-cover, then all three of the* C**-algebras are maximal* C**-covers.*

Proof. Suppose there is a covering extension, call it $\mathcal{E}_1$, with first and third C*-algebras $\mathcal{D}$ and $\mathcal{F}$ respectively and the middle C*-algebra $\mathrm{C}^*_{\max}(B)$. Let $\alpha : \mathcal{D} \to \mathrm{C}^*_{\max}(B)$. By Lemma 2.2.4, the C*-algebra generated by $\alpha(A)$ in $\mathrm{C}^*_{\max}(B)$ is $\mathrm{C}^*_{\max}(A)$. This then implies that $\mathrm{C}^*_{\max}(A) \cong \mathrm{C}^*_{\max}(\alpha(A)) \subset \alpha(\mathcal{D})$. This gives a morphism from $\mathrm{C}^*_{\max}(A) \to \mathcal{D}$ indicating $\mathcal{D}$ is a maximal C*-cover of A. With $\mathcal{F} \cong \mathrm{C}^*_{\max}(B)/\mathrm{C}^*_{\max}(A)$, again by Lemma 2.2.4, $\mathcal{F}$ is a maximal C*-cover of C. $\square$

Lemma 5.3.6. *If $\mathcal{D}$ is a* C**-algebra then there is a canonical bijection from* $\mathbf{Ext}(C, \mathcal{D})$ *onto* $\mathbf{Ext}(\mathrm{C}^*_{\max}(C), \mathcal{D})$ *which takes the split extensions onto the split extensions.*

Proof. Define $\gamma : \mathbf{Ext}(C, \mathcal{D}) \to \mathbf{Ext}(\mathrm{C}^*_{\max}(C), \mathcal{D})$ by $\gamma(\mathrm{E}_\tau) = \mathrm{E}_{\tau_*}$ where τ_* is the extension of τ to $\mathrm{C}^*_{\max}(C)$. By the uniqueness of τ_*, the mapping is well defined on $\mathbf{Ext}(C, \mathcal{D})$. To see the mapping is bijective, let B be the middle algebra in an extension of $\mathcal{D}$ by C. With $\mathcal{D}$ its own maximal C*-cover, by Proposition 5.3.4, the middle algebra is necessarily a maximal C*-cover of B. Every extension in $\mathbf{Ext}(\mathrm{C}^*_{\max}(C), \mathcal{D})$ has this form. Given an extension in $\mathrm{E}_{\tau'} \in \mathbf{Ext}(\mathrm{C}^*_{\max}(C), \mathcal{D})$,

τ' restricts to C into $\mathcal{Q}(\mathcal{D})$. This gives an extension in $\mathbf{Ext}(C,\mathcal{D})$ as the pre-image of $\mathrm{E}_{\tau'}$ under γ as well as an inverse mapping of γ. Define $\gamma^{-1}(\mathrm{E}_\tau) = \mathrm{E}_{\tau_{|C}}$. For injectivity it will be shown $\gamma^{-1}\circ\gamma = I$, the identity on $\mathbf{Ext}(C,\mathcal{D})$. This is easy since $(\gamma^{-1}\circ\gamma)(\mathrm{E}_\tau) = \gamma^{-1}(\mathrm{E}_{\tau_*}) = \mathrm{E}_{\tau_{*_{|C}}} = \mathrm{E}_\tau$.

To see it takes split extensions to split extensions, note that if E is a split extension in $\mathbf{Ext}(C,\mathcal{D})$ there exists a morphism $\gamma : C \to B)$. This extends to an $*$-homomorphism from $\mathrm{C}^*_{\max}(C) \to \mathrm{C}^*_{\max}(B)$ making the extension split. □

Proposition 5.3.7. *For any separable operator algebra C,and stable approximately unital operator algebra A, there exists an essential split extension of C by A. The middle term may be chosen to be nonunital if desired, or to be unital if C is unital.*

Proof. Let $\pi : \mathrm{C}^*_{\max}(C) \to B(\ell^2)$ be a faithful $*$-representation which is possible since $\mathrm{C}^*_{\max}(C)$ is separable. Furthermore, it can be assumed that $\pi(\mathrm{C}^*_{\max}(C)) \cap \mathbb{K} = 0$ by replacing π with $\pi\oplus\pi\oplus\cdots$ (unital case) or by $0\oplus\pi\oplus\pi\oplus\cdots$ (nonunital case). Now

$$B(\ell^2) = \mathcal{M}(\mathbb{K}) \subset \mathcal{M}(A\otimes\mathbb{K}) \cong \mathcal{M}(A),$$

and so a faithful completely contractive representation $\theta : \mathrm{C}^*_{\max}(C) \to \mathcal{M}(A)$ is obtained for which it is easy to see that $\theta(\mathrm{C}^*_{\max}(C)) \cap A = (0)$. Consider the maps

$$\mathrm{C}^*_{\max}(C) \to \mathcal{M}(A) \to \mathcal{Q}(A) \to \mathcal{Q}(\mathrm{C}^*_{\max}(A)).$$

These compose to a $*$-homomorphism, by [5, Proposition 1.2.4.] Note that if $\theta(x) \in$

$C^*_{\max}(A)$, then

$$\theta(x) \in \mathcal{M}(A) \cap C^*_{\max}(A) \subset A^{\perp\perp} \cap C^*_{\max}(A) = A,$$

by [5, Lemma A.2.3(4)], so that $\theta(x) = 0$. Thus the composition of the maps in the last centered sequence is faithful, and so completely isometric. Hence the associated morphisms $C^*_{\max}(C) \to \mathcal{Q}(A)$ and $C \to \mathcal{Q}(A)$ are completely isometric and the factor through $\mathcal{M}(A)$. $\square$

Example. There are many very interesting and topical examples of C^*-enveloped extensions, for example coming from the generalization of Gelu Popescu's noncommutative disc algebra A_n which have attracted much interest lately. The way in which these are usually obtained is to find a "Toeplitz-like' C^*-algebra $\mathcal{E}$ with a quotient 'Cuntz-like' C^*-algebra $\mathcal{F}$ which in turn is generated by a non-selfadjoint operator algebra A. In Popescu's original setting the picture is:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K} & \longrightarrow & C^*(S_1, \cdots, S_n) & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{K} & \longrightarrow & \circ & \longrightarrow & A_n & \longrightarrow & 0.
\end{array}$$

(When $n = 1$, A_n is just the disk algebra, and the top row is just the Toeplitz extension by the compacts.) In any such setting, by Theorem 4.1.5, thee is a unique completion of the diagram to a subextension. Indeed, in the example above the missing term in the diagram is the inverse image under the top right arrow $\beta : C^*(S_1, \cdots, S_n) \to \mathcal{O}$ of the bottom right algebra A_n which is the closure in

$C^*(S_1, \cdots, S_2)$ of $\mathbb{K} + A_n$. If one can show that the top right C*-algebra ($\mathcal{F}$ in the language above) is a C^*-envelope of the bottom right algebra, and doing this is currently quite an industry (initiated by Muhly and Solel), it follows from Proposition 5.3.4 that the covering extension is C^*-enveloping.

Chapter 6

Diagrams II, III, and IV

6.1 Diagram II

Most of what follows in this section are generalizations of results for C*-algebras in [17]. In all Diagram completions the use of covering extensions will be helpful.

Diagram II completions have the following form:

$$\begin{array}{ccccccccc}
E_2 : 0 & \longrightarrow & A_1 & \dashrightarrow & \circ & \dashrightarrow & \circ & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow & & \downarrow & & \\
E_1 : 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0.
\end{array}$$

With the pullback algebra critical to solving these completions, there is an added wrinkle compared to the C*-algebra theory. That is the pullback need not stay in the category of AUOA even when all three algebras are in this category. (See

Example 3.1.1.) After the discussion of the universal Diagram II completion, a criteria for this completion to stay in the same category will be given.

The universal completion of type II diagrams follows the same construction as in the C*-algebra case. For now a typical Diagram II completion will be considered without regard for a given pullback having a cai. Given a Diagram II form as above with μ proper, the universal completion is accomplished by two successive pullback constructions. The first defines $C_1 = C \oplus_{\mathcal{Q}(A)} \mathcal{Q}(A_1)$ by the following commutative diagram where τ is the Busby invariant of the bottom extension and $\tilde{\mu}$ is the unital morphism induced by μ being proper:

$$\begin{array}{ccc} C_1 & \xrightarrow{\tau_1} & \mathcal{Q}(A_1) \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \tilde{\mu}} \\ C & \xrightarrow{\tau} & \mathcal{Q}(A). \end{array} \tag{6.1.1}$$

The morphisms ν and τ_1, the canonical projections onto the first or second coordinate respectively, both are complete quotient morphisms onto their ranges. By commutativity of the above diagram, $\tilde{\mu}$ is also a complete quotient morphism.

The second pullback construction is the universal one defined by the completely contractive morphism τ_1 from C_1 into $\mathcal{Q}(A_1)$ as in Theorem 3.4.3. With this construction, $B_1 = \mathcal{M}(A_1) \oplus_{\mathcal{Q}(A_1)} C_1$. Assuming the pullback algebra C_1 has a cai, then B_1 has a cai by Lemma 3.4.1 and so this first pullback construction will be the key for the universal completion staying in the category AUOA. To finish the Diagram II completion, the morphism χ exists by Theorem 3.5.5 since by design $\tilde{\mu} \circ \tau_1 = \tau \circ \nu$.

$$
\begin{array}{ccccccccc}
E_2 : 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow \mu & & \downarrow \chi & & \downarrow \nu & & \\
E_1 : 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0.
\end{array} \tag{6.1.2}
$$

To show this completion construction is universal, assume the following is another completion with Busby invariant η.

$$
\begin{array}{ccccccccc}
E_3 : 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1'} & B_1' & \xrightarrow{\beta_1'} & C_1' & \longrightarrow & 0 \\
 & & \downarrow \mu & & \downarrow \chi' & & \downarrow \nu' & & \\
E_1 : 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0.
\end{array}
$$

Given the existence of the middle morphism, by Theorem 3.5.5, $\nu' \circ \tau(c) = \tilde{\alpha} \circ \eta(c)$ for all $c \in C_1'$. This indicates ν' and η are coherent morphisms with τ and $\tilde{\mu}$ in the construction of C_1 as the first pullback. The universal property of the pullback gives a unique morphism $\nu'' : C_1' \to C_1$ such that $\tau_1 \circ \nu'' = \eta$ and $\nu \circ \nu'' = \nu'$ giving the following commutative diagram:

$$
\begin{array}{ccccccccc}
E_3 : 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1'} & B_1' & \xrightarrow{\beta_1'} & C_1' & \longrightarrow & 0 \\
 & & \| & & \downarrow \chi'' & & \downarrow \nu'' & & \\
E_2 : 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 & \longrightarrow & 0.
\end{array}
$$

The unique morphism χ'' exists by Theorem 3.5.5 since by above $\tau_1 \circ \nu'' = \eta$. This shows the second completion factors through the universal one.

The following lemma gives conditions for the universal completion E_1 to stay in the same category as E_1, although the hypotheses are strong.

Lemma 6.1.1. *Suppose the middle algebra of E_1 above is unital, then B_1 of the universal completion shown in Diagram 6.1.1 will be unital. Also, if $\mu : A_1 \to A$ is a proper complete isometry and E_1 is trivial, that is $B \cong A \oplus^\infty C$ and has a cai, then B_1 will have a cai.*

Proof. First suppose B is unital. With β a complete quotient morphism, it must take a unit of B to a unit of C making C unital. Then if $\tau, \sigma,$ and π are the canonical morphisms associated with an extensions, $\tau \circ \beta = \pi \circ \sigma$, with the latter composition of morphisms necessarily unital, meaning $Y = \mathrm{Ran}(\pi \circ \sigma$ is a subalgebra of $\mathcal{Q}(A)$ with a unit, then τ is unital. The unit of C_1 is $(1_C, 1_Y)$ and C_1 is unital. Since $\tau : C_1 \to \mathcal{Q}(A_1)$ would then be unital, by Theorem 3.4.4 B_1 is unital.

Now suppose that $B \cong A \oplus^\infty C$ and that B is approximately unital and μ is a proper complete isometry. For B to have this form requires that $\tau = 0$. In this case, $C_1 = C \oplus_{\mathcal{Q}(A)} \mathcal{Q}(A_1) = C \oplus^\infty \mathrm{Ker}(\tilde{\mu})$. With μ a proper complete isometry, then $\mathrm{Ker}(\tilde{\mu}) = 0$ so that $C_1 \cong C$ and has a cai. $\square$

The following proposition is from [17]. Another proof is offered.

Proposition 6.1.2. *In the universal solution of Diagram 6.1.2, the square*

$$\begin{array}{ccc} B_1 & \xrightarrow{\sigma_1} & \mathcal{M}(A_1) \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \hat{\mu}} \\ B & \xrightarrow{\sigma} & \mathcal{M}(A). \end{array}$$

is a pullback. In other words, $B_1 = C_1 \oplus_{\mathcal{Q}(A_1)} \mathcal{M}(A_1) \cong B \oplus_{\mathcal{M}(A)} \mathcal{M}(A_1)$.

Proof. As indicated above, $B_1 = \mathcal{M}(A_1) \oplus_{\mathcal{Q}(A_1)} C_1$, the pullback constructed from τ_1. If $\pi : \mathcal{M}(A) \to \mathcal{Q}(A)$, and taking B to be the pullback due to τ so that

$$B = \{(m, c) : m \in \mathcal{M}(A), c \in C \text{ and } \pi(m) = \tau(c)\}.$$

Let $PB_2 = B \oplus_{\mathcal{M}(A)} \mathcal{M}(A_1)$, the pullback constructed from the above diagram. With B also a pullback,

$$PB_2 = \{((m, c), m_1) : (m, c) \in B, \sigma(m, c) = \hat{\mu}(m_1)\}.$$

Taking B to be the pullback, then σ is the projection onto the first coordinate giving $m = \hat{\mu}(m_1)$. For $(m_1, c_1) \in B_1$, define $\gamma : B_1 \to PB_2$ by $\gamma((m_1, c_1)) = ((\hat{\mu}(m_1), \nu(c_1)), m_1)$. Let $\pi_1 : \mathcal{M}(A_1) \to \mathcal{Q}(A_1)$. With $(m_1, c_1) \in B_1$, then $\pi_1(m_1) = \tau_1(c_1) = \tau_1(c, m + A_1)$ such that $\tau(c) = \tilde{\mu}(m_1 + A_1) = \hat{\mu}(m_1) + A$. This gives $(\hat{\mu}(m_1), c) \in PB$. Since $m = \hat{\mu}(m_1)$, we have $((\hat{\mu}(m_1), c), m_1) \in PB_2$. To show γ is a complete contraction, from Diagram 6.1.2, $\chi : B_1 \to B$ would need to be defined by $\chi((m_1, c_1)) = (\hat{\mu}(m_1), \nu(c_1))$. This makes sense by the above discussion. With this understanding, $\gamma((m_1, c_1)) = (\chi((m_1, c_1)), m_1)$ and so is a completely contractive morphism. To see it is a complete isometry, first recognize that elements of B_1 can be written as $(m_1, (c, m_1 + A_1))$, viewing C_1 as a pullback. Taking the norms,

$$\|(m_1, (c, m_1 + A_1))\| = \max\{\|m_1\|, \|c\|, \|m_1 + A_1\|\} = \max\{\|\hat{\mu}(m_1)\|, \|c\|, \|m_1\|\} =$$

$$\|\gamma((m_1, c, m_1 + A))\|.$$

This would be true at all matrix levels giving γ is a complete isometry. To see it is surjective, if $((m, c), m_1) \in PB_2$, then $m = \hat{\mu}(m_1)$ so that $(c, m_1 + A_1) \in C_1$ and there is an element in B_1, namely $(m_1, (c, m_1 + A_1))$, such that $\gamma((m_1, (c, m_1 + A_1))) = (m, c, m_1)$. □

Proposition 6.1.3. *Let B be a* C**-cover for A, an operator algebra with a cai. Then the following is a Type II diagram universal completion.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \mathcal{M}(A) & \xrightarrow{\pi_1} & \mathcal{Q}(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \hat{j}} & & \downarrow{\scriptstyle \tilde{j}} & & \\ 0 & \longrightarrow & B & \longrightarrow & \mathcal{M}(B) & \xrightarrow{\pi} & \mathcal{Q}(B) & \longrightarrow & 0. \end{array}$$

Proof. Since j is a complete isometry, the induced maps into the multiplier and corona algebras of $B = C^*_j(A)$ are also complete isometries, the latter by Lemma 2.1.2. The first pullback, call it PB_1, is completely isometrically isomorphic to $\mathcal{Q}(A)$. The second pullback, call it PB_2, is completely isometrically isomorphic to $\mathcal{M}(A)$. This can be seen by the respective commutative diagram.

$$\begin{array}{ccc} PB_1 & \xrightarrow{\tau_1} & \mathcal{Q}(A_1) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tilde{j}} \\ \mathcal{Q}(B) & = & \mathcal{Q}(B) \end{array}$$

Here τ_1 and β are the projections on the associated component and π is the canonical morphism. With the two algebras on the bottom equal and $\tilde{j}$ a complete isometry, $PB_1 \cong \mathcal{Q}(A)$. By Proposition 6.1.2, the middle algebra of the universal completion would be $\mathcal{M}(B) \oplus_{\mathcal{M}(B)} \mathcal{M}(A)$. This gives the middle algebra as $\mathcal{M}(A)$. The Diagram II completion can be formed as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & PB_2 & \xrightarrow{\tau_1} & \mathcal{Q}(A) & \longrightarrow & 0 \\ & & \Big\downarrow j & & \Big\downarrow \chi & & \Big\downarrow \tilde{j} & & \\ 0 & \longrightarrow & B & \longrightarrow & \mathcal{M}(B) & \xrightarrow{\pi} & \mathcal{Q}(B) & \longrightarrow & 0. \end{array}$$

By Lemma 3.3.3 the middle morphism is a complete isometry. By definition of extension the range of χ contains A as an ideal so $\chi(PB_2) \subset \hat{j}(\mathcal{M}(A))$ by Lemma 7.1.5. To see $\hat{j}(\mathcal{M}(A)) \subseteq \chi(BP_2)$, let $m \in \hat{j}(\mathcal{M}(A))$, $\pi(m) = m + B$. However, with $m \in \mathcal{M}(A)$ (suppressing $\hat{j}$), there exists a coset $m+A \in \mathcal{Q}(A)$ with $\tilde{j}(m+A) = m+B$ so that $(m, m+A) \in PB_2$ and $m \in \chi(PB_2)$ and $PB_2 \cong \mathcal{M}(A)$. □

The above lemma gives a sense of connecting the type II diagram completions with the Busby invariant of Chapter 2. Note that if C is an approximately unital operator algebra and there is an extension of C by A, this extension can be considered to be a Diagram II completion of the diagram in Proposition 6.1.3. This must factor through the universal completion, giving a complete contraction from C to $\mathcal{Q}(A)$. Additionally by Proposition 6.1.2, $\mathcal{M}(A) \cong \mathcal{M}(B) \oplus_{\mathcal{Q}(B)} \mathcal{Q}(A)$, which is easily seen to be true.

