THE OPERATOR SYSTEM GENERATED BY CUNTZ ISOMETRIES AND ITS APPLICATIONS

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

By
Da Zheng
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Dedicated to Jianing, with my deepest love.

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Abstract

In this thesis, we focus on the operator system $\mathscr{S}_n$ generated by n $(2 \leq n < \infty)$ Cuntz isometries, i.e. $\mathscr{S}_n = \text{span}\{I, S_i, S_i^* : 1 \leq i \leq n\}$.

We first study the properties of $\mathscr{S}_n$, such as the uniqueness, universal property, embedding property, etc. Then we construct an operator subsystem $\mathscr{E}_n$ in M_n—the n by n matrix algebra and prove that $\mathscr{S}_n$ is completely order isomorphic to an operator system quotient of $\mathscr{E}_n$. This result also leads to a characterization of positive elements in $\mathscr{S}_n$.

Next, we study the tensor products and related properties of $\mathscr{S}_n$, which is motivated by the nuclearity of the Cuntz algebra $\mathscr{O}_n$. In contrast with $\mathscr{O}_n$, $\mathscr{S}_n$ is not nuclear in the operator system category. However, we can show that it is C^*-nuclear by using the nuclearity of $\mathscr{O}_n$ and some dilation theoerems. This implies an Ando-type theorem for dual row contractions. With the help of shorted operator techniques, we are able to show that $\mathscr{S}_n$ is C^*-nuclear without using the nuclearity of $\mathscr{O}_n$. And this provides us with a new proof of the nuclearity of the $\mathscr{O}_n$.

Finally, we turn our attention to the dual operator system $\mathscr{S}_n^d$ of $\mathscr{S}_n$. By considering $\mathscr{S}_n^d$, we are able to derive an alternative characterization of the dual row contractions as well as an equivalent condition for unital completely positive maps on $\mathscr{S}_n^d$. Moreover, it is a little surprising to see that $\mathscr{S}_n^d$ is completely order isomorphic to $\mathscr{E}_n'$, an operator subsystem in M_{n+1}. The last result is a lifting theorem about the joint numerical radius, which is implied by the C^*-nuclearity of $\mathscr{S}_n^d$.

Contents

Chapter 1

Overview and Preliminaries

1.1 Overview

In the field of operator algebras, operator systems play a very important role and their theory has seen a great deal of development since the 1970s. On the other hand, although operator systems are a very important tool that has been used a lot in the study of C^*-algebras and operator spaces, the theory of tensor products of operator systems was not systematically studied until [12], where the foundations of the theory of operator system tensor products was laid, together with discussions of various types of tensor products. Since then, lots of research on the tensor products of operator systems was done. In particular, the operator system quotient was rigorously defined and studied, which led to various characterizations of some important classes of C^*-algebras using the operator system tensor products. To be concrete, Farenick and Paulsen [9] studied the operator system $\mathfrak{S}_n$

generated by n universal unitaries in $C^*(F_n)$, where F_n is the discrete free group on n generators, and they showed that $\mathfrak{S}_n$ is completely order isomorphic to a quotient of the operator system $\mathfrak{T}_n$ of tri-diagonal matrices [9, Theorem 4.2]. By using this isomorphism, they gave a new equivalent condition for a C^*-algebra to have WEP (weak expectation property), which they called property $\mathfrak{S}$ [9, Theorem 6.2]. Moreover, they defined an operator system $\mathfrak{W}_n \subseteq C^*(F_n)$ and proved that $\mathfrak{W}_n$ is completely order isomorphic to a quotient of the matrix algebra M_n and this leads to a new proof of Kirchberg's theorem [9, Corollary 3.2]. In their proofs, the universality is one of the key elements to the construction of the isomorphism.

This is the motivation for this thesis. The Cuntz algebra $\mathscr{O}_n$ is the universal C^*-algebra generated by n ($2 \leq n \leq \infty$) isometries $S_1, \ldots, S_n$ with relation $\sum_{i=1}^n S_i S_i^* = I$, where I is the identity operator (see [6] for details). We call such S_i's Cuntz isometries throughout the paper. Moreover, $\mathscr{O}_n$ enjoys many nice properties (such as nuclearity, universality, etc) and plays a very important role in the study of C^*-algebras (such as every separable and exact C^* algebra can be embedded in $\mathscr{O}_2$). Thus, one may hope that the universal property of these isometries can give us some analogous results as mentioned in the last paragraph. So we begin studying the operator system generated by Cuntz isometries, which will be denoted by $\mathscr{S}_n$ ($2 \leq n \leq \infty$) throughout, and utilize the universal property of the isometries to derive some properties of $\mathscr{S}_n$.

In Chapter 2, we first define the operator system $\mathscr{S}_n$ generated by n Cuntz isometries as:

$$\mathscr{S}_n := \operatorname{span}\{I, S_i, S_i^* : 1 \leq i \leq n\}.$$

We can show that $\mathscr{S}_n$ does not depend on the choice of the set of isometries, i.e., if isometries $\{T_i\}_{i=1}^n$ with $\sum_{i=1}^n T_i T_i = I$ span the operator system $\mathscr{T}_n$, then $\mathscr{S}_n = \mathscr{T}_n$ completely order isomorphically.

A subsequent result is the universality of $\mathscr{S}_n$:

Theorem 1.1.1. *Let $(A_1, \cdots, A_n)$ be a row contraction on some Hilbert space $\mathscr{H}$ and denote*

$$\mathscr{T}_n = \text{span}\{I_{\mathscr{H}}, A_i, A_i^* : 1 \leq i \leq n\}$$

so that $\mathscr{T}_n$ is an operator system, then there exists a unital completely positive map ϕ : $\mathscr{S}_n \to \mathscr{T}_n$ such that $\phi(S_i) = A_i$.

The main result in Chapter 2 is that $\mathscr{S}_n$ is completely order isomorphic to an operator system quotient of an operator subsystem in the matrix algebra M_{n+1}. We define an operator system $\mathscr{E}_n \subseteq M_{n+1} := M_{n+1}(\mathbb{C})$ as follows:

$$\mathscr{E}_n = \text{span}\{E_{00}, E_{0i}, E_{i0}, \sum_{i=1}^n E_{ii} : 1 \leq i \leq n\}.$$

Next, we define a map $\phi : \mathscr{E}_n \to \mathscr{S}_n$ by $\phi(E_{0i}) = \frac{1}{2} S_i$, $1 \leq i \leq n$, $\phi(E_{00}) = \phi(\sum_{i=1}^n E_{ii}) = \frac{1}{2} I$. We can check that $\ker \phi = \text{span}\{E_{00} - \sum_{i=1}^n E_{ii}\}$ and therefore we can form the quotient operator system $\mathscr{E}_n / \ker \phi$ and we prove:

Theorem 1.1.2. *[18] We have that $\mathscr{E}_n / \ker \phi$ is completely order isomorphic to $\mathscr{S}_n$ and hence ϕ is a complete quotient map,.*

Theorem 2.3.1 plays an important role in my subsequent paper with Paulsen [15] and it also implies the following corollary which gives a characterization of positive elements in $M_p(\mathscr{S}_n)$.

Corollary 1.1.3. *[18] We have that* $A_0 \otimes I + \sum_{i=1}^n A_i \otimes S_i + \sum_{i=1}^n A_i^* \otimes S_i^* \in M_p(\mathscr{S}_n)^+$ *if and only if there exists* $B \in M_p$ *such that*

$$\begin{pmatrix} A_0 & 2A_1^* & \cdots & 2A_n^* \\ 2A_1 & A_0 & & \\ \vdots & & \ddots & \\ 2A_n & & & A_0 \end{pmatrix} + \begin{pmatrix} B & & & \\ & -B & & \\ & & \ddots & \\ & & & -B \end{pmatrix} \in M_{n+1}(M_p)^+.$$

Next, in Chapter 3 we turn our attention to tensor products and nuclearity-related properties of $\mathscr{S}_n$, which is motivated by the well-known fact that $\mathscr{O}_n$ ($2 \le n \le \infty$) is nuclear in the sense that for every unital C^*-algebra $\mathscr{A}$,

$$\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{O}_n \otimes_{\max} \mathscr{A}.$$

Since $\mathscr{S}_n$ contains all the generators of $\mathscr{O}_n$ and its C^*-envelope coincides with $\mathscr{O}_n$, it is natural to study tensor properties of $\mathscr{S}_n$ ($2 \le n \le \infty$) in the operator system category.

Of course, we expect that $\mathscr{S}_n$ is nuclear in the operator system category. Unfortunately, we can show that $\mathscr{S}_n$ is not (min, max)-nuclear by constructing a counter-example. However, (min, max)-nuclearity is quite a strong condition for an operator system, as it has been shown that a finite-dimensional operator system is (min, max)-nuclear if and only if it is completely order isomorphic to a C^*-algebra if and only if it is injective [13, Theorem 6.11]. So we make a concession and ask whether $\mathscr{S}_n$ is C^*-nuclear. Fortunately, the answer is affirmative for this case. This fact follows from a refined version of Bunce's dilation theorem for row-contractions [3, Proposition 1] and the fact that $\mathscr{O}_n$ is nuclear. Thus, the operator system $\mathscr{S}_n$ enjoys many nice properties such as WEP, OSLLP, DCEP, exactness, etc (See [13]).

On the other hand, it is tempting to show directly that $\mathscr{S}_n$ is C^*-nuclear, that is, without using the nuclearity of $\mathscr{O}_n$. This is motivated by our result that $\mathscr{O}_n$ is nuclear if and only if $\mathscr{S}_n$ is C^*-nuclear. We are able to show the latter directly by using operator system techniques together with the theory of shorted operators. This provides us with a new proof of the nuclearity of the Cuntz algebras. Moreover, it motivates us to approach some important properties of the Cuntz algebra via operator system techniques.

This direct proof of the C^*-nuclearity of $\mathscr{S}_n$ also yields a dual row contraction version of Ando's theorem characterizing operators of numerical radius 1.

Theorem 1.1.4. *Let $\mathscr{A}$ be a unital C*-algebra and $(a_1,\ldots,a_n) \in \mathscr{A}$ be a strict dual row contraction, then there exists $a,b \in \mathscr{A}_{-1}^{+}$ with $a+b=1$ such that*

$$\begin{bmatrix} a & a_1 & \cdots & a_n \\ a_1^* & b & & \\ \vdots & & \ddots & \\ a_n^* & & & b \end{bmatrix}$$

is in $M_{n+1}(\mathscr{A})_{-1}^{+}$. Moreover, if $\mathscr{M}$ is a von Neumann algebra and $(a_1,\ldots,a_n) \in \mathscr{M}$ is a *dual row contraction, then there exists* $a, b \in \mathscr{M}^+$ *with* $a + b = 1$ *such that*

$$\begin{bmatrix} a & a_1 & \cdots & a_n \\ a_1^* & b & & \\ \vdots & & \ddots & \\ a_n^* & & & b \end{bmatrix}$$

is in $M_{n+1}(\mathscr{M})^+$.

Moreover, Kavruk showed in [11] that for a finite-dimensional operator system, C^*-nuclearity passes to its dual operator system, and vice versa. This motivates us to study the dual operator system of $\mathscr{S}_n$, which we denote by $\mathscr{S}_n^d$. We show that $\mathscr{S}_n^d$ is completely order isomorphic to an operator subsystem of M_{n+1}. By Kavruk's result, we know that this operator system is also C*-nuclear. However, we were unable to give a direct proof that this operator subsystem is C^*-nuclear, although an operator system in the matrix algebras seems easier to deal with.

Finally, from the general theory of operator system tensor products, we know that C^*-nuclearity is stronger than a lifting property, the OSLLP. Since $\mathscr{S}_n^d$ is C^*-nuclear, it has the OSLLP and we use this fact to prove a lifting property for Popescu's joint numerical radius for n-tuples of operators.

Theorem 1.1.5. *Let* $\mathscr{A}$ *be a unital* C^**-algebra and* $J \lhd \mathscr{A}$ *be an ideal. Suppose* $T_1 + J, \ldots, T_n + J \in \mathscr{A}/J$*, then there exist* $W_1, \ldots, W_n \in \mathscr{A}$ *with* $W_i + J = T_i + J$ *for each* $1 \leq i \leq n$*, such that* $w(W_1, \ldots, W_n) = w(T_1 + J, \ldots, T_n + J)$.