Lemma 6.1.4. *Suppose μ, the first vertical arrow in type II diagram, is a complete isometry. Then the universal completion will be strongly isomorphic to a subextension of E_1.*

Proof. By Lemma 3.3.3 it is enough to show that $\nu : C_1 \to C$ is a complete isometry. Recall that $C_1 = C \oplus_{\mathcal{Q}(A)} \mathcal{Q}(A_1)$. With μ a complete isometry, $\tilde{\mu}$ is also by Lemma 3.5.3. The morphism ν is the projection onto the first coordinate of the ordered pair $(c, m + A_1)$. With $\tilde{\mu}$ a complete isometry, $\|c\| \geq \|\tau(c)\| = \|m + A_1\|$. Hence, $\|(c, m + A_1)\| = \|c\|$ and at all matrix levels, giving that ν is a complete isometry. □

Remarks. 1.) Suppose the universal complete of a Type II diagram is given as in Diagram 6.1.2. An interesting question is, given a covering extension for E_1 in Diagram 6.1.2, E_{1*} say, and a C*-cover for A_1, $(\mathcal{D}_1, i)$ with a proper morphism from $\mathcal{D}_1$ into $\mathcal{D}$, what are the necessary hypotheses for the universal completion of Diagram 6.1.3 below to involve a covering extension for the universal completion. In other words, is E_3 below strongly isomorphic to a covering extension for E_2.

$$\begin{array}{ccccccccc} E_3 : 0 & \longrightarrow & \mathcal{D}_1 & \xrightarrow{\alpha'} & \mathcal{E}' & \xrightarrow{\beta'} & \mathcal{F}' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \chi'} & & \downarrow{\scriptstyle \nu'} & & \\ E_{1*} : 0 & \longrightarrow & \mathcal{D} & \xrightarrow{\alpha_*} & \mathcal{E} & \xrightarrow{\beta_*} & \mathcal{F} & \longrightarrow & 0. \end{array} \tag{6.1.3}$$

In the above diagram the $*$ subscripts would indicate the morphisms are the extensions of μ, α and β from Diagram 6.1.2. Even in the case μ is a complete isometry, where it is most likely the answer would be in the affirmative, a proof is quite involved. An outline of such a proof would first require verifying the relationship of the Busby invariants of the E_{1*} and E_3. With E_3 a subextension of E_{1*} by Lemma 6.1.4, Lemma 4.1.3 would need to be satisfied as well as the the canonical Busby invariant for E_3 as the universal completion. This, in fact, is the case. The difficulty comes in using this relationship to show that $\mathcal{F}'$ is generated by $\nu(C')$. After determining $\mathcal{F}'$ is a C*-cover for $\nu(C_1)$, finally using properties of sub-extensions and covering extensions, it would need to be determined that $\mathcal{E}'$ is a C*-cover for $\chi(B_1)$.

2.) One can also consider a maximal covering extension of Diagram 6.1.3 as below:

$$\begin{array}{ccccccccc} E_2' : 0 & \longrightarrow & \mathrm{C}^*_{\max}(A_1) & \xrightarrow{\alpha_{1*}} & \mathrm{C}^*_{\max}(B_1) & \xrightarrow{\beta_{1*}} & \mathrm{C}^*_{\max}(C_1) & \longrightarrow & 0 \\ & & \downarrow \mu_* & & \downarrow \chi_* & & \downarrow \nu_* & & \\ E_{1*} : 0 & \longrightarrow & \mathcal{D} & \xrightarrow{\alpha_*} & \mathcal{E} & \xrightarrow{\beta_*} & \mathcal{F} & \longrightarrow & 0. \end{array}$$

This is certainly a Diagram II completion, but not likely to be a universal such one. However, it has possibilities, using some of the previous results in this section, to give new identifications for maximal C*-covers.

6.2 Amalgamated Free Products and Type III Diagram Completions

A Diagram III completion has the form:

$$
\begin{array}{ccccccccc}
E_1 : 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mu} & & \vdots & & \vdots & & \\
E_2 : 0 & \longrightarrow & A_1 & \dashrightarrow & \circ & \dashrightarrow & \circ & \longrightarrow & 0.
\end{array}
$$

Taking μ to be proper, the universal completion utilizes repeating C as the third algebra and is of the form:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \chi} & & \Big\| & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C & \longrightarrow & 0.
\end{array}
$$

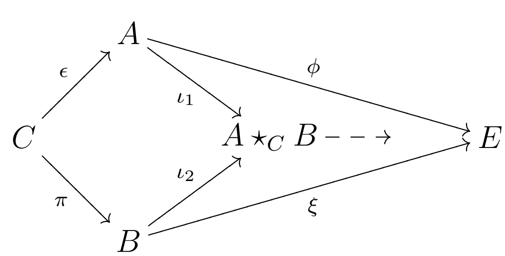
For this top line to make sense, there must be a morphism $\tau_1 : C \to \mathcal{Q}(A_1)$. Recall that with μ proper, there is a canonical morphism $\tilde{\mu} : \mathcal{Q}(A) \to \mathcal{Q}(A_1)$ by Lemma 3.5.3. Forming the composition of the Busby invariant τ from the top line with $\tilde{\mu}$ accomplishes this. By Theorem 3.5.5, with $\tau_1 = \tilde{\mu} \circ \tau$, the middle morphism χ exists. Also in this case B_1 has a cai by Lemma 3.4.1 if C is approximately unital.

Suppose there is another Diagram III completion. This would require an operator algebra C', an extension of C' by A_1 and a completely contractive morphism $\nu' : C \to C'$, which would form a commuting diagram with E_1. The existence of the middle vertical arrow requires that $\tau' \circ \nu' = \tilde{\mu} \circ \tau$ by Theorem 3.5.5. This means $\tau' \circ \nu' = \tau_1$, giving a middle morphism from $B_1 \to B'$ and the following commutative diagram. This shows any other Diagram III completion factors through the universal completion.

$$
\begin{array}{ccccccccc}
E_\tau : 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0. \\
 & & \downarrow \mu & & \downarrow \chi & & \downarrow \nu & & \\
E_{\tau_1} : 0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C & \longrightarrow & 0 \\
 & & \| & & \downarrow \chi' & & \downarrow \nu' & & \\
E_{\tau'} : 0 & \longrightarrow & A_1 & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 0.
\end{array}
$$

We will see that in the Diagram III completion above B_1 is the operator algebra amalgamated free product $A_1 \star_A B$. In this section, results regarding Type III diagram completions will be extended to operator algebras in the category of AUOA. Since the results for the C*-algebra case involve C*-algebra amalgamated free products, a description for the general class of operator algebra amalgamated free products will be given.

In the C*-algebra theory of amalgamated free products, an amalgamated free product is formed using the following diagram:



(6.2.1)

The morphisms ϵ and π are called linking morphisms. Given morphisms ϕ and ξ which are coherent morphisms if $\varphi \circ \epsilon = \xi \circ \pi$, then there exists a morphism represented above by the dotted line.

For operator algebras A, B, and C let i, j, and k be the complete isometries of each algebra respectively into its maximal C^*-algebra, let $\mathrm{C}^*_{\max}(A) \star_{\mathrm{C}^*_{\max}(C)} \mathrm{C}^*_{\max}(B)$ be the C^*-algebra amalgamated free product with linking $*$-homomorphism $\epsilon : \mathrm{C}^*_{\max}(C) \to \mathrm{C}^*_{\max}(A)$ and $\pi : \mathrm{C}^*_{\max}(C) \to \mathrm{C}^*_{\max}(B)$ and ι_1 and ι_2 the injection maps of $\mathrm{C}^*_{\max}(A)$ and $\mathrm{C}^*_{\max}(B)$ into the C^*-amalgamated free product. The amalgamated free product $A \star_C B$ will be defined as the closed subalgebra generated by $\iota_1 \circ i(A) + \iota_2 \circ j(B)$ with linking morphisms $\epsilon|_{k(C)}$ and $\pi|_{k(C)}$. It will now be shown this algebra has the required universal property.

Assume $\phi : A \to E$ and $\psi : B \to E$ are a coherent set of morphisms. By the universal property of maximal C^*-algebras, there are morphisms $\tilde{\phi} : \mathrm{C}^*_{\max}(A) \to \mathrm{C}^*_{\max}(E)$ and $\tilde{\psi} : \mathrm{C}^*_{\max}(B) \to \mathrm{C}^*_{\max}(E)$ such that $\tilde{\phi} \circ i = \phi$ on A and $\tilde{\psi} \circ j = \psi$ on B. Applying the universal property of $\mathrm{C}^*_{\max}(A) \star_{\mathrm{C}^*_{\max}(C)} C^*_M(B)$, the morphism $\tilde{\phi} \star \tilde{\psi} : \mathrm{C}^*_{\max}(A) \star_{\mathrm{C}^*_{\max}(C)} \mathrm{C}^*_{\max}(B) \to \mathrm{C}^*_{\max}(E)$ exists. It remains to see that $\tilde{\phi} \star \tilde{\psi}|_{A \star_C B}$ maps into E as a subalgebra of $\mathrm{C}^*_{\max}(E)$. By the fact that $\tilde{\phi} \circ i = \phi$ on A and $\tilde{\psi} \circ j = \psi$ on B, it is clear (suppressing some of the injection maps into the amalgamated free product) that elements of $i(A) \cup j(B)$ map into the copy of E in $\mathrm{C}^*_{\max}(E)$. By linearity, multiplicity, and continuity, the result follows on the completion of the algebraic free product. Diagram 6.2.1 will be considered to be an amalgamated free product diagram for operator algebras with the inclusion morphisms to the respective maximal C^*-covers suppressed.

Lemma 6.2.1. *With the above definition,* $\mathrm{C}^*_{\max}(A \star_C B) = \mathrm{C}^*_{\max}(A) \star_{\mathrm{C}^*_{\max}(C)} \mathrm{C}^*_{\max}(B)$.

Proof. To prove the lemma it is only necessary to show the amalgamated free product has the necessary universal maximal C*-algebra property. Suppose $\gamma : A \star_C B \to \mathcal{E}$, where $\mathcal{E}$ is a C*-algebra. The restriction of γ to $\iota_1(A)$ and $\iota_2(B)$ are a coherent pair of maps into $\mathcal{E}$ and there exists the morphism $\gamma|_{\iota_1(A)} \star \gamma|_{\iota_2(B)} : \mathrm{C}^*_{\max}(A) \star_{\mathrm{C}^*_{\max}(C)} \mathrm{C}^*_{\max}(B) \to \mathcal{E}$ demonstrating the universal property. □

In [26] extensions of C*-algebra were used to identity amalgamated free products. Here is a generalization of [26, Theorem 2.5] which utilizes the definition above of the operator algebra amalgamated free product.

Theorem 6.2.2. *In a commutative diagram of extensions of operator algebras in the category of AUOA as follows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & C & \xrightarrow{\beta} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & J & \longrightarrow & A & \xrightarrow{\delta} & X & \longrightarrow & 0, \end{array}$$

the right square is a pushout if and only if $\alpha(I)$ generates J as an ideal in the sense of [26]. Thus,

$$X = A \star_C B = A/Id(\alpha(\mathrm{Ker}(\beta))).$$

Proof. The proof follows exactly as in [26, Theorem 2.5]. □

To guarantee the amalgamated free product has a cai it would suffice to require that A and B each has a cai and the linking morphisms be proper. With these requirements, given (e_t) is a cai for C, then $\epsilon(e_t)$ is a cai for A, $\pi(e_t)$ is a cai for B.

By the construction, $\epsilon(e_t) = \pi(e_t)$ is in the amalgamated free product. It is more likely that the linking morphisms will not be proper. Even for C*-algebras, with A, B and C unital, the amalgamated free product may not be unital. As an example, let $C = \mathbb{C}$ and both $A = B = \ell_2^\infty$ with $\mathbb{C} \in A, B$ as the second coordinate $(0, z)$. Forming the amalgamated free product gives $\mathbb{C} \oplus^\infty X$, where X is the C*-algebra generated by 2 projections. A result in [28] shows this X is not only infinite dimensional, but also non-unital.

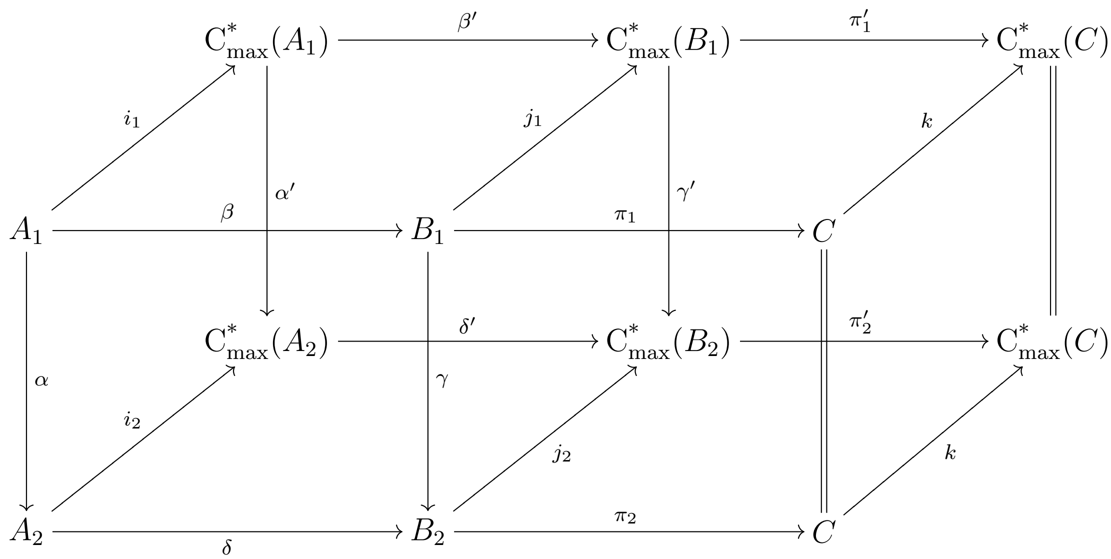
Guaranteeing the amalgamated free product will have to be done indirectly. In extending the next result for Type III diagram completions to the AUOA category, the next result will lead to another criteria for the amalgamated free product staying inside the AUOA category.

Theorem 6.2.3. *In the category of operator algebras with a cai, given the commutative diagram of extensions with α proper,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{\beta} & B_1 & \xrightarrow{\pi_1} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \| & & \\ 0 & \longrightarrow & A_2 & \xrightarrow{\delta} & B_2 & \xrightarrow{\pi_2} & C & \longrightarrow & 0, \end{array}$$

the left square is a pushout.

Proof. Since α is proper, there is a complete contraction $\tilde{\alpha} : \mathcal{Q}(A_1) \to \mathcal{Q}(A_2)$ is induced by α. Let τ be the Busby invariant of the bottom sequence. There is a completely contractive morphism $\tilde{\alpha} \circ \tau : C \to \mathcal{Q}(A_2)$. The morphism γ is from Theorem 3.5.5 making the diagram commute. Using the maximal C^*-covering extensions, we can form the following commutative box diagram using the universal property of the maximal C*-algebras.



Since α is proper, we claim that α' is proper. Let (e_t) be a cai for A_1. Then $\alpha(e_t)$ is a cai for A_2 and also a cai for $C^*_{max}(A_2)$. The diagram is commutative and this implied that $(i \circ \tilde{\alpha})(e_t)$ is cai for $C^*_{max}(A_2)$. By the corresponding result for C*-algebra Diagram III completions, $C^*_{max}(B_2) = C^*_{max}(A_2) \star_{C^*_{max}(A_1)} C^*_{max}(B_1)$ $*$-isomorphically. By definition, $B_2 = A_2 \star_{A_1} B_1$. $\square$

A welcome consequence of Theorem 6.2.3 is that it gives an example where an amalgamated free product has a cai.

Corollary 6.2.4. *Let A_1, B_1 and A_2 be in the category AUOA such that the linking morphism from $\alpha : A_1 \to A_2$ is a proper morphism and the linking morphism from $\beta : A_1 \to B_1$ is a complete isometry with the image an ideal in B_1. Then the* *amalgamated free product has a cai.*

Proof. With $\beta(A_1)$ an ideal in B_1, an exact sequence can be formed as in the first line of the diagram below. The second line can be formed via a Diagram III completion.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{\beta} & B_1 & \xrightarrow{\pi_1} & B_1/A_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \Vert & & \\
0 & \longrightarrow & A_2 & \xrightarrow{\delta} & B_2 & \xrightarrow{\pi_2} & B_1/A_1 & \longrightarrow & 0.
\end{array}
$$

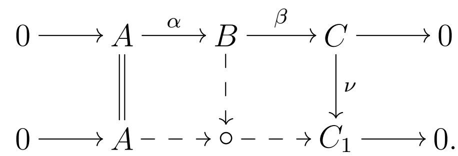
By Theorem 6.2.3 $B_2 \cong A_2 \star_{A_1} B_1$. Since it was shown B_1 has a cai in the discussion of Diagram III completions, it can be concluded $A \star_{A_1} B$ is approximately unital. $\square$

6.3 Corona Extendibility and Type IV Diagram Completions

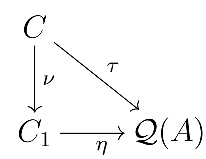
The basic diagram form is:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\nu} & & \\
0 & \longrightarrow & \circ & \dashrightarrow & \circ & \dashrightarrow & C_1 & \longrightarrow & 0.
\end{array}
$$

Consider the following completion form:



From Theorem 3.5.5 a completion will exist only if $\tau_1 : C_1 \to A$ has the property that $\tau_1 \circ \nu = \tau$ where τ is the Busby invariant from the top extension. Define a morphism $\nu : C \to C_1$ to be *corona extendible* if every morphism τ taking C into the corona algebra of a σ-unital algebra A may be factored through C_1 via a completely contractive morphism $\eta : C_1 \to \mathcal{Q}(A)$ as in the following diagram.

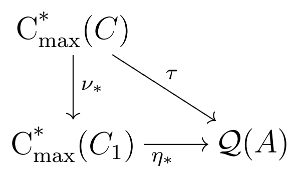


In order to extend results from C*-algebra theory to operator algebras, there would need to be a way to connect corona extendability of complete contractions between operator algebras in the category AUOA to coronal extendability of C*-algebras. It would be tempting to try and do this via the maximal C*-cover. One direction of such a connection is quite straightforward as shown in the next lemma.

Lemma 6.3.1. *Let ν be a complete contraction between operator algebras C and C_1. Then $\nu_* : \mathrm{C}^*_{\max}(C) \to \mathrm{C}^*_{\max}(C_1)$, the extension of ν to the maximal* C**-covers is corona extendible if ν, is corona extendible.*

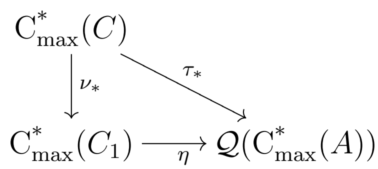
Proof. Let $\nu : C \to C_1$ be corona extendible. To show ν_* is corona extendible, let

$\tau : \mathrm{C}^*_{\max}(C) \to \mathcal{Q}(\mathcal{A})$ for a σ-unital C*-algebra $\mathcal{A}$. The restriction of τ to C is a complete contraction, so there exists a morphism $\eta : C_1 \to \mathcal{Q}(\mathcal{A})$ by virtue of ν being corona extendible. Appealing to the universal property of maximal C*-covers, ν and η can be extended to the maximal C^*-covers giving the following diagram:



implying the morphism ν_* is corona extendible. □

The difficulty for the necessary direction of Lemma 6.3.1 lies in how restricting the associated $*$-homomorphisms to the underlying algebras can be assumed to behave. Suppose that $\nu : C \to C_1$ is a completely contractive morphism with $\nu_* : \mathrm{C}^*_{\max}(C) \to \mathrm{C}^*_{\max}(C_1)$ corona extendible. Suppose there exists a completely contractive morphism $\tau : C \to \mathcal{Q}(A)$ for a σ-unital algebra A. With $\mathcal{Q}(A) \subset \mathcal{Q}(\mathrm{C}^*_{\max}(A))$, completely isometrically, there exists a $*$-homomorphism $\tau_* : \mathrm{C}^*_{\max}(C) \to \mathcal{Q}(\mathrm{C}^*_{\max}(A))$. A diagram similar to the one above can be constructed. It is shown below with the morphisms reflecting this case.



The morphism $\eta : \mathrm{C}^*_{\max}(C_1) \to (\mathcal{Q}(\mathrm{C}^*_{\max}(A)))$ exists due to ν_* being corona extendible. For ν to be corona extendible, it would need to shown that $\eta_{|C_1}$ takes C_1 to the completely isometric copy of $\mathcal{Q}(A)$ in $\mathcal{Q}(\mathrm{C}^*_{\max}(A))$. With the only interesting cases occurring when ν is not surjective, it cannot be assumed that $\eta(C_1) \subset \nu_*(C)$. Also, since corona algebras are linked to the multiplier algebras, and it can not be assumed $\mathcal{M}(A)$ generates $\mathcal{M}(\mathrm{C}^*_{\max}(A))$ the range of $\eta(C_1)$ cannot even be confirmed to stay in the C^*-algebra generated by $\mathcal{Q}(A)$. There is no hope of assuring oneself that the restriction behaves as needed and results concerning corona extendability will not be covered in this dissertation.

Chapter 7

Multiplier Algebras

The notations will be slightly different from the previous chapters. In this chapter the morphism $\pi : B \to C$ will be a complete quotient morphism such that $\text{Ker}(\pi)$ has a cai. Usually B will be required to have a cai. It will be noted if this is not the case. One then gets an exact sequence with π as the second morphism as follows:

$$0 \longrightarrow \text{Ker}(\pi) \hookrightarrow B \xrightarrow{\pi} C \longrightarrow 0.$$

The notation β will be used to denote the strict topology. If A has a cai, the β-closure is the convergence of all nets of the form $(a_t)_t$ such that when multiplied on the left or right by a fixed element of A form a norm convergent net. This gives that $\overline{A}^{\beta} = \mathcal{M}(A)$ when B is approximately unital which will be proved in Lemma 7.1.2. The morphisms α, σ, and τ will retain the same meanings as in the previous chapters. For an approximately unital algebra A, the complete quotient morphism

$q : \mathcal{M}(A) \to \mathcal{Q}(A)$ will be the canonical such morphism.

7.1 Morphisms Between Multiplier Algebras

The multiplier algebra of a C*-algebra plays a critical role in extension theory. Some of the results in [17] and [29] will be extended for an approximately unital operator algebra A. When the proofs follow directly as in the C^*-algebraic theory, it will be noted. A C*-algebra form of the following proposition can be found in [29, Proposition 2.3.7] and is a case where the proof for operator algebras in the category of AUOA is essentially the same as in [29]. The strict topology will be referred to as the β-topology.

Proposition 7.1.1. *Let A, B be approximately unital operator algebras and $\psi : \mathcal{M}(A) \to \mathcal{M}(B)$ a norm continuous homomorphism for which $B \subset \psi(A)$. Then ψ is strictly continuous, i.e. $\psi(x_\lambda) \to^\beta \psi(x)$ when $x_\lambda \to^\beta x$ in $\mathcal{M}(A)$.*

Proof. Let (x_λ) be a strictly convergent net in $\mathcal{M}(A)$ and let $b \in B$ be arbitrary. There is a lift of b in A since by hypothesis $B \subset \psi(A)$. Let a be such a lift. This gives convergent nets for all $a \in A$ of the form (ax_λ) and $(x_\lambda a)$. Since ψ is norm continuous, the images of these nets under ψ in B are also convergent nets. Since b was arbitrary, $\psi(x_\lambda)$ is strictly convergent in $\mathcal{M}(B)$. □

Lemma 7.1.2. *Suppose that A has a cai, call it (f_t), and let $m \in \mathcal{M}(A)$. Then strictly converging nets can be formed as follows:*

$$m f_t \to^{\beta} m \text{ and } f_t m \to^{\beta} m.$$

Proof. To show the above nets converge in the strict topology it needs to be shown that for any $a \in A$, $m f_t a$ and $a m f_t$ are convergent nets. With f_t a cai for A, then $m(f_t)a \to ma$ in norm. Let $am = a'$. Then $amf_t = a' f_t \to a'$ in norm. Similar arguments can be made for $f_t m$. □

Lemma 7.1.3. *Suppose that I and A are operator algebras with cais and I is a completely essential ideal in A. Then $\mathcal{M}(A) \subset \mathcal{M}(I)$.*

Proof. The canonical morphism $\sigma : A \to \mathcal{M}(I)$ is multiplier nondegenerate and by assumption is a complete isometry. By Theorem 3.5.2 there exists a unital complete isometry $\hat{\sigma} : \mathcal{M}(A) \to \mathcal{M}(I)$ which is a complete isometry. □

Alternately the above lemma could be proved in the following manner. With I an ideal in A, I is also an ideal in $\mathcal{M}(A)$. Let $\theta : \mathcal{M}(A) \to \mathcal{M}(I)$ be the canonical morphism, which is a complete isometry on A since I is completely essential in A. By Proposition 2.2.3, σ is a complete isometry on $\mathcal{M}(A)$ giving the result.

Lemma 7.1.4. *Let A be an operator algebra and $\mathcal{D}$ a* C**-algebra containing A. If $i : A \to \mathcal{D}$ is the inclusion morphism, then, working inside $\mathcal{D}^{**}$, $i^{**}(A^{**}) \cap \mathcal{D} = i^{**}(A)$. Additionally, if A and $\mathcal{D}$ share a cai, then $i^{**}(A^{**}) \cap \mathcal{M}(\mathcal{D}) = \mathcal{M}(i^{**}(A))$.*

Proof. Let $i^{**} : \mathcal{D}^{**} \to A^{**}$ be the canonical second dual morphism of i which is a complete isometry. Note that if $J = \text{Ker}(i^*)$, then $J = A^{\perp}$ and $A = J_{\perp}$. Looking inside the second dual, if $k : \mathcal{D} \to \mathcal{D}^{**}$, $i^{**}(A^{**}) = (k(A))^{\perp\perp} = J^{\perp}$. Since A is closed, $i^{**}(A^{**}) \cap k(\mathcal{D}) = k(J_{\perp}) = i^{**}(A)$. For the second assertion, since A has a cai, using the definition of $\mathcal{M}(A)$ from [5, Theorem 2.6.2 (1), Section 2.6.7], $\mathcal{M}(A) \subset A^{**} \cong A^{\perp\perp} \subset \mathcal{D}^{**}$. Also with A and $\mathcal{D}$ sharing a cai, $\mathcal{M}(A) \subset \mathcal{M}(\mathcal{D})$ by Lemma 4.1.4. This gives $\mathcal{M}(A) \subset \mathcal{M}(\mathcal{D}) \cap A^{\perp\perp}$. Conversely, if $x \in \mathcal{M}(\mathcal{D}) \cap A^{\perp\perp}$, then let $(e_t)_t$ be a cai shared by A and $\mathcal{D}$. Suppose $a \in A$. Since $\mathcal{D}$ is an algebra, $xa = y \in \mathcal{D}$. It is also the case that $xe_ta \to y \in A^{\perp\perp}$ so that $xa \in \mathcal{D} \cap A^{\perp\perp} = A$ by the first assertion. A similar calculation can be made for right multiplication giving $x \in \mathcal{M}(A)$. □

The next lemma will be used toward the end of this chapter.

Lemma 7.1.5. *Suppose that A and B each have a cai, with A a completely essential ideal in B. Let $x \in \mathcal{M}(A)$ such that $xB \subset B$ and $Bx \subset B$, then x is contained in the isometrically isomorphic copy of $\mathcal{M}(B)$ in $\mathcal{M}(A)$.*

Proof. By Lemma 7.1.3 $\mathcal{M}(B) \subset \mathcal{M}(A)$ completely isometrically as a unital subalgebra. Let $x \in \mathcal{M}(A)$ such that $xB \subset B$ and $Bx \subset B$. To see $x \in \mathcal{M}(B) \subset \mathcal{M}(A)$, let Θ be a nondegenerate completely isometric unital representation of A on $B(H)$ for some Hilbert space H. Using the definition from [5, Proposition 2.6.2 (2) and Proposition 2.6.8] of the multiplier algebra of A as:

$$\{T \in B(H) : T\Theta(A) \subset \Theta(A) \text{ and } \Theta(A)T \subset \Theta(A)\}.$$

With $\mathcal{M}(A) \subset B(H)$ completely isometrically isomorphically, and with $A \subset B \subset$

$\mathcal{M}(A)$ completely isometrically isomorphically, Θ is also a nondegenerate representation of B. Using the same definition for the $\mathcal{M}(B)$ as the one above for $\mathcal{M}(A)$, $\Theta(x)$ is in $\mathcal{M}(B) \subset B(H)$. It is also clear that Θ takes the completely isometric copy of $\mathcal{M}(B) \subset \mathcal{M}(A)$ onto the multiplier algebra of $\Theta(A)$ in $B(H)$. Since Θ is a complete isometry, $x \in \mathcal{M}(B) \subset \mathcal{M}(A)$. □

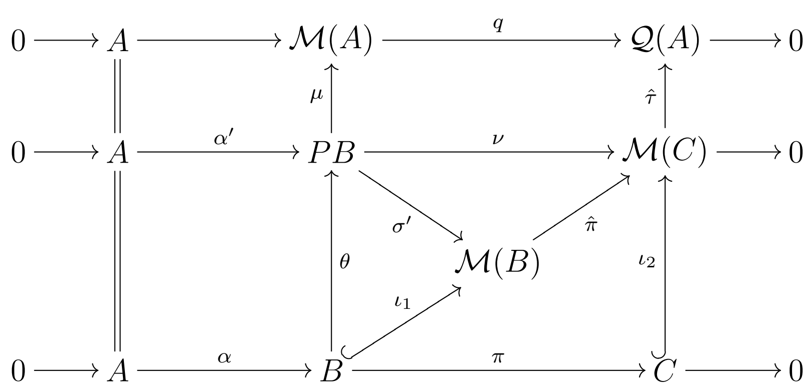
7.2 First Tietze Extension Result for Operator Algebras

Proposition 7.2.1. *Let B, C be in the category AUOA and let $\pi : B \to C$ be a complete quotient morphism such that $Ker(\pi) = A$ has a cai. Then π extends to a complete quotient morphism $\hat{\pi} : \mathcal{M}(B) \to \mathcal{M}(C)$ if the canonical completely contractive Busby morphism $\tau : C \to \mathcal{Q}(A)$ extends to a completely contractive morphism $\hat{\tau} : \mathcal{M}(C) \to \mathcal{Q}(A)$.*

Proof. With $\hat{\tau} : \mathcal{M}(C) \to \mathcal{Q}(A)$, there exists an extension $\mathcal{M}(C)$ by A with middle algebra

$$PB = \{(z, y) : z \in \mathcal{M}(A), y \in \mathcal{M}(C) \text{ and } q(z) = \hat{\tau}(y)\}.$$

Here $q : \mathcal{M}(A) \to \mathcal{Q}(A)$ is the canonical morphism. This extension contains a strongly isomorphic copy of the original extension as a sub-extension by Proposition 4.1.1 and noting that $\hat{\tau}_{|C} = \tau \circ \iota_2$ as in the diagram below. The completely isomorphic copy of B inside PB will be denoted as PB_0. The following morphism between extensions can be formed. The diagram includes canonical morphisms $\mu : PB \to \mathcal{M}(A)$ and $\nu : PB \to \mathcal{M}(C)$, the projections from the first and second coordinates of PB respectively. Not shown is $\sigma : B \to \mathcal{M}(A)$ and its canonical extension $\hat{\sigma} : \mathcal{M}(B) \to \mathcal{M}(A)$ utilizing that A is an ideal in $\mathcal{M}(B)$. These morphisms will be important in the discussion below.