1.2 Preliminaries

In this section, we provide some basics of operator systems, operator system quotients and operator system tensor products that will be constantly used in this paper. We suggest the the reader refer to [14] or [10] for more details.

1.2.1 Operator Systems

Definition 1.2.1 (Concrete Operator System)**.** A **concrete operator system** $\mathscr{S}$ is a unital $*$-closed subspace of some unital C^*-algebra $\mathscr{A}$, that is, $\mathscr{S} \subseteq \mathscr{A}$ is a subspace of $\mathscr{A}$ such that $a \in \mathscr{S} \Rightarrow a^* \in \mathscr{S}$ and $1 \in \mathscr{S}$, where 1 denotes the unit of $\mathscr{A}$.

Definition 1.2.2 (Abstract Operator System)**.** An **abstract operator system** $\mathscr{S}$ is a matrix-ordered $*$-vector space with an Archimedean matrix order unit.

We write $M_n(\mathscr{S})^+$, $n \in \mathbb{N}$ for the positive cones of $\mathscr{S}$ and $(a_{ij}) \geq 0$ if $(a_{ij}) \in M_n(\mathscr{S})^+$ and such elements will be called **positive**.

Definition 1.2.3 (Completely Positive Maps)**.** Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems. A linear map $\phi : \mathscr{S} \to \mathscr{T}$ is called **completely positive** if

$$\phi^{(n)}((a_{ij})) := (\phi(a_{ij})) \geq 0, \quad \text{for each } (a_{ij}) \in M_n(\mathscr{S})^+ \text{ and for all } n \in \mathbb{N}.$$

Definition 1.2.4 (Complete Order Isomorphism, Complete Order Injection)**.** Let $\mathscr{S}$ and $\mathscr{T}$ be operators systems. A map $\phi : \mathscr{S} \to \mathscr{T}$ is called a **complete order isomorphism** if ϕ is a unital linear isomorphism and both ϕ and ϕ^{-1} are completely positive, and we say that

$\mathscr{S}$ is completely order isomorphic to $\mathscr{T}$ if such ϕ exists. A map ϕ is called a **complete order injection** if it is a complete order isomorphism onto its range with $\phi(1_{\mathscr{S}})$ being an Archimedean order unit. We shall denote this by $\mathscr{S} \subseteq_{\text{c.o.i}} \mathscr{T}$.

Theorem 1.2.5. *Let $\mathscr{S}$ be an abstract operator system, then there exists a Hilbert space $\mathscr{H}$, a concrete operator system $\mathscr{S}_1 \subseteq B(\mathscr{H})$, and a unital complete order isomorphism $\varphi : \mathscr{S} \to \mathscr{S}_1$. Conversely, a concrete operator system is also an abstract operator system.*

Due to this theorem, we can always identify an abstract operator system with a concrete one.

Definition 1.2.6 (Dual Operator System)**.** Let $\mathscr{S}$ be an operator system and $\mathscr{S}^d$ be the space of all bounded linear functionals on it. We define an order structure on $\mathscr{S}^d$ by

$$(f_{ij}) \in M_p(\mathscr{S}_n^d)^+ \Longleftrightarrow (f_{ij}) : \mathscr{S}_n \to M_p \text{ is completely positive .}$$

We call $\mathscr{S}^d$ with the above operator system structure the **dual operator system** of $\mathscr{S}$.

Remark 1.2.7. It is a well-known result by Choi and Effros [5, Theorem 4.4] that with the order structure defined above, the dual space of a finite-dimensional operator system is again an operator system with an Archimedean order unit, and indeed, any strictly positive linear functional is an Archimedean order unit.

1.2.2 Operator System Quotients

Definition 1.2.8 (Kernel in an Operator System)**.** Given an operator system $\mathscr{S}$, we call $J \subseteq \mathscr{S}$ a **kernel**, if $J = \ker\phi$ for an operator system $\mathscr{T}$ and some (unital) completely positive map $\phi : \mathscr{S} \to \mathscr{T}$.

Proposition 1.2.9. *[13, Proposition3.4] Let $\mathscr{S}$ be an operator system and $J \subseteq \mathscr{S}$ be a kernel. If we define a family of matrix cones on $\mathscr{S}/J$ by setting*

$$C_n = \{(x_{ij} + J) \in M_n(\mathscr{S}/J) : \text{ for each } \varepsilon > 0, \text{ there exists } (k_{ij}) \in M_n(J)$$
$$\text{such that } \varepsilon \otimes I_n + (x_{ij} + k_{ij}) \in M_n(\mathscr{S})^+\},$$

then $(\mathscr{S}/J, \{C_n\}_{n=1}^{\infty})$ is a matrix ordered $*$-vector space with an Archimedean matrix unit $1 + J$, and the quotient map $q : \mathscr{S} \to \mathscr{S}/J$ is completely positive.

Definition 1.2.10 (Operator System Quotient)**.** Let $\mathscr{S}$ be an operator system and $J \subseteq \mathscr{S}$ be kernel. We call the operator system $(\mathscr{S}/J, \{C_n\}_{n=1}^{\infty}, 1 + J)$ the **quotient operator system**.

Definition 1.2.11 (Complete Quotient Map)**.** Let $\mathscr{S}$, $\mathscr{T}$ be operator systems and $\phi : \mathscr{S} \to \mathscr{T}$ be a completely positive map, then ϕ is called a **complete quotient map** if $\mathscr{S}/\ker\phi$ is complete order isomorphic to $\mathscr{T}$.

Definition 1.2.12 (Completely Order Proximinal)**.** Let J be a kernel and define

$$D_n = \{(x_{ij} + J) \in M_n(\mathscr{S}/J) : \text{ there exists } y_{ij} \in J$$
$$\text{such that } (x_{ij} + y_{ij}) \in M_n(\mathscr{S})^+\}.$$

Then J is **completely order proximinal** if $C_n = D_n$ for all $n \in \mathbb{N}$.

Remark 1.2.13. This means that if a kernel J is complete order proximinal, then

$$M_n(\mathscr{S}/J)^+ = M_n(\mathscr{S})^+ + M_n(J).$$

1.2.3 Operator System Tensor Products

Definition 1.2.14. Given a pair of operator systems $(\mathscr{S}, \{P_n\}_{n=1}^{\infty}, e_1)$, $(\mathscr{T}, \{Q_n\}_{n=1}^{\infty}, e_2)$, by an **operator system structure** on the algebraic tensor product $\mathscr{S} \otimes \mathscr{T}$, we mean a family of cones $\tau = \{C_n\}_{n=1}^{\infty} \subseteq M_n(\mathscr{S} \otimes \mathscr{T})$, such that:

1. $(\mathscr{S} \otimes \mathscr{T}, C_n, e_1 \otimes e_2)$ is an operator system denoted by $\mathscr{S} \otimes_\tau \mathscr{T}$, and

2. $P_n \otimes Q_m \subseteq C_{nm}$, for all $n, m \in \mathbb{N}$, and

3. If $\phi : \mathscr{S} \to M_n$ and $\psi : \mathscr{T} \to M_m$ are unital completely positive maps, then $\phi \otimes \psi : \mathscr{S} \otimes_\tau \mathscr{T} \to M_{nm}$ is a unital completely positive map.

Definition 1.2.15 (Operator System Tensor Product)**.** By an **operator system tensor product**, we mean a mapping $\tau : \mathscr{O} \times \mathscr{O} \to \mathscr{O}$, where $\mathscr{O}$ denotes the the category of operator systems, such that for every pair of operator systems $\mathscr{S}$ and $\mathscr{T}$, $\tau(\mathscr{S}, \mathscr{T})$ is an operator system structure on $\mathscr{S} \otimes \mathscr{T}$, denoted $\mathscr{S} \otimes_\tau \mathscr{T}$.

1. We call $\tau_1 \leq \tau_2$ if $M_n(\mathscr{S} \otimes_{\tau_2} \mathscr{T})^+ \subseteq M_n(\mathscr{S} \otimes_{\tau_1} \mathscr{T})^+$ for every $n \in \mathbb{N}$.

2. We call τ **functorial** if for any four operator system $\mathscr{S}_1$, $\mathscr{S}_2$, $\mathscr{T}_1$, $\mathscr{T}_2$, we have that if $\phi : \mathscr{S}_1 \to \mathscr{S}_2$, $\psi : \mathscr{T}_1 \to \mathscr{T}_2$ are unital completely positive, then $\phi \otimes \psi : \mathscr{S}_1 \otimes_\tau \mathscr{T}_1 \to \mathscr{S}_2 \otimes_\tau \mathscr{T}_2$ is unital completely positive.

3. We call τ **symmetric** if $\theta : x \otimes y \mapsto y \otimes x$ extends to a complete order isomorphism between $\mathscr{S} \otimes_\tau \mathscr{T}$ and $\mathscr{T} \otimes_\tau \mathscr{S}$.

4. We call τ **associative** if for any three operator systems $\mathscr{S}$, $\mathscr{T}$, $\mathscr{R}$, the natural Isomorphism from $(\mathscr{R}\otimes_\tau\mathscr{S})\otimes_\tau\mathscr{T}$ onto $\mathscr{R}\otimes_\tau(\mathscr{S}\otimes_\tau\mathscr{T})$ is indeed a complete order isomorphism.

5. Let α and β be two operator system tensor products. An operator system $\mathscr{S}$ is called **(α,β)-nuclear** if the identity map between $\mathscr{S}\otimes_\alpha\mathscr{T}$ and $\mathscr{S}\otimes_\beta\mathscr{T}$ is complete order isomorphic for every operator system $\mathscr{T}$.

There are several different types of tensor products introduced in [12], but we will mainly use the following three:

Definition 1.2.16 (The Min Tensor Product)**.** The **minimal operator system structure** on $\mathscr{S}\otimes\mathscr{T}$ is defined as

$$C_n^{\min} = \{(p_{ij})\in M_n(\mathscr{S}\otimes\mathscr{T}) : \Big((\phi\otimes\psi)(p_{ij})\Big)\in M_{nkm}^+,$$
$$\text{for all } \phi\in S_k(\mathscr{S}), \psi\in S_m(\mathscr{T}), \text{ for all } k,m\in\mathbb{N}\},$$

where $S_k(\mathscr{S})$ denotes the set of all completely positive maps from $\mathscr{S}$ to M_k. We call the operator system $(\mathscr{S}\otimes\mathscr{T},(C_n^{\min})_{n=1}^\infty, 1\otimes 1)$ the **minimal tensor product** of $\mathscr{S}$ and $\mathscr{T}$ and denote it by $\mathscr{S}\otimes_{\min}\mathscr{T}$.

It can be shown that the min-tensor product is injective, associative, symmetric and functorial. Moreover, it coincide with the operator system arising from the embedding $\mathscr{S}\otimes\mathscr{T}\subseteq_{\text{c.o.i}} B(\mathscr{H}\otimes\mathscr{K})$.

Definition 1.2.17 (The Max Tensor Product)**.** The **maximal operator system structure**

on $\mathscr{S} \otimes \mathscr{T}$ is defined as the Archimedeanization of the following cones:

$$D_n^{\max} = \{a(P \otimes Q)a^* : P \in M_k(\mathscr{S})^+, Q \in M_m(\mathscr{T})^+, a \in M_{n,km}, k, m \in \mathbb{N}\}.$$

We denote the Archimedeanization of $D_n^{\max}$ as $C_n^{\max}$, then the **maximal tensor product** of $\mathscr{S}$ and $\mathscr{T}$, denoted by $\mathscr{S} \otimes_{\max} \mathscr{T}$, is the operator system $(\mathscr{S} \otimes \mathscr{T}, (C_n^{\max})_{n=1}^{\infty}, 1 \otimes 1)$.

The max-tensor product is symmetric, associative and functorial. We will also see later that it is projective.

Definition 1.2.18 (The Commuting Tensor Product)**.** Let $\{\mathscr{S}, \mathscr{T}\}$ be operator systems. We set

$$\begin{aligned} \mathrm{CP}(\mathscr{S}, \mathscr{T}) = & \{(\phi, \psi) : \phi \text{ is CP from } \mathscr{S} \text{ to } B(\mathscr{H}), \\ & \psi \text{ is CP from } \mathscr{T} \text{ to } B(\mathscr{H}), \text{ and } \phi(\mathscr{S}) \text{ commutes with } \phi(\mathscr{T})\} \end{aligned}$$

We define $\phi \cdots \psi : \mathscr{S} \otimes \mathscr{T} \to B(\mathscr{H})$ as $\phi \cdot \psi(x \otimes y) = \phi(x)\psi(y)$.