(7.2.1)

Let $\theta : B \to PB_0$ be the canonical completely isometric isomorphism from $B \to PB_0$. Suppressing the inclusion morphisms ι_1 and ι_2, for all $b \in B$, $\theta(b) = (\hat{\sigma}(b), \hat{\pi}(b)) = (\sigma(b), \pi(b)) \in PB_0$ since $q \circ \sigma(b) = \tau \circ \pi(b)$. With $\hat{\sigma}, \hat{\pi}$ and $\hat{\tau}$ the extension of σ, π and τ respectively, this also shows how PB_0 sits inside PB. The inverse $\theta^{-1} : \theta(B) \to B$ takes $(\hat{\sigma}(b), \hat{\pi}(b)) \to b$. After it is shown that PB_0 is an ideal in PB, there will be a unique canonical completely contractive morphism, call it $\sigma' : PB \to \mathcal{M}(B)$, which extends θ^{-1}. This will in turn be the unique morphism which makes the top triangle in the center of the above diagram commute.

Let $(y_0, z_0) \in PB$ with $y_0 \in \mathcal{M}(A),\ z_0 \in \mathcal{M}(C)$, so that $q(y_0) = \hat{\tau}(z_0)$. Form the product of $(y_0, z_0)(\hat{\sigma}(b), \hat{\pi}(b)) = (y_0\hat{\sigma}(b), z_0\hat{\pi}(b))$, which must stay in PB since it is an algebra. Using the definition of PB, $q(y_0\hat{\sigma}(b)) = \hat{\tau}(z_0\hat{\pi}(b))$. The second component of the product, $z_0\hat{\pi}(b) = c \in C$ since $z_0 \in \mathcal{M}(C)$. It will be shown that $y_0\hat{\sigma}(b) \in \hat{\sigma}(B)$. Let $m \in \mathcal{M}(A)$ such that $y_0\hat{\sigma}(b) = m$ giving that $q(m) = q(\hat{\sigma}(b')) = \hat{\tau}(b)$. Certainly there is a $b' \in B$, by the existence of the bottom extension, such that $(q \circ \hat{\sigma})(b') = \tau(c) = \hat{\tau}(c)$. The second equality is due to $\hat{\tau}$ being the extension of τ. With $A = \mathrm{Ker}(q)$, then $m - \hat{\sigma}(b') \in A$. Let $a \in A$ with $m - \hat{\sigma}(b') = a$. The following equation can be formed using the properties of a morphism, and the fact that $(\sigma \circ \alpha)(a) = a$:

$$m = a + \hat{\sigma}(b') = \hat{\sigma}(\alpha(a)) + \hat{\sigma}(b') = \hat{\sigma}(\alpha(a) + b') \in \hat{\sigma}(B).$$

This implies that $y_0\hat{\sigma}(b) \in \hat{\sigma}(B)$. Let $b_1, b_2 \in B$ with $\hat{\sigma}(b_1) = y_0\hat{\sigma}(b)$ and $\hat{\pi}(b_2) = z_0\hat{\pi}(b)$. By definition of the pullback combined with the properties of an extension (here we will use the morphisms without the hat designation),

$$q(\sigma(b_1)) = \tau(\pi(b_2)) = (\tau \circ \pi)(b_2) = (q \circ \sigma)(b_2) = q(\sigma(b_2)).$$

Putting the first term together with the last, and noting $A = \mathrm{Ker}(q)$, indicates $\sigma(b_1) - \sigma(b_2) = a'$ for some $a' \in A$. By linearity of σ, $\sigma(b_1 - b_2) = a'$. With σ a complete isometry on A, this indicates $b_1 - b_2 = a'$ and $\pi(b_1) = \pi(b_2)$. The ordered pair $(\sigma(b_1), \pi(b_2))$ can equivalently be rewritten as $(\sigma(b_1), \pi(b_1))$. This indicates the product is in $PB_0 \cong B$, which is thus an ideal in PB.

Having shown that PB_0 is an ideal in PB, $\sigma' : PB \to \mathcal{M}(B)$ will be defined canonically. That is for all $(y, z) \in PB$, $\sigma'((y, z)) = m$ for which,

$$\theta(mb) = \theta(b') = (\sigma(b'), \pi(b')) = (y\sigma(b), z\tau(b)) = (y, z)(\sigma(b), \pi(b)),$$

and similarly for right multiplication. With σ' multiplier nondegenerate and $\sigma'_{|_{\theta(B)}}$ a complete isometry, by Theorem 3.5.2, σ' will also be a complete isometry. It needs be shown that if $\sigma'((y, z)) = m$, then $\hat{\pi}(m) = z$. From the above displayed equation it can be deduced that $\pi(mb) = zc$ and $\pi(bm) = cz$ for all $c \in C$. Since π and $\hat{\pi}$ agree in B, then $\hat{\pi}(mb) = \hat{\pi}(m)\hat{\pi}(b) = \hat{\pi}(m)c = zc$. A similar result can be found for right multiplication, giving $\hat{\pi}(m) = z$ and $\hat{\pi}$ is surjective.

To see it is also a complete quotient morphism, note that the above calculations indicate the top triangular part of the above diagram commutes with $\nu = \hat{\pi} \circ \sigma'$. As a projection, ν is a complete quotient morphism. With σ' a complete isometry, then $\hat{\pi}$ must be a complete quotient morphism. □

Unfortunately the existence of $\hat{\tau}$ is a rather strong condition, but does not require a σ-unital hypothesis. An application is split extensions. We recall that a split extension is an extension of C by A in which there exists $\gamma : C \to B$, where B is the middle algebra, in which $\pi \circ \gamma$ gives the identity morphism on B. By Theorem 3.4.4, there is an associated morphism $\eta : C \to \mathcal{M}(A)$ with $\tau = q \circ \eta$. The next lemma addresses the existence of $\hat{\tau}$ from Proposition 7.2.1 in light of split extensions.

Corollary 7.2.2. *Suppose an extension of C by A in $AUOA$ is split. If γ can be chosen such that the associated morphism $\eta : C \to \mathcal{M}(A)$ is multiplier nondegenerate, then $\hat{\tau}$ exists and by Proposition 7.2.1, $\hat{\pi} : \mathcal{M}(B) \to \mathcal{M}(C)$ is a complete quotient morphism. Additionally the extension of $\mathcal{M}(C)$ by A is split.*

Proof. Suppose that $\eta : C \to \mathcal{M}(A)$ is multiplier nondegenerate. By Theorem 3.5.2 η extends to a completely contractive morphism $\hat{\eta} : \mathcal{M}(C) \to \mathcal{M}(A)$. Composing ρ with q gives a completely contractive morphism $q \circ \rho : \mathcal{M}(C) \to \mathcal{Q}(A)$. This gives an extension of $\mathcal{M}(C)$ by A. Let PB be the middle algebra of this extension. In Theorem 3.4.4 it was shown if $\tau = q \circ \eta$ then there exists $\gamma : \mathcal{M}(C) \to PB$ showing the extension splits. By Proposition 7.2.1, $\hat{\pi}$ is a complete quotient morphism. □

7.3 Structure of Multiplier Extensions

For the rest of this chapter it will be helpful to list the notations and conventions for specific algebras and morphisms, many of which are from Lemma 3.6.2 from Chapter 3. The morphism $\pi : B \to C$ will be a complete quotient morphism with $\text{Ker}(\pi) = A$ and A having a cai. The canonical morphisms $\sigma : B \to \mathcal{M}(A)$ and $\tau : C \to \mathcal{Q}(A)$ will be as in the previous sections. Then $\text{Ker}(\sigma) = J$ and $\text{Ker}(\tau) = K$ with no assumptions about either being approximately unital. The morphisms $\theta_1 : B \to B/J$ and $\theta_2 : C \to C/K$ are the canonical complete quotient morphisms. The completely isometric morphism $\tilde{\sigma} : B/J \to \mathcal{M}(A)$ and $\tilde{\tau} : C/K \to \mathcal{Q}(A)$ are the canonical ones due to the Factor Theorem. The extensions to the respective multiplier algebras of any of the above morphisms, if they exist, will be designated with a hat. For instance, $\hat{\sigma} : \mathcal{M}(B) \to \mathcal{M}(A)$ is the extension of σ to the multiplier algebra of B. If a morphism is an extension to a $*$-homomorphism from a C*-cover, then it will be given a asterisk as a subscript. For instance $\pi_* : \mathrm{C}^*_{\max}(B) \to \mathrm{C}^*_{\max}(C)$ is the extension of π to its maximal C*-cover. The asterisk designation will also be used for a $*$-homomorphism if the relationship to the operator algebra morphism parallels what would be the relationship if it were an extension. For example, it was proved in Lemma 5.1.4 that, if $(\mathcal{D}, i)$ and $(\mathcal{E}, j)$ are C*-covers of A and B in a covering extension, and if $\hat{\sigma}' : \mathcal{M}(\mathcal{E}) \to \mathcal{M}(\mathcal{D})$ is the canonical morphism for a covering extension, and the similar canonical morphism from the operator algebra extension is $\hat{\sigma} : \mathcal{M}(B) \to \mathcal{M}(A)$, then $\sigma' \circ j = \hat{i} \circ \sigma$. With the notational conventions given above, we will want to redesignate $\hat{\sigma}' = \hat{\sigma}_*$. If a $*$-homomorphism is not necessarily related to another respective morphism, it will be given a new name if there is a possibility for confusion. Algebras that are not necessarily C*-algebras will be designated with roman letters, and C*-algebras with scripted letters. While it will be assumed that an algebra will be contained in the C*-algebra of the same letter designation, at least isometrically isomorphically, there is no assumption that the C*-algebra is a C*-cover unless it is proven to be.

Another approach has, as its inspiration, [17, Theorem 3.1] which is stated below.

Theorem 7.3.1. *[17, Theorem 3.1] Let $\mathcal{Q}(\mathcal{A})$ denote the corona algebra of a σ-unital* C*-*algebra $\mathcal{A}$, and let $\mathcal{D}$ and $\mathcal{N}$ be separable* C*-*subalgebras of $\mathcal{Q}(\mathcal{A})$. For every morphism*

$$\tau : \mathcal{C} \to \mathcal{Q}(\mathcal{A}) \cap \mathcal{D}' \cap \mathcal{N}^{\perp},$$

where $\mathcal{C}$ ia a σ-unital C*-*algebra, and every element m in $\mathcal{M}(\mathcal{C})$, there is a z in $\mathcal{Q}(\mathcal{A}) \cap \mathcal{D}' \cap \mathcal{N}^{\perp}$ such that $z\tau(c) = \tau(mc)$ and $\tau(c)z = \tau(cm)$ for each c in $\mathcal{C}$. If $0 \leq m \leq 1$, we can choose $0 \leq z \leq 1$.*

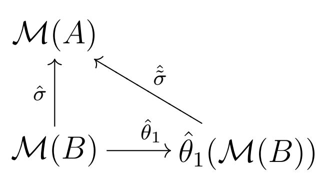
Building on Lemma 3.6.2 from Chapter 3, by Theorem 3.5.2 there exist completely contractive extensions to the respective multiplier algebras. Let $\hat{\pi}, \hat{\theta}_1, \hat{\theta}_2,$ and $\hat{\lambda}$ respectively be these extensions. The diagram below is just Diagram 3.6.1 with each morphism the expansion to the respective multiplier algebras. Usually $\hat{\lambda}$, as shown below, will be restricted to the range of $\hat{\theta}_1$.

$$\begin{array}{ccc} \hat{\theta}_1(\mathcal{M}(B)) & \xrightarrow{\hat{\lambda}} & \hat{\theta}_2(\mathcal{M}(C)) \\ {\scriptstyle \hat{\theta}_1}\uparrow & & {\scriptstyle \hat{\theta}_2}\uparrow \\ \mathcal{M}(B) & \xrightarrow{\hat{\pi}} & \mathcal{M}(C) \end{array} \tag{7.3.1}$$

With each morphism the unique extension of morphisms from Diagram 3.6.1, the diagram commutes with $\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(B))}$ surjective if $\hat{\pi}$ is surjective. Below it will be shown that if $\hat{\lambda}$ is restricted as above and satisfies certain conditions, then $\hat{\pi}$ will be a complete quotient morphism, but this is not so obvious. The only other point in question is whether or not the vertical maps have closed range. For the time being $\mathrm{Ran}(\hat{\theta}_1)$ will be assumed to be closed, which will be stated in the hypothesis of the next several results. For the next lemma and proposition, the extensions of $\sigma : B \to \mathcal{M}(A)$, and $\tilde{\sigma} : B/J \to \mathcal{M}(A)$ to the associated multiplier algebra will be required. These extensions will be denoted $\hat{\sigma} : \mathcal{M}(B) \to \mathcal{M}(A)$ and $\hat{\tilde{\sigma}} : \mathcal{M}(B/J) \to \mathcal{M}(A)$ respectively. Here is a useful lemma based on the above discussion.

Lemma 7.3.2. *Using notation from Lemma 3.6.2 and the above discussion, if σ is a complete quotient morphism onto its range, then $\hat{\sigma}$ has closed range if and only if $\hat{\theta}_1$ has closed range. Additionally, $\hat{\sigma}$ is a complete quotient morphism onto its range if and only if $\hat{\theta}_1$ is a complete quotient morphism onto its range.*

Proof. For this proof each algebra will be considered to be contained in its respectively multiplier algebra so that, for instance, $\hat{\sigma}_{|_B} = \sigma$. With σ a complete quotient morphism onto its range, $B/J \subset \mathcal{M}(A)$ completely isometrically. By Theorem 3.5.2 $\hat{\tilde{\sigma}} : \mathcal{M}(B/J) \to \mathcal{M}(A)$ is a complete isometry. That the following diagram commutes can first be noted by the fact that $(\tilde{\sigma} \circ \theta_1)(b) = \sigma(b)$ for all $b \in B$. Also, each extension to the multiplier algebras is unique and continuous in the respective strict topologies by Proposition 7.1.1. If $(e_t)_t$ is a cai for B, then if $m \in \mathcal{M}(B)$, by Lemma 7.1.2, $me_t \to^{\beta} m$. Since $\hat{\sigma}(me_t) = (\hat{\tilde{\sigma}} \circ \hat{\theta}_1)(me_t)$, $\hat{\sigma}(m) = (\hat{\tilde{\sigma}} \circ \hat{\theta}_1)(m)$.