The **commuting operator system structure** on $\mathscr{S} \otimes \mathscr{T}$ is defined as:

$$C_n^c = \{u \in M_n(\mathscr{S} \otimes \mathscr{T}) : (\phi \cdot \psi)^{(n)}(u) \geq 0, \text{ for all } (\phi, \psi) \in \mathrm{CP}(\mathscr{S}, \mathscr{T}).$$

We call the operator system $(\mathscr{S} \otimes \mathscr{T}, (C_n^c)_{n=1}^{\infty}, 1 \otimes 1)$ the **commuting tensor product** of $\mathscr{S}$ and $\mathscr{T}$ and denote it by $\mathscr{S} \otimes_c \mathscr{T}$.

The commuting tensor product is symmetric and functorial.

Given operator systems $\mathscr{S}$ and $\mathscr{T}$ and two possibly different operator system structures $\mathscr{S} \otimes_\alpha \mathscr{T}$ and $\mathscr{S} \otimes_\beta \mathscr{T}$ on their tensor product, we shall write $\mathscr{S} \otimes_\alpha \mathscr{T} = \mathscr{S} \otimes_\beta \mathscr{T}$ to mean that the identity map is a complete order isomorphism.

The tensor products of operator systems we will use in this paper are: min, max, c (See [12] for their definitions). The relationship between these tensor products is min $\leq c \leq$ max, that is, the identity maps $\mathrm{id} : \mathscr{S} \otimes_{\max} \mathscr{T} \to \mathscr{S} \otimes_c \mathscr{T}$, $\mathrm{id} : \mathscr{S} \otimes_c \mathscr{T} \to \mathscr{S} \otimes_{\min} \mathscr{T}$ are completely positive. Also, please note that the "=" signs in the following propositions and definitions all mean completely order isomorphic.

Definition 1.2.19. An operator system $\mathscr{S}$ is called **(min, max)-nuclear** if $\mathscr{S} \otimes_{\min} \mathscr{T} = \mathscr{S} \otimes_{\max} \mathscr{T}$, for every operator system $\mathscr{T}$.

Definition 1.2.20. An operator system $\mathscr{S}$ is called **C^*-nuclear** if $\mathscr{S} \otimes_{\min} \mathscr{A} = \mathscr{S} \otimes_{\max} \mathscr{A}$ for every unital C^*-algebra $\mathscr{A}$.

Proposition 1.2.21. *[10, Proposition 4.11] An operator system $\mathscr{S}$ is C^*-nuclear if and only if $\mathscr{S} \otimes_{\min} \mathscr{T} = \mathscr{S} \otimes_c \mathscr{T}$ for every operator system $\mathscr{T}$.*

Proposition 1.2.22. *[12, Corollary 4.10 and Theorem 5.12] Let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-algebras, then $\mathscr{A} \otimes_{\min} \mathscr{B} \subseteq_{\text{c.o.i}} \mathscr{A} \otimes_{C^*-\min} \mathscr{B}$ and $\mathscr{A} \otimes_{\max} \mathscr{B} \subseteq_{\text{c.o.i}} \mathscr{A} \otimes_{C^*-\max} \mathscr{B}$, where the $\otimes_{C^*-\min}$, $\otimes_{C^*-\max}$ denote the the tensor products in the C^*-algebra category.*

Proposition 1.2.23. *[12, Theorem 6.7] Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{S}$ be an operator system, then $\mathscr{S} \otimes_c \mathscr{A} = \mathscr{S} \otimes_{\max} \mathscr{A}$.*

Proposition 1.2.24. *[12, Theorem 4.6][Injectivity of the min tensor product] The min tensor product is injective in the sense that for every choices of four operator systems $\mathscr{S}$ and*

$\mathscr{T}$, $\mathscr{S}_1$, $\mathscr{T}_1$ with inclusions $\mathscr{S} \subseteq_{\text{c.o.i}} \mathscr{S}_1$ and $\mathscr{T} \subseteq_{\text{c.o.i}} \mathscr{T}_1$, we have that

$$\mathscr{S} \otimes_{\min} \mathscr{T} \subseteq_{\text{c.o.i}} \mathscr{S}_1 \otimes_{\min} \mathscr{T}_1.$$

Proposition 1.2.25. *[9, Proposition 1.6][Projectivity of the max tensor product] The max tensor product is projective in the following sense: Let $\mathscr{S}$, $\mathscr{T}$, $\mathscr{R}$ be operator systems and suppose $\psi : \mathscr{S} \to \mathscr{R}$ is a complete quotient map, then the map $\psi \otimes \mathrm{id}_{\mathscr{T}} : \mathscr{S} \otimes_{\max} \mathscr{T} \to \mathscr{R} \otimes_{\max} \mathscr{T}$ is also a complete quotient map.*

Chapter 2

The Operator System Generated by Cuntz Isometries

This chapter is based on [18].

2.1 Properties of the Generated by Cuntz Isometries

The readers can refer to [6] for the properties of the Cuntz algebra . In this paper, we will mainly use its universal property.

Let $S_1, \ldots, S_n$ be n $(2 \leq n < +\infty)$ Cuntz isometries that generate $\mathscr{O}_n$, I be the identity, and we define the operator system $\mathscr{S}_n$ as:

$$\mathscr{S}_n := \operatorname{span}\{I, S_i, S_i^* : 1 \leq i \leq n\}.$$

Similarly, we denote $\mathscr{S}_\infty$ as the operator system generated by the isometries that generate $\mathscr{O}_\infty$. Also, for $n = 1$, we let $\mathscr{S}_1$ be the three-dimensional operator system generated by the a universal unitary (for example, $z \in C(\mathbb{T})$).

Also, let $\hat{S}_1, \cdots, \hat{S}_n$ be n ($1 \leq n < +\infty$) Toeplitz-Cuntz isometries (that is, $\hat{S}_i^* \hat{S}_j = 0$ if $i \neq j$ and $\sum_{i=1}^n \hat{S}_i \hat{S}_i^* < I$), and we set

$$\hat{\mathscr{S}_n} = \operatorname{span}\{I, \hat{S}_1, \ldots, \hat{S}_n, \hat{S}_1^*, \ldots, \hat{S}_n^*\}.$$

Remark 2.1.1. By the uniqueness of the Cuntz algebra, we know that if $\mathscr{S}_n$ and $\mathscr{T}_n$ are two operator systems generated by n Cuntz isometries, then $\mathscr{S}_n$ is unitally completely order isomorphic to $\mathscr{T}_n$, that is, any n isometries satisfying the Cuntz relation give rise to the same $\mathscr{S}_n$. Similarly, the universal property of Toeplitz-Cuntz algebra also implies that any n isometries satisfying the Toeplitz-Cuntz relation give rise to the same $\hat{\mathscr{S}_n}$. Conversely, it is not hard to show by using Choi's multiplicative domain techniques that if $\mathscr{S}_n$ is unitally completely order isomorphic to $\mathscr{T}_n$, then $C^*(\mathscr{S}_n) \cong C^*(\mathscr{T}_n)$, which would lead to a new proof of the uniqueness of the Cuntz algebra. This reveals that we may prove some properties of the Cuntz algebra via $\mathscr{S}_n$. It is also interesting to explore how many nice properties $\mathscr{S}_n$ can inherit from $\mathscr{O}_n$.

Lemma 2.1.2. *Suppose $T_1, \ldots, T_n$ are n ($2 \leq n \leq +\infty$) isometries on a Hilbert space $\mathscr{H}$ with $\sum_{i=1}^n T_i T_i^* < I_{\mathscr{H}}$, then they can be dilated to n isometries $\tilde{T}_1, \ldots, \tilde{T}_n$ on some Hilbert space $\mathscr{K}$ with $\sum_{i=1}^n \tilde{T}_i \tilde{T}_i^* = I_{\mathscr{K}}$.*

Proof. Let $M = \operatorname{Ran}(\sum_{i=1}^n T_i T_i^*)$ and hence $M^\perp = \operatorname{Ran}(I_{\mathscr{H}} - \sum_{i=1}^n T_i T_i^*)$. Note that since T_i's are isometries, $\dim M^\perp \leq \dim M$. Let $P_{M\perp}$ be the projection onto $M^\perp$ and we define operators $X_i : \mathscr{H} \to \mathscr{H}$ as the following:

$$X_i = \begin{cases} P_{M^\perp}, & i = 1 \\ 0, & 2 \le i \le n \end{cases}.$$

Correspondingly, we choose n operators $Y_i : \mathscr{H} \to \mathscr{H}$ where Y_1 is a partial isometry with initial space M and Y_i $(2 \le i \le n)$ are isometries such that $\sum_{i=1}^n Y_i Y_i^* = I_{\mathscr{H}}$.

Next, let $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}$, we define $\tilde{T}_i : \mathscr{K} \to \mathscr{K}$ by

$$\tilde{T}_i = \begin{pmatrix} T_i & X_i \\ 0 & Y_i \end{pmatrix},$$

and it is easy to check that

$$\tilde{T}_i^* \tilde{T}_i = \begin{pmatrix} T_i^* T_i & T_i^* X_i \\ X_i^* T_i & X_i^* X_i + Y_i^* Y_i \end{pmatrix} = \begin{pmatrix} I_{\mathscr{H}} & 0 \\ 0 & I_{\mathscr{H}} \end{pmatrix}$$

$$\sum_{i=1}^n \tilde{T}_i \tilde{T}_i^* = \sum_{i=1}^n \begin{pmatrix} T_i T_i^* + X_i X_i^* & X_i Y_i^* \\ Y_i X_i^* & Y_i Y_i^* \end{pmatrix} = \begin{pmatrix} I_{\mathscr{H}} & 0 \\ 0 & I_{\mathscr{H}} \end{pmatrix}.$$

Hence, $\tilde{T}_1, \ldots, \tilde{T}_n$ are the desired dilations. □

Corollary 2.1.3. *The operator system $\hat{\mathscr{S}}_n$ is unitally completely isomorphic to $\mathscr{S}_n$ in the canonical way, that is, there exists a unital complete order isomorphism $\phi : \hat{\mathscr{S}}_n \to \mathscr{S}_n$ such that $\phi(\hat{S}_i) = S_i$.*

Proof. Let S_i's $(1 \le i \le n)$ be Cuntz isometries and $\hat{S}_i$'s $(1 \le i \le n)$ both in $B(\mathscr{H})$, then it is easy to see that $S_i \oplus \hat{S}_i$'s on $\mathscr{H} \oplus \mathscr{H}$ are Toeplitz-Cuntz isometries which dilate S_i's. So the corollary follows by the fact that compressions are completely positive. □

Corollary 2.1.4. *We have that $\mathscr{S}_n \subseteq_{\text{c.o.i}} \mathscr{S}_m$ via the natural embedding, for $1 \le n < m \le +\infty$.*

Corollary 2.1.5. *Let $1 \le n \le +\infty$, $1 \le i \le n$, and define $\rho_i : \mathscr{S}_n \to \mathscr{S}_n$, $X \mapsto S_i^* X S_i$, then we have that ρ_i is a completely positive projection whose range is completely order isomorphic to $\mathscr{S}_1$.*

Proof. Clearly ρ_i is completely positive. Due to the Cuntz relation, it is easily seen that $\rho_i \circ \rho_i = \rho_i$, and $\operatorname{Ran}\rho_i = \operatorname{span}\{I, S_i, S_i^*\}$. However, S_i is a single Toeplitz-Cuntz isometry and hence $\operatorname{span}\{I, S_i, S_i^*\} = \mathscr{S}_1$ completely order isomorphically by the above corollary. □

Definition 2.1.6. The n-tuple of operators $(A_1, \dots, A_n)$ is called a **row contraction** if $\sum_{i=1}^n A_i A_i^* \le I$,where I is the identity operator.

Bunce proved in [3] that any family of n operators $\{A_i\}_{i=1}^n$ with $\sum_{i=1}^n A_i^* A_i \le I$ can be dilated to n coisometries with orthogonal initial spaces. Here, we rephrase that proposition for isometric dilation of row contractions.