From the above diagram, with $\hat{\tilde{\sigma}}$ a complete isometry, the restriction to $\hat{\theta}_1(\mathcal{M}(B))$ is a complete isometry. It is clear that $\hat{\sigma}$ has closed range (respectively is a complete quotient morphism) if and only if $\hat{\theta}_1$ has closed range (respectively is a complete quotient morphism). □

Proposition 7.3.3. *Using notations for the morphisms and algebras in Lemma 3.6.2 and the above discussion, let $\tau : C \to \mathcal{Q}(A)$ be a complete quotient morphism onto* *its range with $B \cong PB$ the pullback due to τ. Then if $\hat{\pi} : \mathcal{M}(B) \to \mathcal{M}(C)$ is surjective onto $\mathcal{M}(C)$ and the associated morphism $\hat{\theta}_1$ has closed range, there exist closed subalgebras X and Y of $\mathcal{M}(A)$ and $\mathcal{Q}(A)$, in particular $X = \hat{\tilde{\sigma}}(\hat{\theta}_1(\mathcal{M}(B)))$ and $Y = \hat{\tilde{\sigma}}(\hat{\theta}_1(\mathcal{M}(B)))/A$, and a surjective homomorphism $\rho : Y \to \hat{\theta}_2(\mathcal{M}(B))$ such that the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\quad q \quad} & Y \\ \Big\uparrow{\scriptstyle \hat{\tilde{\sigma}}} & & \Big\downarrow{\scriptstyle \rho} \\ \hat{\theta}_1(\mathcal{M}(B)) & \xrightarrow{\hat{\lambda}} & \hat{\theta}_2(\mathcal{M}(C)) \end{array} \tag{7.3.2}$$

Proof. The bottom horizontal arrow is surjective onto $\hat{\theta}(\mathcal{M}(C))$ since by hypothesis $\hat{\pi}$ is surjective onto $\mathcal{M}(C)$. To see this, if $\eta \in \mathcal{M}(C)$, then η has a pre-image under $\hat{\pi}$, ζ say. By commutativity of Diagram (7.3.1), and noting the definition of $\hat{\lambda}$, which must agree with the definition of λ in the proof of Lemma 3.6.2, $\hat{\lambda}(\hat{\theta}_1(\zeta)) = \hat{\theta}_2(\eta)$, giving that each element of $\hat{\theta}_2(\mathcal{M}(C))$ has a pre-image under $\hat{\lambda}$ in $\hat{\theta}_1(\mathcal{M}(B))$. Also from the proof of Lemma 3.6.2, A/J is completely essential in B/J with τ being a complete quotient morphism. The extension of $\tilde{\sigma}$ to $\mathcal{M}(B/J)$ is also a complete isometry and $\mathcal{M}(B/J) \subset \mathcal{M}(A)$ completely isometrically by Lemma 4.1.4. This gives that $X = \hat{\tilde{\sigma}}(\hat{\theta}_1(\mathcal{M}(B)))$ is closed and Y is closed by Lemma 2.1.2 since X contains $A = \mathrm{Ker}(q)$ which has a cai. Looking at Diagram 7.3.2, the morphisms in that diagram are restrictions of the morphisms in the following diagram once ρ has been defined as a completely contractive morphism.

$$\begin{array}{ccc} \hat{\tilde{\sigma}}(\mathcal{M}(B/J)) & \xrightarrow{q} & \hat{\tilde{\sigma}}(\mathcal{M}(B/J))/A \\ {\scriptstyle \hat{\tilde{\sigma}}}\uparrow & & \downarrow{\scriptstyle \rho} \\ \mathcal{M}(B/J) & \xrightarrow{\hat{\lambda}} & \mathcal{M}(C/K) \end{array} \qquad (7.3.3)$$

With $\hat{\tilde{\sigma}}$ a complete isometry, $\mathrm{Ker}(\hat{\tilde{\sigma}}) = 0$, and with $\tilde{\sigma}(A/J) = A$, $\mathrm{Ker}(q \circ \hat{\tilde{\sigma}}) = A/J$. As in Lemma 3.6.2, $A \cong A/J \subset B/J \subset \mathcal{M}(B/J)$. Define, for $m + A/J \in \mathcal{M}(B/J)/(A/J)$, $\psi : \mathcal{M}(B/J)/(A/J) \to \hat{\tilde{\sigma}}(\mathcal{M}(B/J)/A$ by $\psi(m+A/J) = \hat{\tilde{\sigma}}(m)+A$. With $A/J = \mathrm{Ker}(q \circ \hat{\tilde{\sigma}})$, by the Factor Theorem ψ is a completely isometric isomorphism. If $q' : \mathcal{M}(B/J) \to \mathcal{M}(B/J)/(A/J)$, the following commutative diagram exists.

$$\begin{array}{ccc} & & \hat{\tilde{\sigma}}(\mathcal{M}(B/J))/A \\ & \nearrow^{\psi \circ q'} & \downarrow{\scriptstyle \rho} \\ \mathcal{M}(B/J) & \xrightarrow{\hat{\lambda}} & \mathcal{M}(C/K) \end{array}$$

With $\hat{\tilde{\sigma}}(\mathcal{M}(B/J))/A \cong \mathcal{M}(B/J)/(A/J)$ via ψ, ρ exists by the Factor Theorem as a completely contractive morphism with $\mathrm{Ran}(\rho) = \mathrm{Ran}(\hat{\lambda}) = \hat{\theta}_2(\mathcal{M}(C))$. With all the morphisms in Diagram 7.3.2 the restrictions of the morphisms in Diagram 7.3.3, the result is proven. $\square$

Remarks. 1). If $I = \mathrm{Ker}(\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(B))})$, then it is easy to see that $\hat{\tilde{\sigma}}(I)/A = \mathrm{Ker}(\rho)$ by commutativity of Diagram 7.3.2.

2). Considering Theorem 7.3.1, suppose $\mathcal{A}$ and $\mathcal{C}$ are both σ-unital C*-algebras. The pullback, $\mathcal{PB}$, due to τ is σ-unital. By the C*-algebra Tietze extension theorem, the extension of the canonical surjective $*$-homomorphism $\nu : \mathcal{PB} \to \mathcal{C}$ extends to $\hat{\nu} : \mathcal{M}(\mathcal{PB}) \to \mathcal{M}(\mathcal{C})$ as a surjective $*$-homomorphism. By Proposition 7.3.3, ρ exists giving that for all $m \in \mathcal{M}(\mathcal{C})$, there exists a $z \in (q \circ \hat{\hat{\sigma}} \circ \hat{\theta}_1)(\mathcal{M}(\mathcal{PB}))$ such that $\rho(z) = \hat{\theta}(m)$. In other words, z acts on $\tau(\mathcal{C})$ in the similar manner to the way $\hat{\theta}_2(m)$ acts on $\theta_2(\mathcal{C})$.

3). Putting Proposition 7.3.3 together with Proposition 7.2.1, by Proposition 7.2.1, if $\hat{\tau}$ extends from $\mathcal{M}(C) \to \mathcal{Q}(A)$, then $\hat{\pi}$ is a complete quotient morphism. If, in addition, τ is a complete quotient morphism and $\hat{\theta}_1$ has closed range, Proposition 7.3.3 applies. The surjective morphism $\rho : Y \to \hat{\theta}_2$ as well as $\hat{\hat{\tau}} : \mathcal{M}(C/K) \to \mathcal{Q}(A)$ exist with $\hat{\hat{\sigma}}$ a complete isometry. By Lemma 3.6.1, $\hat{\hat{\tau}}$ a complete isometry giving the following commutative diagram.

$$
\begin{array}{ccc}
\hat{\hat{\sigma}}(\hat{\theta}_1(\mathcal{M}(B))) & \xrightarrow{\ q\ } & Y \\
\Big\uparrow{\scriptstyle \hat{\hat{\sigma}}} & & {\scriptstyle \hat{\hat{\tau}}}\Big\uparrow\Big\downarrow{\scriptstyle \rho} \\
\hat{\theta}_1(\mathcal{M}(B)) & \xrightarrow{\ \hat{\lambda}\ } & \hat{\theta}_2(\mathcal{M}(C))
\end{array}
$$

From the above diagram it is easy to see the ρ is a complete isometry. This also gives that $\hat{\theta}_2$ is surjective with closed range.

Above it was noted that is not obvious that if $\hat{\lambda}$ is a complete quotient morphism onto its range, then the same will be true for $\hat{\pi}$. The next proposition addresses this. Although not stated in Proposition 7.3.4, by Proposition 7.3.3, the morphism ρ will exist as a consequence.

Proposition 7.3.4. *Using the notations from Proposition 7.3.3 suppose that* $\hat{\theta}_1$ *has* *closed range and the restriction of $\hat{\lambda}$ to $\hat{\theta}_1(\mathcal{M}(B))$ is a complete quotient morphism onto $\hat{\theta}_2(\mathcal{M}(C))$ such that the kernel of this restriction has a cai. Further suppose that as in Proposition 7.3.3, τ is a complete quotient morphism onto its range. Then $\hat{\pi}$ is a complete quotient morphism with Ker$(\hat{\pi}) \cong$ Ker$(\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B))}})$.*

Proof. Let $I = \mathrm{Ker}(\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B))}}) \subset \mathcal{M}(B/J)$ which has a cai by hypothesis. Also by hypothesis, this restriction is a complete quotient morphism onto $\hat{\theta}_2(\mathcal{M}(C))$, indicating $\hat{\theta}_2$ has closed range. With $J = \mathrm{Ker}(\sigma)$, note that $A \cong A/J \subset I$ by Lemma 3.6.2. With $\hat{\hat{\sigma}}$ a complete isometry also by Lemma 3.6.2, the restriction to I is a complete isometry. Hence, by definition, A/J is completely essential in I. By Lemma 4.1.4, $\mathcal{M}(I) \subset \mathcal{M}(A)$ completely isometrically isomorphically. On the other hand, with I an ideal in $\hat{\theta}(\mathcal{M}(B))$, the following morphism between extensions exists with the prime designation the canonical morphisms of this new extension.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \xrightarrow{q'} & \mathcal{Q}(I) & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \sigma'} & & \uparrow{\scriptstyle \tau'} & & \\
0 & \longrightarrow & I & \hookrightarrow & \hat{\theta}_1(\mathcal{M}(B)) & \xrightarrow{\hat{\lambda}} & \hat{\theta}_2(\mathcal{M}(C)) & \longrightarrow & 0
\end{array}$$

The morphism σ' is the canonical such one taking $\hat{\theta}_1(\mathcal{M}(B))$ into $\mathcal{M}(I)$ and it will now be proved that σ' is a complete isometry. By the observation in the previous paragraph that $\mathcal{M}(I) \subset \mathcal{M}(A)$, define the complete isometry $\zeta : \mathcal{M}(I) \to \mathcal{M}(A)$ as the extension to $\mathcal{M}(I)$ of $\hat{\hat{\sigma}}_{|_I} : I \to \mathcal{M}(A)$. This will map $\mathcal{M}(I)$ onto a completely isometrically isomorphic copy of $\mathcal{M}(I) \subset \mathcal{M}(A)$. To see that $\hat{\hat{\sigma}}(\hat{\theta}_1(\mathcal{M}(B))) \subset \zeta(\mathcal{M}(I))$, apply Lemma 7.1.5 noting that $\hat{\hat{\sigma}}(\hat{\theta}_1(\mathcal{M}(B)))$ contains $\hat{\hat{\sigma}}(I)$ as an ideal. Since

$\zeta_{|_I} = \hat{\hat{\sigma}}(I)$ and σ' is the identity morphism on $I \subset \hat{\theta}_1(\mathcal{M}(B))$, the composition $\zeta \circ \sigma'_{|_I} = \hat{\hat{\sigma}}_{|_I}$, or more particularly $\sigma'_{|_I} = \zeta^{-1}_{|_I} \circ \hat{\hat{\sigma}}_{|_I}$. The completely contractive morphism σ' is the unique morphism taking $\hat{\theta}_1(\mathcal{M}(B)) \to \mathcal{M}(I)$ which when composed with (in this case) the inclusion morphism taking $I \to \hat{\theta}_1(\mathcal{M}(B))$ is the identity on I. By the previous observation, $\sigma' = \zeta^{-1} \circ \hat{\hat{\sigma}}_{|_{\hat{\theta}(\mathcal{M}(B))}}$ and is a complete isometry. This also makes the extension of $\hat{\theta}_2(\mathcal{M}(C))$ by I completely essential and $\tau' : \hat{\theta}_2(\mathcal{M}(C)) \to \mathcal{Q}(I)$ a complete isometry. (Note that $\mathcal{Q}(I)$ is not generally a subset of $\mathcal{Q}(A)$, but there is a canonical morphism $\delta : \zeta(\mathcal{M}(I))/A \to \mathcal{Q}(I)$ which is surjective by the Factor Theorem. This morphism will be used in a later section.) Now the following morphisms between extensions will be considered substituting B/A for C.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \xrightarrow{q'} & \mathcal{Q}(I) & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \sigma'} & & \uparrow{\scriptstyle \tau'} & & \\
0 & \longrightarrow & I & \longrightarrow & \hat{\theta}_1(\mathcal{M}(B)) & \xrightarrow{\hat{\lambda}} & \hat{\theta}_2(\mathcal{M}(B/A)) & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \mu} & & \uparrow{\scriptstyle \hat{\theta}_2} & & \\
0 & \longrightarrow & I & \xrightarrow{\alpha} & PB & \xrightarrow{\nu} & \mathcal{M}(B/A) & \longrightarrow & 0 \\
 & & \| & & \hookuparrow & & \hookuparrow & & \\
0 & \longrightarrow & I & \xrightarrow{\alpha} & PB_0 & \xrightarrow{\nu'} & B/A & \longrightarrow & 0 \\
 & & \hookuparrow & & \uparrow{\scriptstyle \gamma} & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\pi} & B/A & \longrightarrow & 0
\end{array} \qquad (7.3.4)$$

With $(\tau' \circ \hat{\theta}_2) : \mathcal{M}(B/A) \to \mathcal{Q}(I)$, there exists an extension of $\mathcal{M}(B/A)$ by I with $PB \subset \mathcal{M}(I) \oplus_{\mathcal{Q}(I)} \mathcal{M}(B/A)$. By definition,

$$PB = \{(m_1, m_2) : m_1 \in \mathcal{M}(I), m_2 \in \mathcal{M}(B/A) \text{ and } q'(m_1) = (\tau' \circ \hat{\theta}_2)(m_2)\}.$$

The intention from this point on will be to give more refined definitions of the pullbacks PB and PB_0 above. The morphisms μ and ν are the canonical projections from the first and second coordinates respectively. Note that $I \subset PB$ as $(I, 0)$ since $I = \text{Ker}(q')$ and for all $y \in I$, $q'(y) = 0 = (\tau' \circ \nu)(y)$. Using the fact that σ' and τ' are complete isometries and considering each respective algebra as a subalgebra of $\mathcal{M}(I)$ and $\mathcal{Q}(I)$, the second line is a sub-extension of the first line. To redefine the pullback using this sub-extension, it needs to be shown that given an $m \in \mathcal{M}(I)$ such that $q'(m) = \tau'(m_0)$ for $m_0 \in \hat{\theta}_2(\mathcal{M}(B/A))$, then $m \in \sigma'(\hat{\theta}_1(\mathcal{M}(B)))$. Let $(m, m_0) \in PB$. With $\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B))}}$ onto $\hat{\theta}_2(\mathcal{M}(B/A))$, there is an $x \in \hat{\theta}_1(\mathcal{M}(B))$ such that $(q' \circ \sigma')(x) = \tau'(m_0)$. With $I = \text{Ker}(q')$, then $m - \sigma'(x) \in I$, or $m = \sigma'(x) + y$ for some $y \in I$. Since $I \subset \sigma'(\hat{\theta}_1(\mathcal{M}(B)))$, $m \in \sigma(\hat{\theta}_1(\mathcal{M}(B)))$. The definition of the pullback can be amended to

$$PB = \{(m_1, m_2) : m_1 \in \hat{\theta}_1(\mathcal{M}(B)), m_2 \in \mathcal{M}(B/A) \text{ and } \hat{\lambda}(m_1) = \hat{\theta}_2(m_2)\}.$$

A specific definition of PB_0 will be given again considering $\hat{\theta}_1(\mathcal{M}(B)) \subset \mathcal{M}(I)$ and $\hat{\theta}_2(\mathcal{M}(B/A)) \subset \mathcal{Q}(I)$. Define the pullback due to $(\tau' \circ \theta_2)_{|_{B/A}}$ by

$$PB_0 = \{(m_1, b + A) : m_1 \in \hat{\theta}_1(\mathcal{M}(B)), b + A \in B/A \text{ and } \hat{\lambda}(m_1) = \hat{\theta}_2(b + A)\}.$$

The fourth line from the top would need to be a sub-extension of the third line by Proposition 4.1.1 with ν' being the restriction of ν to PB_0 and a complete quotient morphism by Lemma 2.1.2.

For the last line, define for all $b \in B$, $\gamma : B \to PB_0$ by $\gamma(b) = (\theta_1(b)), \pi(b))$. Evidently $\hat{\theta}_1 : B \to B/J \subset \mathcal{M}(B/J)$ since it agrees with θ_1 on B. By Diagram (3.6.1),

$$\hat{\lambda}(\hat{\theta}_1(b)) = \lambda(\theta_1(b)) = \theta_2(\pi(b)) = \hat{\theta}_2(\pi(b)).$$

By the revised definition of PB, takes $B \to PB$. The last line contains A as the first algebra which is completely isometrically isomorphic to $A/J \subset I$. The right vertical arrow is a complete isometry, so by Lemma 2.1.2 the middle arrow is a complete isometry, and the last line can be considered to be a sub-extension of the fourth line. Let $\mathrm{Ran}(\gamma) = PB_{00}$ with a typical element having the form $(b + J, b + A)$. Also note that γ is a complete isometry so it has an inverse morphism.