Proposition 2.1.7. *Let $(A_1, \dots, A_n)$ be a row contraction on some Hilbert space $\mathscr{H}$, then there exists isometries $W_1, \dots, W_n$ with $W_i^* W_j = 0$ if $i \ne j$, such that $P_{\mathscr{H}} W_i|_{\mathscr{H}} = A_i$.*

The above proposition implies the following universal property of $\mathscr{S}_n$:

Theorem 2.1.8. *The operator system $\mathscr{S}_n$ has the following universal property:*

Let $(A_1, \cdots, A_n)$ be a row contraction on some Hilbert space $\mathscr{H}$ and denote

$$\mathscr{T}_n = \text{span}\{I_{\mathscr{H}}, A_i, A_i^* : 1 \le i \le n\}$$

so that $\mathscr{T}_n$ is an operator system, then there exists a unital completely positive map $\phi : \mathscr{S}_n \to \mathscr{T}_n$ such that $\phi(S_i) = A_i$.

Proof. By Proposition 2.1.7, we can dilate $A_1, \cdots, A_n$ to $W_1, \ldots, W_n$ with orthogonal ranges, which implies $\sum_{i=1}^n W_i W_i^* \le I$. Let $W_n = \text{span}\{I, W_i, W_i^* : 1 \le i \le n\}$, then the universal property of the Teoplitz-Cuntz algebra implies that there exists a unital completely positive map from $\hat{\mathscr{S}}_n$ to W_n which sends $\hat{S}_i$ to W_i. As compressions are always completely positive, we have a unital completely positive map from $\hat{\mathscr{S}}_n$ to $\mathscr{T}_n$ mapping $\hat{S}_i$ to A_i. Now the conclusion follows from Remark 2.1.3. □

2.2 The Operator System $\mathscr{E}_n$ and its Quotient

We define an operator system $\mathscr{E}_n \subseteq M_{n+1} := M_{n+1}(\mathbb{C})$ as the following,

$$\mathscr{E}_n = \text{span}\{E_{00}, E_{0i}, E_{i0}, \sum_{i=1}^n E_{ii} : 1 \le i \le n\},$$

where E_{ij}'s are matrix units in M_{n+1}. So every element in $\mathscr{E}_n$ is of the form,

$$\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & b & & \\ \vdots & & \ddots & \\ a_{n0} & & & b \end{pmatrix}.$$

By representing the Cuntz isometries $S_1, \ldots, S_n$ on some Hilbert space $\mathscr{H}$, we define an operator $R : \mathscr{H}^{(n+1)} \to \mathscr{H}$ by $R := (\frac{\sqrt{2}}{2}I, \frac{\sqrt{2}}{2}S_1^*, \ldots, \frac{\sqrt{2}}{2}S_n^*)$. So we know that

$$R^*R = \begin{pmatrix} \frac{1}{2}I & \frac{1}{2}S_1^* & \cdots & \frac{1}{2}S_n^* \\ \frac{1}{2}S_1 & & & \\ \vdots & & \left(\frac{1}{2}S_iS_j^*\right) & \\ \frac{1}{2}S_n & & & \end{pmatrix},$$

is positive in $M_{n+1}(B(\mathscr{H}))$.

Now, we can define a map $\psi : M_{n+1} \to B(\mathscr{H})$ by

$$\psi(E_{ij}) = (R^*R)_{ij},$$

where $(R^*R)_{ij}$ denotes the i, j-th entry of R^*R, and extend it linearly to M_{n+1}. It is straightforward that ϕ is unital. Then we know that ψ is unitally completely positive by a theorem of Choi (see [14, Theorem 3.14]).

Next, we can easily calculate that $\ker \psi = \operatorname{span}\{E_{00} - \sum_{i=1}^{n} E_{ii}\}$, which will be denoted as J throughout the rest of the paper.

Proposition 2.2.1. *[10] Let J be a finite-dimensional $*$-subspace in a operator system $\mathscr{S}$ which contains no positive elements other than* 0*, then it is a completely order proximinal kernel.*

Lemma 2.2.2. *The kernel J is completely order proximinal.*

Proof. According to Proposition 2.2.1, we just need to show that J contains no positive elements other than 0. This is clear by considering the first and second diagonal entries of any nonzero element in J. Hence, J is completely order proximinal. □

Now, let $\phi = \psi|_{\mathscr{E}_n}$, then $\phi : \mathscr{E}_n \to \mathscr{S}_n$ is unitally completely positive whose kernel is also J. By Proposition 1.2.9 we can form the operator system quotient $\mathscr{E}_n/J$, and the induced map

$$\tilde{\phi} : \mathscr{E}_n/J \to \mathscr{S}_n, \quad x + J \mapsto \phi(x)$$

is unitally completely positive and bijective.

2.3 A Characterization of Positive Elements in $M_p(\mathscr{S}_n)$

We now state the main result of this section:

Theorem 2.3.1. *We have that $\mathscr{E}_n/J$ is completely order isomorphic to $\mathscr{S}_n$ and hence ϕ is a complete quotient map, where $\mathscr{S}_n$ is the operator system generated by Cuntz isometries and* $J = \text{span}\{E_{00} - \sum_{i=1}^{n} E_{ii}\}$.

To prove the theorem, we need to show that $\tilde{\phi}^{-1}$ is also completely positive, which will imply that $\tilde{\phi}$ is a complete order isomorphism.

To this end, we first notice the following from the definition $\tilde{\psi}$:

$$\begin{aligned}\tilde{\phi}^{-1}(S_i) &= 2E_{i0} + J, \\ \tilde{\phi}^{-1}(I) &= I_{n+1} + J = 2E_{00} + J = 2\sum_{i=1}^{n} E_{ii} + J.\end{aligned}$$

We can embed $\mathscr{E}_n/J$ in $B(\mathscr{K})$ completely order isomorphically. Let the embedding be γ, and denote

$$T_i := \gamma(2E_{i0} + J), \quad I_{\mathscr{K}} := \gamma(I_{n+1} + J),$$

it is equivalent to show that the map

$$\hat{\phi} : \mathscr{S}_n \to B(\mathscr{K}), \quad S_i \mapsto T_i, \quad S_i^* \mapsto T_i^*, \quad I \mapsto I_{\mathscr{K}}$$

is completely positive.

The proof of the following lemma is quite similar to that of Lemma 3.1 in [14], so we omit the proof.

Lemma 2.3.2. *Let $\mathscr{H}, \mathscr{K}$ be Hilbert spaces and $T \in B(\mathscr{H}, \mathscr{K})$. Also, denote $I_{\mathscr{H}}$ and $I_{\mathscr{K}}$ as the identity operators on $\mathscr{H}$ and $\mathscr{K}$ respectively. Then we have $\|T\| \leq 1$ if and only if*

$$\begin{pmatrix} I_{\mathscr{H}} & T^* \\ T & I_{\mathscr{K}} \end{pmatrix}$$

is positive in $B(\mathscr{H} \oplus \mathscr{K})$.

Using this lemma, we can prove the following.

Proposition 2.3.3. *We have that $(T_1,\ldots,T_n)$ is a row contraction, i.e., $\sum_{i=1}^n T_iT_i^* \leq I_{\mathscr{K}}$.*

Proof. We represent T_i's on a Hilbert space $\mathscr{H}$ via some unital injective $*$-homomorphism, so $(T_1,\ldots,T_n) \in B(\mathscr{H}^{(n)},\mathscr{H})$. By the above lemma, equivalently, we show the following,

$$\begin{pmatrix} I_{\mathscr{K}} & T_1 & \cdots & T_n \\ T_1^* & I_{\mathscr{K}} & & \\ \vdots & & \ddots & \\ T_n^* & & & I_{\mathscr{K}} \end{pmatrix} \geq 0.$$

Notice that $E_{00}+\sum_{i=1}^n E_{ii} = I_{n+1}$ and $E_{00}+J = \sum_{i=1}^n E_{ii}+J$, so

$$E_{00}+J = \sum_{i=1}^n E_{ii}+J = \frac{1}{2}I_{n+1}+J.$$

Since the quotient map $q: M_{n+1} \to M_{n+1}/J$ is completely positive, we just need to show

$$\begin{pmatrix} 2\sum_{i=1}^n E_{ii} & 2E_{10} & \cdots & 2E_{n0} \\ 2E_{01} & 2E_{00} & & \\ \vdots & & \ddots & \\ 2E_{0n} & & & 2E_{00} \end{pmatrix} \geq 0.$$

To this end, we write this matrix as a sum of n matrices

$$\begin{pmatrix} \sum_{i=1}^n E_{ii} & E_{10} & \cdots & E_{n0} \\ E_{01} & E_{00} & & \\ \vdots & & \ddots & \\ E_{0n} & & & E_{00} \end{pmatrix} = \begin{pmatrix} E_{nn} & 0 & \cdots & 0 & E_{n0} \\ 0 & \ddots & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \ddots & 0 \\ E_{0n} & 0 & \cdots & 0 & E_{00} \end{pmatrix}$$

$$+ \begin{pmatrix} E_{n-1,n-1} & 0 & \cdots & 0 & E_{n-1,0} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & & & 0 & 0 & \vdots \\ E_{0,n-1} & 0 & \cdots & 0 & E_{00} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} +$$

$$\cdots + \begin{pmatrix} E_{11} & E_{10} & 0 & \cdots & 0 \\ E_{01} & E_{00} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

From the equation above, we can see that each summand on the right is positive, so the block matrix on the left is positive and the conclusion follows. □

Proof of Theorem 2.3.1. By Proposition 2.3.3 and Theorem 2.1.8, we know that there exists a unital completely positive map which sends S_i to T_i. However, this map is necessarily

$\hat{\phi}$. □

Theorem 2.3.1 implies the following corollary which gives a characterization for positive elements in $M_p(\mathscr{S}_n)$.

Corollary 2.3.4. *We have that $A_0 \otimes I + \sum_{i=1}^n A_i \otimes S_i + \sum_{i=1}^n A_i^* \otimes S_i^* \in M_p(\mathscr{S}_n)^+$ if and only if there exists $B \in M_p$ such that*

$$\begin{pmatrix} A_0 & 2A_1^* & \cdots & 2A_n^* \\ 2A_1 & A_0 & & \\ \vdots & & \ddots & \\ 2A_n & & & A_0 \end{pmatrix} + \begin{pmatrix} B & & & \\ & -B & & \\ & & \ddots & \\ & & & -B \end{pmatrix} \in M_{n+1}(M_p)^+$$

Proof. By Theorem 2.3.1, equivalently, we consider

$$M = A_0 \otimes (I_n + J) + \sum_{i=1}^n A_i \otimes (2E_{i0} + J) + \sum_{i=1}^n A_i^* \otimes (2E_{0i} + J) \in M_p(\mathscr{E}_n/J)^+.$$

If $A_k = (a_{ij}^k)_{i,j=1}^p$ for $0 \le k \le n$, then

$$M = \left(a_{ij}^0 (I_{n+1} + J) + \sum_{k=1}^n a_{ij}^k (2E_{k0} + J) + \sum_{k=1}^n \overline{a_{ji}^k} (2E_{0k} + J) \right)_{i,j=1}^p.$$

Lemma 2.2.2 together with definition of positivity in a quotient operator system imply that there exists $(J_{ij}) \in M_p(J)$ such that

$$M = (a_{ij}^0 I_{n+1} + \sum_{k=1}^n 2a_{ij}^k E_{k0} + \sum_{k=1}^n 2\overline{a_{ji}^k} E_{0k}) + (J_{ij}) \in M_p(\mathscr{E}_n)^+.$$

Now we let $J_{ij} = b_{ij}(E_{00} - \sum_{k=1}^{n} E_{kk})$ and M corresponds to

$$(a_{ij}^0) \otimes I_{n+1} + \sum_{k=1}^{n} 2(a_{ij}^k) \otimes E_{k0} + \sum_{k=1}^{n} 2(\overline{a_{ji}^k}) \otimes E_{0k} + (b_{ij}) \otimes (E_{00} - \sum_{k=1}^{n} E_{kk}) \in M_p(\mathscr{E}_n)^+.$$

Finally, using the isomorphism $M_p(\mathscr{E}_n) \cong \mathscr{E}_n(M_p)$, we know that M corresponds to the following positive block matrix in $M_{n+1}(M_p)$,

$$\begin{pmatrix} A_0 & 2A_1^* & \cdots & 2A_n^* \\ 2A_1 & A_0 & & \\ \vdots & & \ddots & \\ 2A_n & & & A_0 \end{pmatrix} + \begin{pmatrix} B & & & \\ & -B & & \\ & & \ddots & \\ & & & -B \end{pmatrix},$$

where $B = (b_{ij})$. Since isomorphism is used in each step, we know that our conclusion is in fact an "if and only if" statement. $\square$

Chapter 3

Tensor Products of the Operator System Generated by Cuntz Isometries

This chapter is based on [15].