It will be shown that for all $m_2 \in \mathcal{M}(B/A)$, there is a pre-image under $\hat{\pi}$ in $\mathcal{M}(B)$ in a manner similar to the proof of Proposition 7.2.1. First it will be shown that PB_{00} is an ideal in PB. Let $(m_1, m_2) \in PB$ and form the product with $\gamma(b)$ for some $b \in B$. This would be $(m_1, m_2)(b + J, b + A) = (b_1 + J, b_2 + A)$ for $b_1, b_2 \in B$ since $m_1 \in \mathcal{M}(B/J)$ and $m_2 \in \mathcal{M}(B/A)$. Using $A = \mathrm{Ker}(\pi) \cong A/J \subset I$, completely isometrically isomorphically, it will be shown that $b_1 + J = b_2 + J$. With $(b_1 + J, b_2 + A) \in PB$, by definition of PB and the various morphisms:

$$\pi(b_1) + K = \lambda(b_1 + J) = \hat{\lambda}(b_1 + J) = \hat{\theta}_2(b_2 + A) = \hat{\pi}(b_2) + K = \pi(b) + K.$$

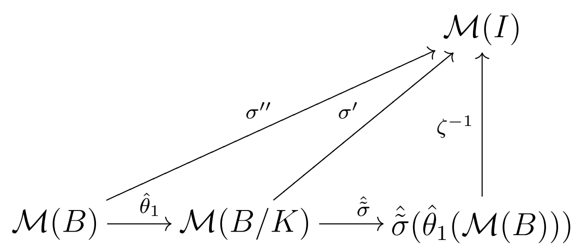
This means $\pi(b_1) - \pi(b_2) = k \in K$. Then, $\pi(b_1 - b_2) = k = \pi(j)$ for some $j \in J$. So let $b_1 - b_2 = j' \in J$ and $b_1 + J = b_2 + J$, or in other words, $\theta_1(b_1) = \theta_1(b_2)$. The ordered pair $(b_1 + J, b_2 + A)$ can be rewritten equivalently as $(b_2 + J, b_2 + A) = \gamma(b_2)$. This shows PB_{00} is an ideal in PB so we may define $\widehat{\gamma^{-1}}$ canonically as in the proof of Proposition 7.2.1. In the same way if $\widehat{\gamma^{-1}}(m_1, m_2) = m$, then $\hat{\pi}(m) = m_2$. This can be done for all $m_2 \in \mathcal{M}(C)$ showing $\hat{\pi}$ is a surjective.

For the last two assertions, to show that $\hat{\pi}$ is a complete quotient morphism and $\text{Ker}(\hat{\pi}) \cong \text{Ker}(\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(B))})$ is equivalent to showing, by Proposition 4.1.1 there is a sub-extension of $\mathcal{M}(C)$ by I with middle algebra $\mathcal{M}(B)$. With $\tau' \circ \hat{\theta}_2 : \mathcal{M}(C) \to \mathcal{Q}(I)$ the pullback extension in line four from the top of Diagram (7.3.4) will be used with $\alpha(I) = \text{Ker}(\nu)$. It needs to be shown that the previously defined morphism $\widehat{\gamma^{-1}} : PB \to \mathcal{M}(B)$ is a completely isometric isomorphism. It is already a complete isometry by Theorem 3.5.2 since $\widehat{\gamma^{-1}}|_{\gamma(B)}$ is a complete isometry. With PB a pullback, and using the second definition of PB, a morphism can be defined from $\mathcal{M}(B) \to PB$ by showing $\hat{\theta}_1$ and $\hat{\pi}$ are coherent morphisms with $\hat{\lambda}$ and $\hat{\theta}_2$. Let this morphism be denoted $\hat{\gamma}$ since it obviously extends γ as defined above giving $PB \cong \mathcal{M}(B)$. $\square$

Several corollaries follow from Proposition 7.3.4.

Corollary 7.3.5. *In the language and notation above, there is a canonical morphism* $\sigma'' : \mathcal{M}(B) \to \mathcal{M}(I)$.

Proof. Let $I_0 = \text{Ker}(\hat{\pi})$. With $\text{Ker}(\hat{\pi}) \cong I$, and $\hat{\theta}_1(I_0) = I$, and $\hat{\theta}_1$ must be a completely isometric isomorphism on I_0. From the commutativity of Diagram 7.3.4, $(\sigma' \circ \hat{\theta}_1)(I_0) = I$. This extends to a completely contractive morphism, call it σ'' : $M(B) \to \mathcal{M}(I)$, which extends $(\sigma' \circ \hat{\theta})$ as shown in the following diagram.



□

Corollary 7.3.6. *The morphism $\zeta : \mathcal{M}(I) \to \mathcal{M}(A)$ can be defined as the extension of $\hat{\sigma} \circ \hat{\theta}^{-1}_{1|_{I_0}}$, where I_0 is the completely isometrically isomorphic copy of I in $\mathcal{M}(B)$.*

Proof. Remember that $\hat{\sigma} = \hat{\hat{\sigma}} \circ \hat{\theta}_1$ and use the above diagram. □

Corollary 7.3.7. *The homomorphism $\delta : \zeta(\mathcal{M}(I))/A \to \mathcal{Q}(I)$ is a surjective completely contractive morphism with Ker$(\delta) = \hat{\hat{\sigma}}(I)/A$.*

Proof. With ζ defined as the extension of $\hat{\hat{\sigma}}(I) \to \mathcal{M}(A)$, then $A \subset \hat{\hat{\sigma}}(I) \subset \zeta(\mathcal{M}(I))$ as an ideal. A composition of morphisms will be done with the following three morphisms: the completely isometric isomorphism $\zeta^{-1}(\zeta(\mathcal{M}(I))) \to \mathcal{M}(I)$, the complete quotient morphism due to $\zeta^{-1}(A)$ being an ideal, $q_{\zeta^{-1}(A)} : \mathcal{M}(I) \to \mathcal{M}(I)/\zeta^{-1}(A)$, and the completely contractive morphism, due to the Factor Theorem, defined as $q'' : \mathcal{M}(I)/\zeta^{-1}(A) \to \mathcal{Q}(I)$. It is easy to see that $q'' \circ q_{\zeta^{-1}(A)} \circ \zeta^{-1} = \delta \circ q$, where $q : \mathcal{M}(A) \to \mathcal{M}(A)/A$, as shown below.

$$\begin{array}{ccccc} \zeta(\mathcal{M}(I)) & \xrightarrow{\quad q \quad} & \zeta(\mathcal{M}(I))/A & \xrightarrow{\quad \delta \quad} & \mathcal{Q}(I) \\ \Big\downarrow{\scriptstyle \zeta^{-1}} & & & & \Big\| \\ \mathcal{M}(I) & \xrightarrow{\quad q_{\zeta^{-1}(A)} \quad} & \mathcal{M}(I)/\zeta^{-1}(A) & \xrightarrow{\quad q'' \quad} & \mathcal{Q}(I) \end{array}$$

It is only necessary to define a morphism $\xi : \zeta(\mathcal{M}(I))/A \rightarrow \mathcal{M}(I)/\zeta^{-1}(A)$ in the obvious way. That is for all $\zeta(m) + A \in \zeta(\mathcal{M}(I))/A$, $\chi(\zeta(m) + A) = m + \zeta^{-1}(A)$. By Lemma 3.3.3, χ is a complete isometry, in fact it is a completely isometric isomorphism. $\square$

It is now possible to give a characterization for $\mathcal{M}(B)$ similar to the one given for B in Corollary 3.6.3

Corollary 7.3.8. *Using notations from Proposition 7.3.4, if the hypotheses of Proposition 7.3.4 are met, then* $\mathcal{M}(B) \cong \mathcal{M}(B/J) \oplus_{\mathcal{M}(C/K)} \mathcal{M}(C)$.

Proof. At the end of the proof of Proposition 7.3.4 $\hat{\gamma} : \mathcal{M}(B) \rightarrow PB$ was defined as the extension of $\gamma : B \rightarrow BP_{00}$. It was shown that its inverse is $\widehat{\gamma^{-1}}$ giving that $\mathcal{M}(B) \cong PB = \mathcal{M}(B/J) \oplus_{\mathcal{M}(C/K)} \mathcal{M}(C)$. $\square$

The next proposition will show that with the hypotheses of Proposition 7.3.4 saying $\hat{\lambda}$ is a complete quotient morphism is equivalent to saying $\hat{\pi}$ is a complete quotient morphism.

Proposition 7.3.9. *Using the notations from Proposition 7.3.4 suppose that $\hat{\theta}_1$ has closed range. Suppose that $I = Ker(\hat{\pi}) \cong Ker(\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(B))})$ and that I has a cai. Then*

$\hat{\pi}$ is a complete quotient morphism onto $\mathcal{M}(C)$ if and only if $\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(B))}$ is a complete quotient morphism onto $\hat{\theta}_2(\mathcal{M}(C))$.

Proof. One direction is done by Proposition 7.3.4. Suppose that $\hat{\pi}$ is a complete quotient morphism onto $\mathcal{M}(C)$. The following diagram can be formed:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \xrightarrow{q'} & \mathcal{Q}(I) & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \sigma'} & & \uparrow{\scriptstyle \tau'} & & \\
0 & \longrightarrow & I & \xrightarrow{\hat{\theta}_1} & \hat{\theta}_1(\mathcal{M}(B)) & \xrightarrow{\hat{\lambda}} & \hat{\theta}_2(\mathcal{M}(C)) & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \hat{\theta}_1} & & \uparrow{\scriptstyle \hat{\theta}_2} & & \\
0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(B) & \xrightarrow{\hat{\pi}} & \mathcal{M}(C) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\pi} & B/A & \longrightarrow & 0.
\end{array}$$

All the morphisms are the same as in Diagram 7.3.4 except now the assumption is that $\hat{\pi}$ is a complete quotient morphism instead of $\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(B))}$ and that $\tau'' : \mathcal{M}(C) \to \mathcal{Q}(I)$ exists instead of τ' above. With $\hat{\pi}$ a complete quotient morphism, $\mathcal{M}(C) \cong \mathcal{M}(B)/I$, and with $\hat{\pi}$ surjective, it is clear $\hat{\lambda}$ is surjective. By assumption, $I \cong \mathrm{Ker}(\hat{\lambda}_{\hat{\theta}_1(\mathcal{M}(B))}) = \hat{\theta}_1(I)$ so it is an ideal in $\hat{\theta}_1(\mathcal{M}(B))$. Now define τ', for $\hat{\theta}_2(m) \in \hat{\theta}_2(\mathcal{M}(C))$, by $\tau'(\hat{\theta}_2(m)) = \tau''(m)$. This makes the diagram below commute:

$$\begin{array}{ccc}
\hat{\theta}_2(\mathcal{M}(C)) & \xrightarrow{\tau'} & \mathcal{Q}(I) \\
\uparrow{\scriptstyle \hat{\theta}_2} & \nearrow{\scriptstyle \tau''} & \\
\mathcal{M}(C) & &
\end{array}$$

showing τ' is completely bounded. Proving it is a completely contractive morphism is a little more complicated since it is not assumed that $\hat{\lambda}_{\hat{\theta}_1(\mathcal{M}(B))}$ is a complete quotient morphism. With σ' still a complete isometry and q' a complete quotient morphism, $\mathrm{Ker}(q' \circ \sigma') = \hat{\theta}_1(I)$. By the Factor Theorem, a complete isometry $\chi : \hat{\theta}_1(\mathcal{M}(B))/\hat{\theta}_1(I) \to (\sigma' \circ \hat{\theta}_1)(\mathcal{M}(B))/I$ can be defined for all $\hat{\theta}_1(m) + \hat{\theta}_1(I) \in \hat{\theta}_1(\mathcal{M}(B))/\hat{\theta}_1(I)$ by $\chi(\hat{\theta}_1(m) + \hat{\theta}_1(I)) = (\sigma' \circ \hat{\theta}_1)(m) + I$. Also, by the Factor Theorem a surjective completely contractive morphism $\tilde{\hat{\lambda}} : \hat{\theta}_1(\mathcal{M}(B))/\hat{\theta}_1(I) \to \hat{\theta}_1(\mathcal{M}(C))$ exists defined in the usual way. This gives the following commutative diagram:

$$\begin{array}{ccc} & & (\sigma' \circ \hat{\theta}_1)(\mathcal{M}(B))/A \\ & \overset{\chi}{\nearrow} & \uparrow{\scriptstyle \tau'} \\ \hat{\theta}_1(\mathcal{M}(B))/\hat{\theta}_1(I) & \xrightarrow{\tilde{\hat{\lambda}}} & \hat{\theta}_1(\mathcal{M}(C)). \end{array}$$

Commutativity comes from that fact that $q' \circ \sigma'' = \tau'' \circ \hat{\pi}$ and $\sigma'' = \sigma' \circ \hat{\theta}_1$. Checking the norms based on the above diagram and using that each element in $\mathcal{M}(C)$ has a pre-image under $\hat{\pi}$, let $m \in \mathcal{M}(B)$:

$$\|(\hat{\theta}_2 \circ \hat{\pi})(m)\| \leq \|\hat{\theta}_1(m) + \hat{\theta}_1(I)\| = \|\tau'(\hat{\theta}_2(\hat{\pi}(m)))\|.$$

With τ' a completely contractive morphism, it will be shown that $\hat{\theta}_1(\mathcal{M}(B)) \cong PB$, where PB is the pullback due to τ'. With σ' and $\hat{\lambda}$ coherent with q' and τ', define $\psi : \hat{\theta}_1(\mathcal{M}(B)) \to PB$ for all $m \in \hat{\theta}_1(\mathcal{M}(B))$ by $\psi(m) = (\sigma'(m), \hat{\lambda}(m))$. With σ' a complete isometry ψ is injective and a complete isometry. That it is surjective comes from the fact that $\hat{\lambda}$ is surjective. This shows that $\hat{\lambda}$ is a complete quotient morphism. The morphism τ' is also a complete isometry by Lemma 3.6.1 since σ' is a complete isometry. $\square$

For C*-algebras, it is the case that all $*$-homomorphisms have closed range and are complete quotient morphisms onto their respective ranges. This gives a nice characterization of the previous results for C*-algebras. For the next result, even though the algebras are C*-algebras, the morphisms will be the same as described at the beginning of this section since all the algebras are C*-algebras and there is no confusion.

Corollary 7.3.10. *Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be C*-algebras with $\pi : \mathcal{B} \to \mathcal{C}$ a surjective $*$-homomorphism and Ker$(\pi) = \mathcal{A}$. Then π extends to a surjective $*$-homomorphism $\hat{\pi} : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{C})$ if and only if the induced $*$-homomorphism $\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(\mathcal{B}))}$ is surjective onto $\hat{\theta}_2(\mathcal{M}(\mathcal{C}))$ if and only if there exists a surjective $*$-homomorphism $\rho : \hat{\hat{\sigma}}(\hat{\theta}_2(\mathcal{M}(\mathcal{B})))/\mathcal{A} \to \hat{\theta}_2(\mathcal{M}(\mathcal{C}))$ such that Diagram 7.3.2 of Proposition 7.3.3 commutes.*

Proof. If $\hat{\pi}$ is surjective, then by Proposition 7.3.3 $\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(\mathcal{B}))}$ and ρ exist and are surjective. Suppose ρ exists as a surjective $*$-homomorphism satisfying the following commutative diagram from Proposition 7.3.3.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;q\;} & \mathcal{Y} \\ {\scriptstyle \hat{\hat{\sigma}}}\uparrow & & \downarrow{\scriptstyle \rho} \\ \hat{\theta}_1(\mathcal{M}(\mathcal{B})) & \xrightarrow{\;\hat{\lambda}\;} & \hat{\theta}_2(\mathcal{M}(\mathcal{C})) \end{array}$$

Here $\mathcal{X} = \hat{\sigma}'(\hat{\theta}_1(\mathcal{M}(\mathcal{B})))$ and $\mathcal{Y}$ is the image of $\mathcal{X}$ under $q : \mathcal{M}(\mathcal{A}) \to \mathcal{Q}(\mathcal{A})$. By commutativity of the diagram, $\hat{\lambda}$ is a complete quotient morphism onto $\hat{\theta}_2(\mathcal{M}(\mathcal{C}))$, and by Proposition 7.3.4, $\hat{\pi}$ is surjective. If $\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(\mathcal{B}))}}$ is a complete quotient morphism onto $\hat{\theta}_2(\mathcal{M}(\mathcal{C}))$, ρ and $\hat{\pi}$ are surjective $*$-homomorphisms by Proposition 7.3.4. $\square$

7.4 Multiplier Extension Results

Theorem 7.4.1. *Suppose that B and C are approximately unital operator algebras and let $\pi : B \to A$ be multiplier extendible such that $\hat{\pi}$ is a complete quotient morphism and all the related morphisms satisfy the hypotheses of Proposition 7.3.4. Suppose B_0 is a subalgebra of B which shares a cai so that $\mathcal{M}(B_o) \subset \mathcal{M}(B)$ completely isometrically isomorphically. Suppose $Ker(\pi_{|_{B_0}})$ shares a cai with $Ker(\pi)$, and that $Ker(\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B_0))}})$ contains a cai for $Ker(\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B))}})$. Let $C_0 = \pi(B_0)$, which will share a cai with C, so that $\mathcal{M}(C_0) \subset \mathcal{M}(C)$ completely isometrically isomorphically. Further suppose that $\tau_{|_{C_0}}$ is a complete quotient morphism onto its range, $B_0/Ker(\hat{\theta}_{1_{|_{B_0}}}) \cong \theta_1(B_0)$, and $\hat{\theta}_2(\mathcal{M}(C_0))$ is closed. Then $\pi_{|_{B_0}}$ is multiplier extendible with $\hat{\pi}_{|_{\mathcal{M}(B_0)}}$ a complete quotient morphism.*