3.1 Tensor Products and C^*-nuclearity of $\mathscr{S}_n$

We begin this section with the following proposition which shows that $\mathscr{S}_n$ $(2 \leq n \leq \infty)$ is not (min, max)-nuclear.

Proposition 3.1.1. *The operator system $\mathscr{S}_n$ is not (min, max)-nuclear, for $2 \leq n \leq \infty$.*

Proof. It is known in that the operator system $\mathfrak{S}_1$ generated by a universal unitary is not (min, max)-nuclear because of the following [8, Theorem 3.7]:

$$\mathfrak{S}_1 \otimes_{\min} \mathfrak{S}_1 \neq \mathfrak{S}_1 \otimes_{\max} \mathfrak{S}_1.$$

On the other hand, we have that $\mathscr{S}_1 = \mathfrak{S}_1$ [18, Corollary 3.3], so we know that

$$\mathscr{S}_1 \otimes_{\min} \mathscr{S}_1 \neq \mathscr{S}_1 \otimes_{\max} \mathscr{S}_1.$$

Now, for $n \geq 2$, if $\mathscr{S}_n \otimes_{\min} \mathscr{S}_n = \mathscr{S}_n \otimes_{\max} \mathscr{S}_n$, then [18, Corollary 3.4 and 3.5] imply that

$$\mathscr{S}_1 \otimes_{\min} \mathscr{S}_1 = \mathscr{S}_1 \otimes_{\max} \mathscr{S}_1,$$

which is a contradiction. Thus, $\mathscr{S}_n$ is not (min, max)-nuclear. □

Lemma 3.1.2. *For $2 \leq n \leq \infty$, we assume that $\hat{\mathscr{S}} \subseteq \mathscr{O}_n$ is an operator system containing $\mathscr{S}_n$ and $\mathscr{A}$ is a C^*-algebra. If we have that*

$$\hat{\mathscr{S}} \otimes_{\min} \mathscr{A} = \hat{\mathscr{S}} \otimes_{\max} \mathscr{A},$$

then

$$\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{O}_n \otimes_{\max} \mathscr{A}.$$

Proof. We first represent $\mathscr{O}_n \otimes_{\max} \mathscr{A}$ on some Hilbert space $\mathscr{H}$. By the definition of the max tensor product of operator systems, the canonical embedding map from $\hat{\mathscr{S}} \otimes_{\max} \mathscr{A}$ into $\mathscr{O}_n \otimes_{\max} \mathscr{A}$ is completely positive. Thus we have a completely positive map $\rho : \hat{\mathscr{S}} \otimes_{\min} \mathscr{A} \to B(\mathscr{H})$, such that $\rho(a \otimes b) = a \otimes b$ for each $a \in \hat{\mathscr{S}}$ and $b \in \mathscr{A}$.

The injectivity of the min tensor product of operator systems implies that $\hat{\mathscr{S}} \otimes_{\min} \mathscr{A} \subseteq_{\text{c.o.i}} \mathscr{O}_n \otimes_{\min} \mathscr{A}$, and we can extend ρ to a completely positive map $\tilde{\rho} : \mathscr{O}_n \otimes_{\min} \mathscr{A} \to B(\mathscr{H})$ by the Arveson's extension theorem.

Next, we use the Stinespring's dilation and obtain a unital $*$-homomorphism $\gamma\colon \mathscr{O}_n \otimes_{\min} \mathscr{A} \to B(\mathscr{K})$ and $V : \mathscr{H} \to \mathscr{K}$ for some Hilbert space $\mathscr{K}$ such that

$$\tilde{\rho}(a) = V^*\gamma(a)V, \text{ for each } a \in \mathscr{O}_n \otimes_{\min} \mathscr{A}.$$

The map ρ being unital implies that $\tilde{\rho}$ is unital and hence $V^*V = I_{\mathscr{H}}$, i.e. V is an isometry. By identifying $\mathscr{H}$ with $V\mathscr{H}$, we can assume that $\mathscr{H} \subseteq \mathscr{K}$.

Now, if we decompose $\mathscr{K} = \mathscr{H} + \mathscr{H}^{\perp}$, then $\tilde{\rho}$ is the $1-1$ corner of γ. Further, we have that

$$\gamma(S_i \otimes 1_{\mathscr{A}}) = \begin{pmatrix} \tilde{\rho}(S_i \otimes 1_{\mathscr{A}}) & C_i \\ B_i & D_i \end{pmatrix}, \quad \text{for every } i \in \{1, \dots, n\}$$

Here, $B_i \in B(\mathscr{H}, \mathscr{H}^{\perp})$, $C_i \in B(\mathscr{H}^{\perp}, \mathscr{H})$, $D_i \in B(\mathscr{H}^{\perp}, \mathscr{H}^{\perp})$. Since $S_i \otimes 1_{\mathscr{A}}$ is an isometry, it follows that $\gamma(S_i \otimes 1_{\mathscr{A}})$ and $\tilde{\rho}(S_i \otimes 1_{\mathscr{A}})$ are isometries, and we immediately have that $B_i = 0$.

Moreover, the condition that $\sum_{i=1}^{n} S_iS_i^* = I_{\mathscr{H}}$ (n finite) or $\sum_{i=1}^{k} S_iS_i^* \leq I_{\mathscr{H}}$ for every $1 \leq k < \infty$ (n infinite) implies that

$$\sum_{i=1}^{n} \gamma(S_i \otimes 1_{\mathscr{A}})\gamma(S_i \otimes 1_{\mathscr{A}})^* = \gamma(\sum_{i=1}^{n} S_iS_i^* \otimes 1_{\mathscr{A}}) = 1_{\mathscr{K}},$$

or

$$\sum_{i=1}^{k}\gamma(S_i\otimes 1_{\mathscr{A}})\gamma(S_i\otimes 1_{\mathscr{A}})^* = \gamma(\sum_{i=1}^{k} S_iS_i^*\otimes 1_{\mathscr{A}}) \leq 1_{\mathscr{K}}, \quad \text{for every } 1\leq k<\infty,$$

which means that

$$\begin{pmatrix} \sum_{i=1}^{n}\tilde{\rho}(S_i\otimes 1_{\mathscr{A}})\tilde{\rho}(S_i\otimes 1_{\mathscr{A}})^* + C_iC_i^* & \sum_{i=1}^{n} C_iD_i^* \\ \sum_{i=1}^{n} C_iC_i^* & \sum_{i=1}^{n} D_iD_i^* \end{pmatrix} = 1_{\mathscr{K}},$$

or

$$\begin{pmatrix} \sum_{i=1}^{k}\tilde{\rho}(S_i\otimes 1_{\mathscr{A}})\tilde{\rho}(S_i\otimes 1_{\mathscr{A}})^* + C_iC_i^* & \sum_{i=1}^{k} C_iD_i^* \\ \sum_{i=1}^{k} C_iC_i^* & \sum_{i=1}^{k} D_iD_i^* \end{pmatrix} \leq 1_{\mathscr{K}}, \quad \text{for every } 1\leq k<\infty.$$

Thus, we have that $C_i = 0$ for every $i\in\{1,\ldots,n\}$ and hence,

$$\gamma(S_i\otimes 1_{\mathscr{A}}) = \begin{pmatrix} \tilde{\rho}(S_i\otimes 1_{\mathscr{A}}) & 0 \\ 0 & D_i \end{pmatrix}.$$

On the other hand, for each unitary $u\in\mathscr{A}$, similarly, we have that

$$\gamma(I_{\mathscr{H}}\otimes u) = \begin{pmatrix} \tilde{\rho}(I_{\mathscr{H}}\otimes u) & 0 \\ 0 & v \end{pmatrix},$$

where v is a unitary in $B(\mathscr{H}^{\perp})$.

Because $\mathscr{A}$ is spanned by its unitaries, and every $X\otimes z\in\mathscr{O}_n\otimes_{\min}\mathscr{A}$ can be written as $X\otimes z = (X\otimes 1_{\mathscr{A}})(1_{\mathscr{H}}\otimes z)$, we see that γ is diagonal on all elementary tensors. Then by continuity of γ, we know it is diagonal on $\mathscr{O}_n \otimes_{\min} \mathscr{A}$.

We now have that the compression of γ onto the $1,1$ corner is a $*$-homomorphism from $\mathscr{O}_n \otimes_{\min} \mathscr{A}$ to $B(\mathscr{H})$, and this compression is exactly $\tilde{\rho}$. Moreover, $\tilde{\rho}(\hat{\mathscr{S}} \otimes_{\min} \mathscr{A}) \subseteq \mathscr{O}_n \otimes_{\max} \mathscr{A}$. Then, $\tilde{\rho}$ being a $*$-homomorphism implies that $\tilde{\rho}(\mathscr{O}_n \odot_{\min} \mathscr{A}) \subseteq \mathscr{O}_n \otimes_{\max} \mathscr{A}$, where $\odot_{\min}$ denotes the algebraic tensor product of $\mathscr{O}_n$ with $\mathscr{A}$ endowed with the minimal tensor norm. The continuity of $\tilde{\rho}$ implies further that $\tilde{\rho}(\mathscr{O}_n \otimes_{\min} \mathscr{A}) \subseteq \mathscr{O}_n \otimes_{\max} \mathscr{A}$. Form this, we can conclude that $\tilde{\rho}(\mathscr{O}_n \otimes_{\min} \mathscr{A}) = \mathscr{O}_n \otimes_{\max} \mathscr{A}$, because by the way $\tilde{\rho}$ is defined, $\operatorname{Ran}\tilde{\rho}$ is dense in $\mathscr{O}_n \otimes_{\max} \mathscr{A}$.

Finally, $\tilde{\rho}(X \otimes z) = X \otimes z$ for every $X \otimes z \in \mathscr{O}_n \otimes_{\min} \mathscr{A}$ forces that the identity map from $\mathscr{O}_n \odot_{\min} \mathscr{A}$ to $\mathscr{O}_n \otimes_{\max} \mathscr{A}$ extends to a $*$-homomorphism from $\mathscr{O}_n \otimes_{\min} \mathscr{A}$ onto $\mathscr{O}_n \otimes_{\max} \mathscr{A}$. Thus, $\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{O}_n \otimes_{\max} \mathscr{A}$. □

Let $T_1, \ldots, T_n$ be the generators of the Toeplitz-Cuntz algebra $\mathscr{T}\mathscr{O}_n$ and $\mathscr{T}_n$ be the operator system generated by T_i's. By Corollary 3.3 in [18], we know that $\mathscr{T}_n = \mathscr{S}_n$ via the natural isomorphism.

Theorem 3.1.3. *Let $\mathscr{A}$ be a unital C^*-algebra, then we have that*

$$\mathscr{S}_n \otimes_{\max} \mathscr{A} \subseteq_{\text{c.o.i}} \mathscr{T}\mathscr{O}_n \otimes_{\max} \mathscr{A},$$

where $\mathscr{T}\mathscr{O}_n$ is the Toeplitz-Cuntz algebra.

Before proving this theorem, we need the following refined version of Bunce's result [3, Proposition 1.].

Lemma 3.1.4. *Let $(A_1,\ldots,A_n) \in B(\mathscr{H})$ be a row contraction, then there exist isometric dilations $W_1,\ldots,W_n \in B(\mathscr{K})$ of $A_1,\ldots,A_n$ such that $W_i^*W_j = 0$ for $i \neq j$, where $\mathscr{K} = \mathscr{H} \oplus (\oplus_{k=1}^{\infty} \mathscr{H}^{(n)})$. Moreover, W_i can be chosen as the following form,*

$$W_i = \begin{pmatrix} A_i & 0 \\ X_i & YZ_i \end{pmatrix},$$

where the entries of X_i, Y and Z_i are all from $C^*(I,A_1,\ldots,A_n)$.