Proof. From the hypotheses that π is a complete quotient morphism and is multiplier extendible and satisfies the hypotheses of Proposition 7.3.4, the following commutative diagram can be assumed with the vertical arrows complete isometries and the horizontal arrows complete quotient morphisms.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \xrightarrow{q'} & \mathcal{Q}(I) & \longrightarrow & 0 \\
 & & \Vert & & \uparrow{\scriptstyle \sigma'} & & \uparrow{\scriptstyle \tau'} & & \\
0 & \longrightarrow & I & \longrightarrow & \hat{\theta}_1(\mathcal{M}(B)) & \xrightarrow{\hat{\lambda}} & \hat{\theta}_2(\mathcal{M}(C)) & \longrightarrow & 0
\end{array}
$$

It can be further deduced from the hypotheses that $\pi_{|_{B_0}}, \hat{\theta}_{1_{|_{B_0}}}$ are complete quotient morphism as are $\hat{\lambda}_{|_{\hat{\theta}_1(B_0)}}$ and $\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B_0))}}$. This gives that $\theta_2(C_0) = \mathrm{Ran}(\hat{\lambda}(\theta_1(B_0)))$ is closed. Let $I_0 = \mathrm{Ker}(\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B_0))}})$, $A_0 = \mathrm{Ker}(\pi_{|_{B_0}})$, $J = \mathrm{Ker}(\theta_1)$ and $X = \mathrm{Ran}\hat{\lambda}(\hat{\theta}_1(\mathcal{M}(B_0)))$. The following commutative diagram can be formed:

$$
\begin{array}{ccccc}
I & \longrightarrow & \mathcal{M}(I) & \xrightarrow{q} & \mathcal{Q}(I) \\
\uparrow & & \uparrow{\scriptstyle \sigma'} & & \uparrow{\scriptstyle \tau'} \\
I_0 & \longrightarrow & \hat{\theta}_1(\mathcal{M}(B_0)) & \xrightarrow{\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B_0))}}} & X \\
\uparrow & & \uparrow & & \uparrow \\
A_0/J & \longrightarrow & \hat{\theta}_1(B_0) & \xrightarrow{\hat{\lambda}_{|_{\hat{\theta}_1(B_0)}}} & \hat{\theta}_2(C_0) \\
\uparrow{\scriptstyle \theta_1} & & \uparrow{\scriptstyle \hat{\theta}_{1_{|_{B_0}}}} & & \uparrow{\scriptstyle \hat{\theta}_{2_{|_{C_0}}}} \\
A_0 & \longrightarrow & B_0 & \xrightarrow{\hat{\pi}_{|_{B_0}}} & C_0.
\end{array}
$$

It will be shown that $\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B_0))}}$ is surjective onto $\hat{\theta}_2(\mathcal{M}(C_0))$ which together with the other hypotheses proves the result. With I_0 sharing a cai with $I = \mathrm{Ker}(\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B))}})$, this means $\mathcal{M}(I_0) \subset \mathcal{M}(I)$ and $\mathcal{Q}(I_0) \subset \mathcal{Q}(I)$ completely isometrically by Theorem 3.5.2. As in the proof of Proposition 7.3.4, $\sigma'(\hat{\theta}_1(\mathcal{M}(B_0)) \subset \mathcal{M}(I_0) \subset \mathcal{M}(I)$. With the intention of showing $X = \hat{\theta}_2(\mathcal{M}(C_0))$, the following diagram can be formed with each morphism a complete quotient morphism or a complete isometry.

$$
\begin{array}{ccc}
\mathcal{M}(I_0) & \xrightarrow{q_{|\mathcal{M}(I_0)}} & \mathcal{Q}(I_0) \\
\sigma' \uparrow & & \uparrow \tau' \\
\hat{\theta}_1(B_0) & \xrightarrow{\hat{\lambda}_{|\hat{\theta}_1(B_0)}} & \hat{\theta}_2(C_0) \\
\hat{\theta}_{1_{|B_0}} \uparrow & & \uparrow \hat{\theta}_{2_{|C_0}} \\
B_0 & \xrightarrow{\hat{\pi}_{|B_0}} & C_0
\end{array}
\tag{7.4.1}
$$

The only morphism that is not clearly a complete quotient morphism is $\hat{\theta}_{2_{|C_0}}$. By the hypothesis that $B_0/\mathrm{Ker}(\hat{\theta}_{1_{|B_0}}) \cong \theta_1(B_0)$, it follows $\hat{\theta}_{1_{|B_0}}$ is a complete quotient morphism. With all other morphisms in the lower box complete quotient morphisms, $\hat{\theta}_{2_{|C_0}}$ must be also.

From the last diagram above, it is evident that $\tau'(\hat{\theta}_2(C_0)) \subset \mathcal{Q}(I_0) \subset \mathcal{Q}(I)$ and with τ' a complete isometry, $\tau'(\hat{\theta}_2(C_0))$ is closed. Next, it will be shown that $\hat{\theta}_2(\mathcal{M}(C_0)) \subset \mathcal{Q}(I_0)$ completely isomorphically. Let $\tau'^{**}_{|_{\hat{\theta}_2(C_0)}} : \hat{\theta}_2(C_0)^{**} \to \mathcal{Q}(I_0)^{**}$ be the canonical completely isometric weak*-continuous extension of this restriction of τ'. With $\hat{\theta}(C_0)^{**}$ containing a completely isometrically isomorphic copy of $\mathcal{M}(\hat{\theta}_2(C_0))$, this indicates that $\mathcal{Q}(I_0)^{**}$ also contains a completely isometrically isomorphic copy of $\mathcal{M}(\hat{\theta}_2(C_0))$. Let $\iota : \mathcal{Q}(I_0) \to \mathcal{Q}(I)$ be the canonical morphism with $\iota^{**} : \mathcal{Q}(I_0)^{**} \to \mathcal{Q}(I)^{**}$ the completely isometrically isomorphically weak*-continuous extension of ι. The following commutative diagram can be formed.

$$
\begin{array}{ccccc}
\mathcal{Q}(I_0)^{**} & & \xrightarrow{\iota^{**}} & & \mathcal{Q}(I)^{**} \\
& \nwarrow \tau'^{**}_{|_{\theta_2(C_0)}} & & \tau'^{**}_{|_{\theta_2(C_0)}} \nearrow & \\
& & \theta_2(C_0)^{**} & &
\end{array}
$$

With $\tau'(\theta_2(C_0)) \subset \iota(\mathcal{Q}(C_0))$, then $\tau'^{**}_{|_{\theta_2(C_0)}}(\theta_2(C_0))^{**} \subset \iota^{**}(\mathcal{Q}(I_0)^{**})$ giving the above diagram is commutative. From this observation ι^{**} can be considered to be the identity morphism on the completely isometrically isomorphic copy of $\mathcal{M}(\theta_2(C_0))$ in $\mathcal{Q}(I)^{**}$. Since π' is a complete quotient morphism, Proposition 7.3.4 implies $\tau'(\hat{\theta}_2(\mathcal{M}(C_0)) \subset \mathcal{Q}(I)$ as well as $\mathcal{Q}(I_0)^{**}$. By Lemma 7.1.4 $\mathcal{Q}(I_0)^{**} \cap \mathcal{Q}(I) = \mathcal{Q}(I_0)$ giving $\tau'(\hat{\theta}_2(\mathcal{M}(C_0))) \subset \mathcal{Q}(I_0)$.

The following commutative diagram is the same as Diagram 7.3.4, but each of the morphisms restrictions as indicated. The inclusion morphisms from $\mathcal{M}(I_0) \to \mathcal{M}(I)$ and $\mathcal{Q}(I_0) \to \mathcal{Q}(I)$ have been suppressed for clarity. However, it is not yet proven that, in particular, $\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B_0))}}$ is surjective onto $\hat{\theta}_2(\mathcal{M}(C_0))$.

$$\begin{array}{ccccc}
I_0 & \longrightarrow & \mathcal{M}(I_0) & \xrightarrow{q_{|_{\mathcal{M}(I_0)}}} & \mathcal{Q}(I_0) \\
\| & & \uparrow{\scriptstyle \sigma'_{|_{\hat{\theta}_1(\mathcal{M}(B_0))}}} & & \uparrow{\scriptstyle \tau'_{|_{\hat{\theta}_2(\mathcal{M}(C_0))}}} \\
I_0 & \longrightarrow & \hat{\theta}_1(\mathcal{M}(B_0)) & \xrightarrow{\hat{\lambda}_{|_{\hat{\theta}_1(\mathcal{M}(B_0))}}} & \hat{\theta}_2(\mathcal{M}(C_0)) \\
\uparrow{\scriptstyle \theta_{1_{|_{A_0}}}} & & \uparrow{\scriptstyle \theta_{1_{|_{B_0}}}} & & \uparrow{\scriptstyle \theta_{2_{|_{C_0}}}} \\
A_0 & \longrightarrow & B_0 & \xrightarrow{\pi_{|_{B_0}}} & C_0
\end{array} \tag{7.4.2}$$

With $\tau'_{|_{\hat{\theta}_2(\mathcal{M}(C_0))}}$ taking $\hat{\theta}_2(\mathcal{M}(C_0))$ into $\mathcal{Q}(I_0)$, there exists an extension of $\hat{\theta}_2(\mathcal{M}(C_0))$ by I_0. As before, the pullback will be shown to contain B_0 as an ideal. The first step is to show A_0 as the first algebra on the bottom row represents more just $\mathrm{Ker}(\pi_{|_{B_0}})$. It needs to be seen that $\theta_1(A_0) = I_0 \cap \theta_1(A)$. One set inclusion is easy. Let $\theta_1(a) \in I_0 \cap \theta(A)$. This means $\theta_1(a) \in \hat{\theta}_1(\mathcal{M}(B_0))$ so that for all $\theta_1(b) \in \theta_1(B_0)$, $\theta_1(ab) \in \theta_1(B_0)$. Since $\theta_1(A)$ is an ideal, $\theta_1(ab) \in \theta_1(A)$, and the product is in $\theta_1(B_0) \cap \theta_1(A)$. Since this intersection is in $\theta_1(A)$, and θ_1 is a complete isometry on A, $\theta_1^{-1}(\theta_1(B_0) \cap \theta_1(A)) = B_0 \cap A$. As the kernel of $\pi_{|_{B_0}}$, $A_0 = B_0 \cap A$ by Proposition 4.1.1. This means $\theta_1(A_0)$ is an ideal in $I_0 \cap \theta_1(A)$. Since A_0 shares a cai with A, then $I_0 \cap \theta_1(A) = \theta_1(A_0)$.

To form the pullback due to $\tau'_{|_{\hat{\theta}_2(\mathcal{M}(C_0))}}$, the definition of the morphisms $\sigma'' : \mathcal{M}(B) \to \mathcal{M}(I)$ from Corollary 7.3.5 will be used. Then define $\tau'' : \mathcal{M}(C) \to \mathcal{Q}(I_0)$ by $\tau'' = \tau' \circ \hat{\theta}_{2_{|_{\mathcal{M}(C_0)}}}$. When the pullback is defined below, it is known that $\mathcal{M}(C_0)$ maps into $\mathcal{Q}(I_0)$, but not which elements of $\mathcal{M}(B)$ map into $\mathcal{M}(I_0)$. The following definition of the pullback reflects this and is simplified as follows:

$$PB_0 = \{(m_1, m_2) : m_1 \in \mathcal{M}(I_0), m_2 \in \mathcal{M}(C_0) : q(m_1) = \tau''(m_2)\}.$$

From the requirement $q(m_1) = \tau''(m_2)$, $m_1 \in \sigma''(\mathcal{M}(B)) \cap \mathcal{M}(I_0)$. A typical element will be more precisely denoted as $(\sigma''(m_1), m_2)$ with $m_1 \in \mathcal{M}(B)$ such that $(q \circ \sigma'')(m_1) = \tau''(m_1) \in \mathcal{Q}(I_0)$. Now we will form the canonical morphism $\gamma : B \to PB$ for all $b \in B$ using the above notations. That is $\gamma(b) = (\sigma''(b), \pi(b))$. Note that $\gamma(B)$ maps into $\mathcal{M}(I) \oplus_{\mathcal{Q}(I)} \mathcal{M}(C) = PB$ from Proposition 7.3.4. The restriction of $\sigma''_{|_{B_0}}$ and $\tau''_{|_{C_0}}$ can give a definition of the completely isometric copy of B_0 in PB. That is, for $b_0 \in B_0$, $\gamma(b_0) = (\sigma''(b_0), \pi(b_0)) \in \mathcal{M}(I_0) \oplus_{\mathcal{Q}(I_0)} \mathcal{M}(C_0)$, which is a subalgebra of $\mathcal{M}(I) \oplus_{\mathcal{Q}(I)} \mathcal{M}(C)$. Form the multiplication by a typical element of PB_0 with a typical element of $\gamma(B_0)$ as:

$$(\sigma''(m_1), m_2)(\sigma''(b_0), \pi(b_0))(\sigma''(m_1 b_0), m_2\pi(b_0)) = (\sigma''(b), \pi(b_0')).$$

Certainly, $(\sigma''(b), \pi(b_0')) \in \gamma(B)$. With γ one-to-one on B, then $(\sigma''(b), \pi(b_0')) = (\sigma''(b_1), \pi(b_1))$ for some $b_1 \in B$. Note that $\sigma''(b_1) = \sigma''(b)$ and $\pi(b_1) = \pi(b_0')$. This gives $\pi(b_0) = \pi(b_1)$ and $b_0 - b_1 \in A$. In addition, $(\sigma''(b_1), \pi(b_1)) \in PB$. By the definition, $q(\sigma''(b_1)) = \tau(\pi(b_1)) = \tau(\pi(b_0'))$. With $q \circ \sigma'' = \tau'' \circ \pi$, then $q \circ \sigma''(b_1) = q \circ \sigma''(b_0')$, so that $\sigma''(b_1) - \sigma''(b_0') = \sigma''(b_1 - b_0') \in I_0$. Putting the two together with the above discussion, namely $\theta_1(A) \cap I_0 = \theta_1(A_0)$, indicates $\theta_1(b_1 - b_0') \in \theta_1(A_0)$. Rewrite the product above as $(\sigma''(b_0 + a_0), \pi(b_0 + a_0)) \in \gamma(B_0)$. There is a canonical completely isometric isomorphism in Proposition 7.3.4 which was designated $\widehat{\gamma^{-1}} : PB \to \mathcal{M}(B)$. The restriction to PB_0, which is a completely isomorphic morphisms, maps into $\mathcal{M}(B_0) \subset \mathcal{M}(B)$ by Lemma 7.1.5. From the definition of $\hat{\gamma} : \mathcal{M}(B) \to PB$ in Proposition 7.3.4, $\hat{\gamma}(m) = (\sigma''(m), \hat{\pi}(m))$. Since $\hat{\pi}(\mathcal{M}(B_0)) \subset \mathcal{M}(C_0)$, and PB_0 is the pullback due to $\tau''_{|\mathcal{M}(C_0)}$, if $m \in \mathcal{M}(B_0)$, $\hat{\gamma}(m) = (\sigma''(m), \hat{\pi}(m)) \in PB_0$. $\square$

Theorem 7.4.1 will be used to give a Tietze extension theorem for σ-unital operator algebras. If $\pi : B \to C$ is a complete quotient morphism, it induces an extension of C by $\mathrm{Ker}(\pi)$ assuming the kernel has a cai. The strategy will be to use a covering extension of this induced extension. The C*-covers of B and C are σ-unital and so the C*-algebra Tietze theorem will apply. First a lemma is required. The conventions for the morphisms differs slightly from that discussed at the beginning of Section 7.3, but the differences are clearly defined.