Proof. The fact that the entries of X_i and Y are from $C^*(I,A_1,\ldots,A_n)$ is directly from Bunce's construction. On the other hand, by his construction, Z_k's can be any set of Cuntz isometries on $\oplus_{k=1}^{\infty} \mathscr{H}^{(n)}$ so we can choose a particular one as:

$$Z_k = (Z_{ij}) = \begin{cases} I^{(n)} & \text{if } i = (j-1)n+k \\ 0 & \text{otherwise,} \end{cases}$$

where $I^{(n)}$ denotes the identity operator on $\oplus_{k=1}^{\infty} \mathscr{H}^{(n)}$. □

Proof of Theorem 3.1.3. By Proposition 1.2.23, we can show instead that $\mathscr{T}_n$ satisfies that

$$\mathscr{T}_n \otimes_c \mathscr{A} \subseteq_{\text{c.o.i}} \mathscr{T}\mathscr{O}_n \otimes_c \mathscr{A}.$$

To this end, it is enough to show that for any pair of unital completely positive maps $\varphi : \mathscr{T}_n \to B(\mathscr{H})$ and $\psi : \mathscr{A} \to B(\mathscr{H})$ with commuting ranges, there always exists an extension $\tilde{\varphi} : \mathscr{T}\mathscr{O}_n$ of φ such that the range of $\tilde{\varphi}$ and ψ commute.

Since φ is unitally completely positive, $(\varphi(T_1),\ldots,\varphi(T_n))$ is a row contraction and hence can be dilated to isometries $W_1,\ldots,W_n$ with orthogonal ranges, by Lemma 3.1.4. Then, there is a $*$-homomorphism $\pi : \mathscr{T}\mathscr{O}_n \to B(\mathscr{K})$ such that $\pi(T_i) = W_i$. Meanwhile, we set $\tilde{\psi} : \mathscr{R} \to B(\mathscr{K})$ as $\tilde{\psi} = \psi \oplus (\oplus_{k=1}^{\infty} \psi^{(n)})$, where $\psi^{(n)}$ denotes the direct sum of n copies of ψ.

It is easy to see that $\tilde{\psi}$ and π have commuting ranges. Clearly, $\psi = P_{\mathscr{H}}\tilde{\psi}|_{\mathscr{H}}$. Now, let $\tilde{\varphi} = P_{\mathscr{H}}\pi|_{\mathscr{H}}$, then it follows that $\tilde{\varphi}$ is a unital completely positive extension of φ and ψ and $\tilde{\varphi}$ has commuting ranges. Thus, we have shown that $\mathscr{T}_n \otimes_c \mathscr{A} \subseteq_{c.o.i} \mathscr{T}\mathscr{O}_n \otimes_c \mathscr{A}$. □

Corollary 3.1.5. *Let $\mathscr{A}$ be a unital C*-algebra. If $\mathscr{T}\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{T}\mathscr{O}_n \otimes_{\max} \mathscr{A}$, then $\mathscr{S}_n \otimes_{\min} \mathscr{A} = \mathscr{S}_n \otimes_{\max} \mathscr{A}$.*

Proof. By the assumption and the injectivity of the min-tensor product, we have the following relations:

$$\begin{array}{ccc} \mathscr{T}_n \otimes_{\min} \mathscr{A} & \subseteq_{\text{c.o.i}} & \mathscr{T}\mathscr{O}_n \otimes_{\min} \mathscr{A} \\ & & \| \\ \mathscr{T}_n \otimes_{c=\max} \mathscr{A} & \subseteq_{\text{c.o.i}} & \mathscr{T}\mathscr{O}_n \otimes_{c=\max} \mathscr{A}. \end{array}$$

This implies that $\mathscr{T}_n \otimes_{\min} \mathscr{A} = \mathscr{T}_n \otimes_{\max} \mathscr{A}$, so, equivalently, we know that $\mathscr{S}_n \otimes_{\min} \mathscr{A} = \mathscr{S}_n \otimes_{\max} \mathscr{A}$.

□

Corollary 3.1.6. *Let $\mathscr{A}$ be a unital C*-algebra. If $\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{O}_n \otimes_{\max} \mathscr{A}$, then $\mathscr{S}_n \otimes_{\min} \mathscr{A} = \mathscr{S}_n \otimes_{\max} \mathscr{A}$.*

Proof. Since $\mathscr{O}_n = \mathscr{T}\mathscr{O}_n/\mathbb{K}$ [6, Proposition 3.1] ($\mathbb{K}$ denotes the algebra of compact operators), we have the following commuting diagram:

$$\begin{array}{ccccc} \mathbb{K} \otimes_{\max} \mathscr{A} & \longrightarrow & \mathscr{T}\mathscr{O}_n \otimes_{\max} \mathscr{A} & \longrightarrow & \mathscr{O}_n \otimes_{\max} \mathscr{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{K} \otimes_{\min} \mathscr{A} & \longrightarrow & \mathscr{T}\mathscr{O}_n \otimes_{\min} \mathscr{A} & \longrightarrow & \mathscr{O}_n \otimes_{\min} \mathscr{A}. \end{array}$$

By assumption, we have that $\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{O}_n \otimes_{\max} \mathscr{A}$. Also, we know that $\mathbb{K}$ is nuclear, so $\mathbb{K} \otimes_{\min} \mathscr{A} = \mathbb{K} \otimes_{\max} \mathscr{A}$. This implies that the first and third vertical map in the above diagram are indeed isomorphisms. Hence, the second vertical map is also an isomorphism, that is, $\mathscr{T}\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{T}\mathscr{O}_n \otimes_{\max} \mathscr{A}$. Now the conclusion follows from the Corollary 3.1.6. □

Combining Corollary 3.1.6 with Lemma 3.1.2, we have

Theorem 3.1.7. *Let $\mathscr{A}$ be a unital C*-algebra. Then $\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{O}_n \otimes_{\max} \mathscr{A}$ if and only if $\mathscr{S}_n \otimes_{\min} \mathscr{A} = \mathscr{S}_n \otimes_{\max} \mathscr{A}$.*

Since $\mathscr{O}_n$ is nuclear, we immediately have the following corollary.

Corollary 3.1.8. *We have that $\mathscr{S}_n$ is C^*-nuclear.*

3.2 Equivalent Conditions of the C^*-Nuclearity of $\mathscr{S}_n$ and the Dual Row Contraction Version of Ando's Theorem

In this section, we prove some necessary and sufficient conditions for $\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{O}_n \otimes_{\max} \mathscr{A}$. We first recall that we denote $\mathscr{E}_n = \mathrm{span}\{E_{00}, \sum_{i=1}^n E_{ii}, E_{i0}, E_{0i} : 1 \le i \le n\}$, where E_{ij}'s are the matrix units in M_{n+1}, and we have proved in Section 2.3.1 that

Theorem 3.2.1. *The map $\phi : \mathscr{E}_n \to \mathscr{S}_n$ defined by the following:*

$$\phi(E_{i0}) = \frac{1}{2}S_i,\ \phi(E_{0i}) = \frac{1}{2}S_i^*,\ \phi(E_{00}) = \frac{1}{2}I,\ \phi(\sum_{i=1}^n E_{ii}) = \frac{1}{2}I, \quad 1 \le i \le n$$

is a complete quotient map, that is, $\mathscr{E}_n/J \cong \mathscr{S}_n$ completely order isomorphically, where $J := \mathrm{Ker}\,\phi = \mathrm{span}\{E_{00} - \sum_{i=1}^n E_{ii}\}$.

The next proposition shows that the operators system $\mathscr{E}_n$ is C^*-nuclear. Before proving it, let us recall a useful result [8, Lemma 1.7].

Lemma 3.2.2. *Let $\mathscr{S}$ and $\mathscr{T}$ be operator systems, then $u \in (\mathscr{S} \otimes_{\max} \mathscr{T})^+$ if and only if for each $\varepsilon > 0$, there exist $(P_{ij}^\varepsilon) \in M_{k_\varepsilon}(\mathscr{S})^+$, $(Q_{ij}^\varepsilon) \in M_{k_\varepsilon}(\mathscr{T})^+$, such that*

$$\varepsilon 1_{\mathscr{S}} \otimes 1_{\mathscr{T}} + u = \sum_{i,j=1}^k P_{ij}^\varepsilon \otimes Q_{ij}^\varepsilon.$$

Proposition 3.2.3. *We have that $\mathscr{E}_n \otimes_{\min} \mathscr{A} = \mathscr{E}_n \otimes_{\max} \mathscr{A}$ for every unital C^*-algebra $\mathscr{A}$.*

Proof. What we need to show is that $M_p(\mathscr{E}_n \otimes_{\min} \mathscr{A})^+ \subseteq M_p(\mathscr{E}_n \otimes_{\max} \mathscr{A})^+$, for each $p \in \mathbb{N}$. By the symmetry and associativity of the min and max tensor products of operator systems, and the nuclearity of M_p, we have that

$$M_p(\mathscr{E}_n \otimes_{\min} \mathscr{A}) = \mathscr{E}_n \otimes_{\min} M_p(\mathscr{A})$$
$$M_p(\mathscr{E}_n \otimes_{\max} \mathscr{A}) = \mathscr{E}_n \otimes_{\max} M_p(\mathscr{A}).$$

Notice that $M_p(\mathscr{A})$ is also a C^*-algebra, so it suffices to show that for each $A \in (\mathscr{E}_n \otimes_{\min} \mathscr{A})^+$, we have that $A \in (\mathscr{E}_n \otimes_{\max} \mathscr{A})^+$.

Since the min tensor product is injective, we have that $\mathscr{E}_n \otimes_{\min} \mathscr{A} \subseteq M_{n+1} \otimes_{\min} \mathscr{A} = M_{n+1}(\mathscr{A})$. So for $A \in (\mathscr{E} \otimes_{\min} \mathscr{A})^+$, we know that it has the form

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1^* & b & & \\ \vdots & & \ddots & \\ a_n^* & & & b \end{pmatrix}.$$

Without loss of generality, by considering $\varepsilon I_{n+1} \otimes 1_{\mathscr{A}} + A$, we can assume that a_0 and b are invertible. According to Cholesky's lemma, we have

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1^* & b & & \\ \vdots & & \ddots & \\ a_n^* & & & b \end{pmatrix} = \begin{pmatrix} a_0 - \sum_{i=1}^n a_i b^{-1} a_i^* & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix} + \begin{pmatrix} \sum_{i=1}^n a_i b^{-1} a_i^* & a_1 & \cdots & a_n \\ a_1^* & b & & \\ \vdots & & \ddots & \\ a_n^* & & & b \end{pmatrix}.$$

The first matrix on the right side is positive in $M_{n+1}(\mathscr{A})$ and is easily seen to be in $(\mathscr{E}_n \otimes_{\max} \mathscr{A})^+$. What we need is to show that the second matrix also lies in $(\mathscr{E}_n \otimes_{\max} \mathscr{A})^+$.

To this end, we use Proposition 3.2.2 and construct the following two matrices,

$$P = (P_{ij}) = \begin{pmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 1 & & & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} & & \\ \vdots & & \ddots & \\ \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} & & & \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} \end{pmatrix},$$

$$Q = (Q_{ij}) = \begin{pmatrix} b & a_1^* & \cdots & a_n^* \\ a_1 & & & \\ \vdots & & B & \\ a_n & & & \end{pmatrix},$$

where $B = (B_{ij}) = (a_i b^{-1} a_j^*)$.

Then it is not hard to check that $P \in M_{n+1}(\mathscr{E}_n)^+$ and $Q \in M_{n+1}(\mathscr{A})^+$. Also, we have that

$$\begin{pmatrix} \sum_{i=1}^n a_i b^{-1} a_i^* & a_1 & \cdots & a_n \\ a_1^* & b & & \\ \vdots & & \ddots & \\ a_n^* & & & b \end{pmatrix} = \sum_{i,j=1}^{n+1} P_{ij} \otimes Q_{ij}.$$

This shows that the matrix on the left side is in $(\mathscr{E}_n \otimes_{\max} \mathscr{A})^+$, and the lemma is proved. □

Lemma 3.2.4. *Let $\mathscr{A}$ be a unital C^*-algebra, then we have that* $\mathrm{id}: \mathscr{S}_n \otimes_{\min} \mathscr{A} \to \mathscr{S}_n \otimes_{\max} \mathscr{A}$ *is completely positive if and only if $\phi \otimes \mathrm{id}_{\mathscr{A}} : \mathscr{E}_n \otimes_{\min} \mathscr{A} \to \mathscr{S}_n \otimes_{\min} \mathscr{A}$ is a complete quotient map.*

Proof. We have the following diagram:

$$\begin{array}{ccc} \mathscr{E}_n \otimes_{\min} \mathscr{A} & \xrightarrow{\cong} & \mathscr{E}_n \otimes_{\max} \mathscr{A} \\ \downarrow{\scriptstyle \phi\otimes \mathrm{id}_{\mathscr{A}}} & & \downarrow{\scriptstyle \phi\otimes \mathrm{id}_{\mathscr{A}}} \\ \mathscr{S}_n \otimes_{\min} \mathscr{A} & \xrightarrow{\mathrm{id}} & \mathscr{S}_n \otimes_{\max} \mathscr{A} \end{array}.$$

By Proposition 1.2.25, we have that $\phi \otimes \mathrm{id}_{\mathscr{A}} : \mathscr{E}_n \otimes_{\max} \mathscr{A} \to \mathscr{S}_n \otimes_{\max} \mathscr{A}$ is a complete quotient map, and hence if $\phi \otimes \mathrm{id}_{\mathscr{A}} : \mathscr{E}_n \otimes_{\min} \mathscr{A} \to \mathscr{S}_n \otimes_{\min} \mathscr{A}$ is a complete quotient map, then every positive element A in $M_p(\mathscr{S}_n \otimes_{\min} \mathscr{A})$ has a positive pre-image in $M_p(\mathscr{E}_n \otimes_{\min} \mathscr{A})$, and therefore A is in $M_p(\mathscr{S}_n \otimes_{\max} \mathscr{A})$. So $\mathrm{id}: \mathscr{S}_n \otimes_{\min} \mathscr{A} \to \mathscr{S}_n \otimes_{\max} \mathscr{A}$ is completely positive. Conversely, if $\mathrm{id}: \mathscr{S}_n \otimes_{\min} \mathscr{A} \to \mathscr{S}_n \otimes_{\max} \mathscr{A}$ is completely positive, then $\mathscr{S}_n \otimes_{\min} \mathscr{A} = \mathscr{S}_n \otimes_{\max} \mathscr{A}$. Also, we have $\mathscr{E}_n \otimes_{\min} \mathscr{A} = \mathscr{E}_n \otimes_{\max} \mathscr{A}$. Thus, $\phi \otimes \mathrm{id}_{\mathscr{A}} : \mathscr{E}_n \otimes_{\max} \mathscr{A} \to \mathscr{S}_n \otimes_{\max} \mathscr{A}$ being a complete quotient map means that $\phi \otimes \mathrm{id}_{\mathscr{A}} : \mathscr{E}_n \otimes_{\min} \mathscr{A} \to \mathscr{S}_n \otimes_{\min} \mathscr{A}$ is a complete quotient map. □

The next theorem is now immediate:

Theorem 3.2.5. *We have that $\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{O}_n \otimes_{\max} \mathscr{A}$ if and only if $\phi \otimes id : \mathscr{E}_n \otimes_{\min} \mathscr{A} \to \mathscr{S}_n \otimes_{\min} \mathscr{A}$ is a complete quotient map, where $\mathscr{A}$ is any unital C*-algebra.*

We now prove a concrete condition on a unital C*-algebra that is equivalent to Theorem 3.2.5. Given an operator system $\mathscr{S}$ we will write $p >> 0$ provided that there exists $\varepsilon > 0$ such that $p - \varepsilon 1 \in \mathscr{S}^+$, and we set $\mathscr{S}^+_{-1} = \{p \in \mathscr{S} : p >> 0\}$.

The reason for this notation is that if $\mathscr{A}$ is a unital C*-algebra and $\psi : \mathscr{S} \to \mathscr{A}$ is a unital completely positive map, then $p \in \mathscr{S}^+_{-1}$ implies $\psi(p)$ is positive and invertible in $\mathscr{A}$. Moreover, $\mathscr{S}^+_{-1}$ is exactly the set of elements of $\mathscr{S}^+$ for which this is true for every unital completely positive map into a C*-algebra. Moreover, if $\psi : \mathscr{T} \to \mathscr{S}$ is a quotient map, then $p \in \mathscr{S}^+_{-1}$ if and only if it has a pre-image, i.e., $\psi(r) = p$ with $r \in \mathscr{T}^+_{-1}$ (see [7, Proposition 3.2]).

Theorem 3.2.6. *Let $\mathscr{A}$ be a unital C*-algebra, then $\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{O}_n \otimes_{\max} \mathscr{A}$ if and only if for all $p \in \mathbb{N}$, whenever $I \otimes 1 + \sum_{j=1}^n S_j \otimes a_j + \sum_{j=1}^n S_j^* \otimes a_j^* >> 0$ in $\mathscr{O}_n \otimes_{\min} M_p(\mathscr{A})$, there exists $a, b \in M_p(\mathscr{A})^+_{-1}$ with $a + b = 1$, such that*

$$\begin{bmatrix} a & a_1^* & \cdots & a_n^* \\ a_1 & b & & \\ \vdots & & \ddots & \\ a_n & & & b \end{bmatrix}$$

is in $M_{n+1}(M_p(\mathscr{A}))^+_{-1}$.

Proof. If $\mathscr{O}_n \otimes_{\min} \mathscr{A} = \mathscr{O}_n \otimes_{\max} \mathscr{A}$ then $q \otimes id : \mathscr{E}_n \otimes_{\min} M_p(\mathscr{A}) \to \mathscr{S}_n \otimes_{\min} M_p(\mathscr{A})$ is a quotient map. Hence, $I \otimes 1 + \sum_{j=1}^n S_j \otimes a_j + \sum_{j=1}^n S_j^* \otimes a_j^* \in q \otimes id\big((\mathscr{E}_n \otimes_{\min} M_p(\mathscr{A}))^+\big)$. Choosing any strictly positive element in the pre-image yields the conclusion.

Conversely, the lifting formula shows that every element of the form $I \otimes 1 + \sum_{j=1}^n S_j \otimes a_j + \sum_{j=1}^n S_j^* \otimes a_j^* >> 0$ has a positive pre-image in $\mathscr{E}_n \otimes_{\min} M_p(\mathscr{A})$.

Let $R = I \otimes r + \sum_{j=1}^n S_j \otimes a_j + \sum_{j=1}^n S_j^* \otimes a_j^*$ be an arbitrary element in $\mathscr{S}_n \otimes_{\min} M_p(\mathscr{A})$ and let $\varepsilon > 0$.

Then $T = I \otimes 1 + \sum_{j=1}^n S_j \otimes (r+\varepsilon 1)^{-1/2} a_j (r+\varepsilon)^{-1/2} + \sum_{j=1}^n S_j^* \otimes (r+\varepsilon 1)^{-1/2} a_j^* (r+\varepsilon 1)^{-1/2} >> 0$, and so by the hypothesis has a lifting. Pre- and post-multiplying the entries of that lifting by $(r+\varepsilon)^{1/2}$ gives a lifting of $R + \varepsilon(I \otimes 1)$. This proves that the mapping $q \otimes id : \mathscr{E}_n \otimes_{\min} M_p(\mathscr{A}) \to \mathscr{S}_n \otimes_{\min} M_p(\mathscr{A})$ is a quotient map, and since p was arbitrary, this map is a complete quotient map. □

Corollary 3.2.7. *The C*-algebra $\mathscr{O}_n$ is nuclear if and only if whenever $\mathscr{A}$ is a unital C*-algebra and $I \otimes 1 + \sum_{j=1}^n S_j \otimes a_j + \sum_{j=1}^n S_j^* \otimes a_j^* >> 0$ in $\mathscr{O}_n \otimes_{\min} \mathscr{A}$ there exists $a, b \in \mathscr{A}_{-1}^+$ with $a + b = 1$ such that*

$$\begin{bmatrix} a & a_1^* & \cdots & a_n^* \\ a_1 & b & & \\ \vdots & & \ddots & \\ a_n & & & b \end{bmatrix} \tag{3.2.1}$$

is in $M_{n+1}(\mathscr{A})_{-1}^+$.

Definition 3.2.8. Let $\mathscr{A}$ be a unital C^*-algebra, then an n-tuple $(a_1, \ldots, a_n)$ in $\mathscr{A}$ is called a **dual row contraction** if

$$I \otimes 1 + \sum_{j=1}^{n} S_j \otimes a_j^* + \sum_{j=1}^{n} S_j^* \otimes a_j \geq 0,$$

where the S_i's are Cuntz isometries. Moreover, it is called a **strict dual row contraction** if

$$I \otimes 1 + \sum_{j=1}^{n} S_j \otimes a_j^* + \sum_{j=1}^{n} S_j^* \otimes a_j >> 0.$$

Remark 3.2.9. Note that a dual row contraction is a row contraction, since

$$I \otimes 1 + \sum_{j=1}^{n} S_j \otimes a_j^* + \sum_{j=1}^{n} S_j^* \otimes a_j \geq 0$$

implies that

$$I \otimes 1 + z\sum_{j=1}^{n} S_j \otimes a_j^* + \bar{z}\sum_{j=1}^{n} S_j^* \otimes a_j \geq 0, \quad \text{for all } z \in \mathbb{T},$$

which is equivalent to

$$w(\sum_{j=1}^{n} S_j \otimes a_j^*) \leq \frac{1}{2},$$

where w means the numerical radius. So, we have that

$$\left\|\sum_{i=1}^{n} a_i a_i^*\right\| = \left\|\left(\sum_{i=1}^{n} S_i \otimes_{\min} a_i^*\right)^* \left(\sum_{i=1}^{n} S_i \otimes_{\min} a_i^*\right)\right\| \leq (2w(\sum_{j=1}^{n} S_j \otimes a_j^*))^2 \leq 1.$$

But not every row contraction is a dual row contraction. A counterexample can be easily constructed. In particular, n ($2 \leq n < \infty$) Cuntz isometries form a row contraction but not dual row contraction, since $\sum_{i=1}^{n} S_i \otimes S_i^*$ is a unitary whose spectrum is the whole unit circle.

Again, since $\mathscr{O}_n$ is nuclear, Corollary 3.2.7 is indeed a (strict) dual row contraction version of Ando's theorem (See [2] for the original version). Moreover, when $\mathscr{M}$ is a von Neumann algebra, we can replace "strict dual row contraction" by "dual row contraction" and "strictly positive" by "positive" by taking weak*-limits. We summarize these statements below. This result is a dual row contraction version of Ando's theorem on numerical radius.

Theorem 3.2.10. *Let $\mathscr{A}$ be a unital C*-algebra and $(a_1,\dots,a_n) \in \mathscr{A}$ be a strict dual row contraction, then there exists $a,b \in \mathscr{A}^+_{-1}$ with $a+b=1$ such that*

$$\begin{bmatrix} a & a_1 & \cdots & a_n \\ a_1^* & b & & \\ \vdots & & \ddots & \\ a_n^* & & & b \end{bmatrix}$$

is in $M_{n+1}(\mathscr{A})^+_{-1}$. Moreover, if $\mathscr{M}$ is a von Neumann algebra and $(a_1,\dots,a_n) \in \mathscr{M}$ is a dual row contraction, then there exists $a,b \in \mathscr{M}^+$ with $a+b=1$ such that

$$\begin{bmatrix} a & a_1 & \cdots & a_n \\ a_1^* & b & & \\ \vdots & & \ddots & \\ a_n^* & & & b \end{bmatrix}$$

is in $M_{n+1}(\mathscr{M})^+$.

3.3 An Alternative Proof of the Nuclearity of $\mathscr{O}_n$

We now give a new proof of the nuclearity of $\mathscr{O}_n$, by showing directly the existence of operators a, b mentioned in Corollary 3.2.7, which will prove that $\mathscr{O}_n$ is nuclear.

To this end, we shall need the notion of "shorted operators", which was introduced in [1]. Here, we briefly quote some results we will need in our proof.

Definition 3.3.1. [1] Let $\mathscr{H}$ be a Hilbert space and $A \in B(\mathscr{H})$ be positive. Assume $S \subseteq \mathscr{H}$ is a closed subspace, then the **shorted operator of** A with respect to S, denoted as $S(A)$ is defined as the maximum of the following set:

$$\{T \in B(\mathscr{H}) : 0 \leq T \leq A, \operatorname{Ran} T \subseteq S\}.$$

Also, we denote $S_0(A) = S(A)|_S$.

The shorted operator always exists [1, Theorem 1]. Moreover, we have that

Proposition 3.3.2. *[1] For each $x \in S$, we have that*

$$\langle S_0(A)x, x\rangle = \inf\left\{ \left\langle A \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle : y \in S^\perp \right\}.$$

We now prove that the condition of Corollary 3.2.7 is met for $n = 2$ without using the nuclearity of $\mathscr{O}_2$.