Lemma 7.4.2. *Let E be an extension of C by A with middle algebra B and E_* a maximal covering extension of E. Let $i : A \to \mathrm{C}^*_{\max}(A)$, $j : B \to \mathrm{C}^*_{\max}(B)$, and $k : C \to \mathrm{C}^*_{\max}(C)$ be the inclusion morphisms. Also let $\hat{i} : \mathcal{M}(A) \to \mathcal{M}(\mathrm{C}^*_{\max}(A))$ and $\tilde{i} : \mathcal{Q}(A) \to \mathcal{Q}(\mathrm{C}^*_{\max}(C))$ be the canonical completely isometric morphisms, and* *similarly for j and k. Furthermore, assume that the canonical morphisms τ and $\hat{\sigma}$ are complete quotient morphisms onto their respective ranges. Then the associated morphisms of the E and E_* as in Proposition 7.3.4 are related in the following ways:*

(i) *If $\pi : B \to C$ is a complete quotient morphism, and $\pi' : \mathrm{C}^*_{\max}(B) \to \mathrm{C}^*_{\max}(C)$ is the canonical extension of π, then $\pi' \circ j = k \circ \pi$ and both π and π' are complete quotient morphisms. Additionally, $\|\pi(b)\| = \|\pi'(j(b))\|$ for all $b \in B$.*

(ii) *If $\tau : C \to \mathcal{Q}(A)$ is the canonical morphism due to the extension E, and if $\tau' : \mathrm{C}^*_{\max}(C) \to \mathcal{Q}(\mathrm{C}^*_{\max}(A))$ is the canonical $*$-homomorphism due to the maximal covering extension, then $\tau' \circ k = \tilde{i} \circ \tau$. Additionally $\|\tau(c)\| = \|\tau'(k(c))\|$ for all $c \in C$.*

(iii) *If $\sigma : B \to \mathcal{M}(A)$ and $\sigma' : \mathrm{C}^*_{\max}(B) \to \mathcal{M}(\mathrm{C}^*_{\max}(A))$ are the canonical morphisms, then both are complete quotient morphisms onto their ranges with $\hat{i} \circ \sigma = \sigma' \circ j$. Additionally $\|\sigma(b)\| = \|\sigma'(j(b))\|$ for all $b \in B$.*

(iv) *If $\hat{\sigma} : \mathcal{M}(B) \to \mathcal{M}(A)$ is the canonical extension of σ to $\mathcal{M}(B)$, and $\hat{\sigma}' : \mathcal{M}(\mathrm{C}^*_{\max}(B)) \to \mathcal{M}(\mathrm{C}^*_{\max}(A))$ is the canonical extension of σ' to $\mathcal{M}(\mathrm{C}^*_{\max}(B))$, then both are complete quotient morphisms onto their respective ranges, and $\hat{i} \circ \hat{\sigma} = \hat{\sigma}' \circ \hat{j}$. Additionally $\|\hat{\sigma}(m)\| = \|\hat{\sigma}'(\hat{j}(m))\|$ for all $m \in \mathcal{M}(B)$.*

(v) *If $\theta_1 : B \to B/J$, where $J = Ker(\sigma)$, is the canonical complete quotient morphism, and $\theta_1' : \mathrm{C}^*_{\max}(B) \to \mathrm{C}^*_{\max}(B)/\mathcal{J}$ where $\mathcal{J} = Ker(\sigma')$ is the canonical surjective $*$-homomorphism, then $B/J \cong B/\mathcal{J}$, and $\mathrm{C}^*_{\max}(B)/\mathcal{J}$ is a C^*-cover for B/J. Let $\iota_1 : B/J \to \mathrm{C}^*_{\max}(B)/\mathcal{J}$ is the canonical inclusion morphism, then $\iota_1 \circ \theta_1 = \theta_1' \circ j$.*

(vi) *If* $\theta_2 : C \to C/K$, *where* $K = Ker(\tau)$, *is the canonical complete quotient morphism, and* $\theta_2' : \mathrm{C}^*_{\max}(C) \to \mathrm{C}^*_{\max}(C)/\mathcal{K}$ *where* $\mathcal{K} = Ker(\tau')$ *is the canonical surjective* $*$*-homomorphism, then* $C/K \cong C/\mathcal{K}$, *and* $\mathrm{C}^*_{\max}(C)/\mathcal{K}$ *is a* C^**-cover for* C/K. *Let* $\iota_2 : C/K \to \mathrm{C}^*_{\max}(C)/\mathcal{K}$ *is the canonical inclusion morphism, then* $\iota_2 \circ \theta_2 = \theta_2' \circ k$.

(vii) *If* $\lambda : B/J \to C/K$ *and* $\lambda' : \mathrm{C}^*_{\max}(B)/\mathcal{J} \to \mathrm{C}^*_{\max}(C)/\mathcal{K}$ *are the morphism from Lemma 3.6.2, then* $\iota_2 \circ \lambda = \lambda' \circ \iota_1$. *Additionally* $\|\lambda(\theta_1(b))\| = \|\lambda'(\theta_1'(j(b)))\|$ *for all* $b \in B$.

(viii) *If* $\hat{\theta}_1 : \mathcal{M}(B) \to \mathcal{M}(B/K)$ *is the canonical morphism from Proposition 7.3.3, and* $\hat{\theta}_1' : \mathcal{M}(\mathrm{C}^*_{\max}(B)) \to \mathcal{M}(\mathrm{C}^*_{\max}(B)/\mathcal{J})$ *is the canonical extension of* θ_1', *then* $\hat{\iota}_1 \circ \hat{\theta}_1 = \hat{\theta}_1' \circ \hat{j}$. *Here the hat notations mean the extensions to the multiplier algebras. Additionally* $\|\hat{\theta}_1(m)\| = \|\hat{\theta}_1'(\hat{j}(m))\|$ *for all* $m \in \mathcal{M}(B)$.

(ix) *If* $\hat{\lambda} : \mathcal{M}(B/J) \to \mathcal{M}(C/K)$ *is the canonical morphism from Proposition 7.3.3, and* $\hat{\lambda}' : \mathcal{M}(\mathrm{C}^*_{\max}(B)/\mathcal{J}) \to \mathcal{M}(\mathrm{C}^*_{\max}(C)/\mathcal{K})$ *is the canonical morphism from Proposition 7.3.3 as it relates to the above* $*$*-homomorphisms, then* $\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(B))}$ *is a complete quotient morphism onto its range if* $Ker(\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(B))})$ *and* $Ker(\hat{\lambda}'|_{\hat{\theta}_1'(\mathcal{M}(\mathrm{C}^*_{\max}(B)))})$ *share a cai. In this case* $\|\hat{\lambda}(\hat{\theta}_1(m))\| = \|\hat{\lambda}'(\hat{\theta}_1'(m))\|$ *for all* $m \in \mathcal{M}(B)$.

(x) *If the condition in* (ix) *is satisfied, and if* $\iota_3 : Ker(\hat{\lambda}) \to Ker(\hat{\lambda}')$ *is the inclusion morphism, then* $\hat{\iota}_3 : \mathcal{M}(Ker(\hat{\lambda})) \to \mathcal{M}(Ker(\hat{\lambda}'))$ *and* $\tilde{\iota}_3 : \mathcal{Q}(Ker(\hat{\lambda})) \to \mathcal{Q}(Ker(\hat{\lambda}'))$ *are complete isometries.*

(xi) *If* $\sigma_1 : \hat{\theta}_1(\mathcal{M}(B)) \to \mathcal{M}(Ker(\hat{\lambda}))$ *and* $\sigma_1' : \hat{\theta}_2'(\mathcal{M}(\mathrm{C}^*_{\max}(B))) \to \mathcal{M}(Ker(\hat{\lambda}'))$

are the associated complete isometries from Proposition 7.3.3 for the E and E_* assuming all the above conditions are met, then $\hat{\iota}_3 \circ \sigma_1 \circ \hat{\theta}_1 = \sigma_1' \circ \hat{\theta}_1' \circ \hat{j}$ for all $m \in \mathcal{M}(B)$. In addition both σ_1 and σ_1' are complete isometries.

Proof. (i) Follows from universal property of maximal C*-cover and Lemma 2.1.2.

(ii) Follows from Lemma 5.1.1 and Corollary 2.1.4.

(iii) That σ is a complete quotient morphism follows from τ being a complete quotient morphism and Lemma 3.5.3. That $\hat{i} \circ \sigma = \sigma' \circ j$ follows from Lemma 5.1.1. With i and j complete isometries, $\|\sigma(b)\| = \|\hat{i}(\sigma(b))\| = \|\sigma'(j(b))\|$ for all $b \in B$.

(iv) By Lemma 5.1.4, $\hat{i} \circ \hat{\sigma} = \hat{\sigma}' \circ \hat{j}$ even if $\hat{\sigma}$ is not a complete quotient morphism. With $\hat{i}$ and $\hat{j}$ complete isometries, as in the proof of (iii), $\|\hat{\sigma}(m)\| = \|\hat{i}(\hat{\sigma}(m))\| = \|\hat{\sigma}'(\hat{j}(m))\|$ for all $m \in \mathcal{M}(B)$.

(v) With σ a complete quotient morphism as in (iii), then by Lemma 3.6.2, for all $b \in B$,

$$\|\sigma(b)\| = \|\theta_1(b)\| = \|b + J\| \leq \|b + \mathcal{J}\| = \|\sigma'(j(b))\| = \|\sigma(b)\|.$$

The last equality is due to (iii). The above holds at all matrix levels, giving $B/J \cong B/\mathcal{J}$. It is easy to see that $B/\mathcal{J}$ generates $C^*_{\max}(B)/\mathcal{J}$ so that the latter C*-algebra is a C*-cover for B/J. Let $\iota_1 : B/J \to B/\mathcal{J}$ be the canonical complete isometry. Then for $b \in B$, $(\iota_1 \circ \theta_1)(b) = \iota_1(b + J) = b + \mathcal{J} = \theta_1'(j(b))$, showing $\iota_1 \circ \theta_1 = \theta_1' \circ j$.

(vi) With τ a complete quotient morphism by hypothesis, by Lemma 3.6.2, for all $c \in C$,

$$\|\tau(c)\| = \|\theta_2(c)\| = \|c + K\| \leq \|c + \mathcal{K}\| = \|\tau'(k(c))\| = \|\tau(c)\|.$$

The last equality due to (ii). The above holds at all matrix levels, giving $C/K \cong C/\mathcal{K}$. It is easy to see that $C/\mathcal{K}$ generates $\mathrm{C}^*_{\max}(C)/\mathcal{K}$ so that the latter C*-algebra is a C*-cover for C/K. Let $\iota_2 : C/K \to C/\mathcal{K}$ be the canonical complete isometry. Then for $c \in C$, $(\iota_2 \circ \theta_2)(c) = \iota_2(c + K) = c + \mathcal{K} = \theta_2'(k(c))$ showing $\iota_2 \circ \theta_2 = \theta_2' \circ k$.

(vii) By Lemma 3.6.1 both λ and λ' are complete quotient morphisms with $\mathrm{Ker}(\lambda) = A/J \cong A$ and $\mathrm{Ker}(\lambda') = \mathrm{C}^*_{\max}(A)/\mathcal{J} \cong \mathrm{C}^*_{\max}(A)$. To see that for all $b + J \in B/J$, $\iota_2 \circ \lambda = \lambda' \circ \iota_1$,

$$(\iota_2 \circ \lambda(b + J) = \iota_2(\tau(b) + K) = \tau(b) + \mathcal{K} = \lambda'(b + \mathcal{J}) = \lambda'(\iota_1(b + J)).$$

To see $\|\lambda(\theta_1(b))\| = \|\lambda'(\theta'(j(b)))\|$, note that A/J and $\mathrm{C}^*_{\max}(A)/\mathcal{J}$ share a cai and the result follows from Lemma 2.1.2.

(viii) That $\hat{\theta}_1$ is a complete quotient morphism onto its range follows from the hypothesis that $\hat{\sigma}$ is a complete quotient morphism onto its range. From Lemma 7.3.2. With ι_1 a complete isometry, then $\hat{\iota}_1 : \mathcal{M}(B/K) \to \mathcal{M}(\mathrm{C}^*_{\max}(B)/\mathcal{J})$ is a complete isometry by Lemma 4.1.4. To see that for all $m \in \mathcal{M}(B)$, $\hat{\iota}_1 \circ \hat{\theta}_1 = \hat{\theta}' \circ \hat{j}$, note that the restrictions to B commute by (v). Each composition is multiplier nondegenerate, and by the uniqueness of the extension of each to $\mathcal{M}(B)$, the composition must also agree on $\mathcal{M}(B)$. That $\|\hat{\theta}_1(m)\| = \|\hat{\theta}_1'(\hat{j}(m))\|$ for all $m \in \mathcal{M}(B)$ will follow from (iv) above and Theorem 3.5.2.

(ix) With the assumption that $\mathrm{Ker}(\hat{\lambda}|_{\hat{\theta}_1(\mathcal{M}(B))})$ and $\mathrm{Ker}(\hat{\lambda}'|_{\hat{\theta}_1'(\mathcal{M}(\mathrm{C}^*_{\max}(B)))})$ share a cai, the result follows from Lemma 2.1.2

(x) Follows from Lemma 4.1.4.

(xi) From (viii), $\hat{\imath}_1 \circ \hat{\theta}_1 = \hat{\theta}_1' \circ \hat{j}$. Now it will be shown that $\sigma_1' \circ \hat{\imath}_1 = \hat{\imath}_3 \circ \sigma_1$ for all $m \in \hat{\theta}_1(\mathcal{M}(B))$. Let $m \in \hat{\theta}_1(\mathcal{M}(B))$, and

$$(\sigma_1' \circ \hat{\imath}_1)(m) = \sigma_1'(\hat{\theta}_1'(\hat{j}(m))) = \hat{\imath}_3(\sigma_1(m)).$$

Put the two equations together to get the result. That σ_1 and σ_1' are complete isometries follows from Proposition 7.3.4. □

Theorem 7.4.3. *Using the notations for algebras and morphisms from Proposition 7.3.4, Theorem 7.4.1, and Lemma 7.4.2, let $\pi : B \to C$ be a complete quotient morphism between σ-unital operator algebras such that $A = Ker(\pi)$ has a cai. Furthermore, suppose that $\tau : C \to \mathcal{Q}(A)$ is a complete quotient morphism onto its range, and $\hat{\sigma} : \mathcal{M}(B) \to \mathcal{M}(A)$ is a complete quotient morphism onto its range. Let $\mathrm{C}^*_{\max}(A)$, $\mathrm{C}^*_{\max}(B)$ and $\mathrm{C}^*_{\max}(C)$ be the maximal C^*-covers for $A, B,$ and C respectively. The extension of π' was defined as $\hat{\pi}' : \mathcal{M}(\mathrm{C}^*_{\max}(B)) \to \mathcal{M}(\mathrm{C}^*_{\max}(C))$. Finally suppose $\hat{\theta}_2(\mathcal{M}(C))$ is closed and $Ker(\hat{\pi})$ and $Ker(\hat{\pi}')$ share a cai. Then π is multiplier extendable and $\hat{\pi}$ is a complete quotient morphism.*

Proof. With B and C σ-unital, by the C^*-algebra Tietze extension theorem, $\hat{\pi}'$:

$\mathcal{M}(\mathrm{C}^*_{\max}(B)) \to \mathcal{M}(\mathrm{C}^*_{\max}(C))$ is a surjective $*$-homomorphism. The result follows if it is shown that the hypotheses of Theorem 7.4.1 are met.

First, B and $\mathrm{C}^*_{\max}(B)$ share a cai. To see that $\mathrm{Ker}(\pi)$ and $\mathrm{Ker}(\pi')$ share a cai, appealing to Proposition 5.1.2 gives a covering extension with middle algebra $\mathrm{C}^*_{\max}(B)$. By Lemma 5.3.5, all the C^*-algebras of this covering extension are maximal C^*-covers. It can be deduced that $\mathrm{C}^*_{\max}(A) = \mathrm{Ker}(\pi')$ and shares a cai with A. The hypothesis that $\tau'_{|C}$ is a complete quotient morphism follows from Lemma 7.4.2 (ii) and the hypothesis that $\tau = \tau'_{|C}$ is a complete quotient morphism. The hypothesis that $B/\mathrm{Ker} \cong \hat{\theta}'_{1_{|B}}(B)$ follows from Lemma 7.4.2 (v) and from the fact that $\hat{\theta}'_1$ extends θ'_1 to $\mathcal{M}(\mathrm{C}^*_{\max}(B))$. Since B is σ-unital, then $\mathrm{C}^*_{\max}(B)/\mathrm{Ker}(\theta'_1)$ is σ-unital. By the C^*-algebra Tietze extension theorem, $\hat{\theta}'_1$ is surjective. With $\hat{\pi}'$ also a surjective $*$-homomorphism, then by Proposition 7.3.4, $\mathrm{Ker}(\hat{\pi}') \cong \mathrm{Ker}(\hat{\lambda}')$. Putting this together with the hypotheses that $\mathrm{Ker}(\hat{\pi})$ and $\mathrm{Ker}(\hat{\pi}')$ share a cai, and that $\hat{\theta}_2(\mathcal{M}(C))$ is closed, then all the hypotheses of Theorem 7.4.1 have been met. By Theorem 7.4.1 $\hat{\pi}$ is a complete quotient morphism. □

Corollary 7.4.4. *Suppose that $\pi : B \to C$ is a complete quotient morphism between σ-unital operator algebras. Further suppose that $A = Ker(\pi)$ has a cai and is a completely essential ideal in B. If $Ker(\hat{\pi})$ contains a cai for $Ker(\hat{\pi}')$, then π is multiplier extendable such that $\hat{\pi}$ is a complete quotient morphism.*

Proof. Automatically τ, σ, and $\hat{\sigma}$ are complete isometries. In this case, θ_1, θ_2, and λ are redundant. The only remaining hypothesis for Theorem 7.4.3 is that $\hat{\pi}$ and $\hat{\pi}'$ share a cai, giving the result. □

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