A proof of the nuclearity of $\mathscr{O}_2$. Let $\mathscr{A} \subseteq B(\mathscr{H})$ be a unital C^*-algebra and $(a_1, \ldots, a_n)$ be a strict dual row contraction in $\mathscr{A}$, that is,

$$A = I \otimes 1 + \sum_{j=1}^{2} S_j \otimes a_j^* + \sum_{j=1}^{2} S_j^* \otimes a_j >> 0, \text{ in } \mathscr{O}_2 \otimes_{\min} \mathscr{A}.$$

By Corollary 3.3 in [18] the operator system spanned by the Toeplitz-Cuntz isometries is completely order isomorphic to the operator system spanned by the Cuntz isometries. Thus, we can take the S_i's to be Toeplitz-Cuntz isometries. Moreover, it suffices to consider the following specific choice of Toeplitz-Cuntz isometries:

$$S_i \in B(l^2), \quad S_i(e_k) = e_{kn+i}, \quad k = 0, 1, 2, \dots, \quad i = 1, 2,$$

where $\{e_i : i = 0, 1, 2, \dots\}$ is the orthonormal basis of l^2.

We write $l^2 \otimes \mathscr{H} = \oplus_{i=0}^{+\infty} \mathscr{H}_i$, where $\mathscr{H}_i = \mathscr{H}$ for all i. Thus, A corresponds to the following operator in $B(l^2 \otimes \mathscr{H})$,

$$A = \begin{pmatrix} 1 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_1^* & 1 & 0 & a_1 & a_2 & 0 & 0 & 0 & 0 & \cdots \\ a_2^* & 0 & 1 & 0 & 0 & a_1 & a_2 & 0 & 0 & \cdots \\ 0 & a_1^* & 0 & 1 & 0 & 0 & 0 & a_1 & a_2 & \cdots \\ 0 & a_2^* & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_1^* & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_2^* & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_1^* & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & a_2^* & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Set $\mathscr{R}_k = \oplus_{i=k}^{+\infty} \mathscr{H}_i$. We then write A as the following block form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in B(\mathscr{H}_0)$, $A_{12} \in B(\mathscr{R}_1, \mathscr{H}_0)$, $A_{21} \in B(\mathscr{H}_0, \mathscr{R}_1)$, $A_{22} \in B(\mathscr{R}_1)$.

Now, let $B = \mathscr{H}_0(A)$, then by Proposition 3.3.2, we have that

$$\langle Bh_0, h_0 \rangle = \inf_{g \in \mathscr{R}_1} \left\{ \left\langle A \begin{pmatrix} h_0 \\ g \end{pmatrix}, \begin{pmatrix} h_0 \\ g \end{pmatrix} \right\rangle \right\}$$

$$= \inf_{h_1\in\mathscr{H}} \inf_{h_2\in\mathscr{H}} \inf_{z\in\mathscr{R}_3} \left\{ \left\langle A \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ z \end{pmatrix}, \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ z \end{pmatrix} \right\rangle \right\}$$

$$= \inf_{h_1\in\mathscr{H}} \inf_{h_2\in\mathscr{H}} \inf_{z\in\mathscr{R}_3} \Bigg\{ \langle h_0 + a_1h_1 + a_2h_2, h_0\rangle + \langle a_1^*h_0, h_1\rangle + \langle a_2^*h_0, h_2\rangle$$

$$+ \left\langle A_{22} \begin{pmatrix} h_1 \\ h_2 \\ z \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \\ z \end{pmatrix} \right\rangle \Bigg\}.$$

We claim that

$$\inf_{z\in\mathscr{R}_3} \left\{ \left\langle A_{22} \begin{pmatrix} h_1 \\ h_2 \\ z \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \\ z \end{pmatrix} \right\rangle \right\} = \langle Bh_1, h_1\rangle + \langle Bh_2, h_2\rangle.$$

Assuming this claim for the moment, we have

$$\langle Bh_0, h_0\rangle = \inf_{h_1\in\mathscr{H}} \inf_{h_2\in\mathscr{H}} \{\langle h_0, h_0\rangle + \langle a_1 f, h_0\rangle + \langle a_2h_2, h_0\rangle + \langle a_1^*h_0, h_1\rangle + \langle a_2^*h_0, h_2\rangle$$

$$+ \langle Bh_1, h_1\rangle + \langle Bh_2, h_2\rangle\}$$

$$= \inf_{h_1\in\mathscr{H}} \inf_{h_2\in\mathscr{H}} \left\{ \left\langle \begin{pmatrix} 1 & a_1 & a_2 \\ a_1^* & B & 0 \\ a_2^* & 0 & B \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} \right\rangle \right\}.$$

So we have that

$$\begin{pmatrix} 1-B & a_1 & a_2 \\ a_1^* & B & 0 \\ a_2^* & 0 & B \end{pmatrix} \geq 0.$$

To justify the claim, we write $\mathbb{N}$ as the disjoint union of $N_1 = \{1+2(2^k-1), 1+3(2^k-1) : k \geq 0\}$ and $N_2 = \{2+3(2^k-1).2+4(2^k-1) : k \geq 0\}$. Set $\mathscr{N}_k = \oplus_{i \in N_k} \mathscr{H}_i$ so that $\mathscr{R}_1 = \mathscr{N}_1 \oplus \mathscr{N}_2$. Observe that both of these subspaces are reducing for A_{22} and that with respect to the obvious identification of $\mathscr{N}_k \sim \oplus_{i=0}^{+\infty} \mathscr{H}_i$ we have that $A_{22} \sim A \oplus A$.

Hence,

$$\begin{aligned} \inf_{z \in \mathscr{R}_3} \left\langle A_{22} \begin{pmatrix} h_1 \\ h_2 \\ z \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \\ z \end{pmatrix} \right\rangle &= \inf_{z_1 \in \mathscr{N}_1 \ominus \mathscr{H}_1} \left\langle A_{22} \begin{pmatrix} h_1 \\ 0 \\ z_1 \end{pmatrix}, \begin{pmatrix} h_1 \\ 0 \\ z_1 \end{pmatrix} \right\rangle \\ &\quad + \inf_{z_2 \in \mathscr{N}_2 \ominus \mathscr{H}_2} \left\langle A_{22} \begin{pmatrix} 0 \\ h_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ h_2 \\ z_2 \end{pmatrix} \right\rangle \\ &= \inf_{z \in \mathscr{R}_1} \left\langle A \begin{pmatrix} h_1 \\ z \end{pmatrix}, \begin{pmatrix} h_1 \\ z \end{pmatrix} \right\rangle + \inf_{z \in \mathscr{R}_1} \left\langle A \begin{pmatrix} h_2 \\ z \end{pmatrix}, \begin{pmatrix} h_2 \\ z \end{pmatrix} \right\rangle \\ &= \langle Bh_1, h_1 \rangle + \langle Bh_2, h_2 \rangle. \end{aligned}$$

It remains to show that $B \in \mathscr{A}$. Since $-S_i$'s are also Cuntz isometries, we have that

$$I \otimes 1 - \sum_{j=1}^{2} S_j \otimes a_j^* - \sum_{j=1}^{2} S_j^* \otimes a_j >> 0.$$

It follows that $\|\sum_{j=1}^{2} S_j \otimes a_j^* - \sum_{j=1}^{2} S_j^* \otimes a_j\| < 1$, and therefore $\|1 - A_{22}\| < 1$. According to the proof of [1, Theorem 1], the shorted operator B has an explicit formula: $B = A_{11} - A_{12}A_{22}^{-1}A_{21}$. So what left for us to show is that all the entries of A_{22}^{-1} are in $\mathscr{A}$. To see this, we first use the Neumann series to write $A_{22}^{-1} = \sum_{n=0}^{\infty}(1 - A_{22})^n$. Since each row and column of $1 - A_{22}$ only has finitely many nonzero entries, we must have that the entries of $(1 - A_{22})^n$ are in $\mathscr{A}$ for each $n \in \mathbb{N}$. Since the Neumann series is norm convergent, we have that each entry of A_{22}^{-1} is in $\mathscr{A}$ and since A_{12} and A_{21} are only non-zero in finitely many entries, $B \in \mathscr{A}$.

Finally, we can repeat the above process for $A - \varepsilon 1 \otimes 1 >> 0$ and see that we can make both B and $1 - B$ strictly positive with

$$\begin{pmatrix} 1-B & a_1 & a_2 \\ a_1^* & B & 0 \\ a_2^* & 0 & B \end{pmatrix} >> 0.$$

□

The proof that $\mathscr{O}_n$ for $n \geq 3$ is nuclear can be done in a similar fashion. In the following we write down the details of the proof.

A proof of the nuclearity of $\mathscr{O}_n$. Let $\mathscr{A} \subseteq B(\mathscr{H})$ be a unital C^*-algebra and $(a_1, \ldots, a_n)$ be a strict dual row contraction in $\mathscr{A}$, that is,

$$A = I \otimes 1 + \sum_{j=1}^{n} S_j \otimes a_j^* + \sum_{j=1}^{n} S_j^* \otimes a_j >> 0, \text{ in } \mathcal{O}_n \otimes_{\min} \mathscr{A}.$$

Then, by Corollary 3.3 in [18], we can take S_i's as Toeplitz-Cuntz isometries. Moreover, it suffices to consider the following specific choice of Toeplitz-Cuntz isometries:

$$S_i \in B(l^2), \quad S_i(e_k) = e_{kn+i}, \quad k = 0, 1, 2, \dots, \quad i = 1, 2, \dots, n,$$

where $\{e_i : i = 0, 1, 2, \dots\}$ is an orthonormal basis of l^2. Thus, A corresponds to the following operator on $B(\mathscr{H}^{(\infty)})$,

$$A = \begin{pmatrix} 1 & a_1 & \cdots & a_n & & & & \\ a_1^* & 1 & & & a_1 & \cdots & a_n & \\ \vdots & & \ddots & & & & & \\ a_n^* & & & 1 & & & & \\ & a_1^* & & & 1 & & & \\ & \vdots & & & & \ddots & & \\ & a_n^* & & & & & 1 & \\ & & & & & & & \ddots \end{pmatrix}.$$

We then write A as the following block form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in B(\mathscr{H}), A_{12} \in B(\mathscr{H}^{(\infty)} \ominus \mathscr{H}, \mathscr{H}), A_{21} \in B(\mathscr{H}, \mathscr{H}^{(\infty)} \ominus \mathscr{H}), A_{22} \in B(\mathscr{H}^{(\infty)})$.

Now, choose $\delta > 0$ such that $A - \delta 1 \otimes 1 \geq 0$ and let $B = \mathscr{H}_0(A - \delta 1 \otimes 1)$, then by Proposition 3.3.2, we have that

$$\langle (B)h, h\rangle = \inf_{g \in \mathscr{H}^\infty \ominus \mathscr{H}} \left\{ \left\langle (A - \delta I) \begin{pmatrix} h \\ g \end{pmatrix}, \begin{pmatrix} h \\ g \end{pmatrix} \right\rangle \right\}$$

$$= \inf_{f_1 \in \mathscr{H}} \cdots \inf_{f_n \in \mathscr{H}} \inf_{z \in \mathscr{H}^\infty \ominus \mathscr{H}^{(n)}} \left\{ \left\langle (A - \delta I) \begin{pmatrix} h \\ f_1 \\ \vdots \\ f_n \\ z \end{pmatrix}, \begin{pmatrix} h \\ f_1 \\ \vdots \\ f_n \\ z \end{pmatrix} \right\rangle \right\}$$

$$= \inf_{f_1 \in \mathscr{H}} \cdots \inf_{f_n \in \mathscr{H}} \inf_{z \in \mathscr{H}^\infty \ominus \mathscr{H}^{(n)}} \left\{ \langle (1 - \delta)h, h\rangle + \left\langle A_{12} \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ z \end{pmatrix}, h \right\rangle \right.$$

$$\left. + \left\langle A_{21} h, \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ z \end{pmatrix} \right\rangle + \left\langle (A_{22} - \delta I) \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ z \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ z \end{pmatrix} \right\rangle \right\}$$

$$= \inf_{f_1\in\mathscr{H}} \cdots \inf_{f_n\in\mathscr{H}} \inf_{z\in\mathscr{H}^\infty\ominus\mathscr{H}^{(n)}} \left\{ \langle (1-\delta)h,h\rangle + \left\langle A_{12}\begin{pmatrix} f_1\\ \vdots \\ f_n \\ 0 \\ \vdots \end{pmatrix}, h\right\rangle \right.$$

$$\left. + \left\langle A_{21}h, \begin{pmatrix} f_1\\ \vdots \\ f_n \\ 0 \\ \vdots \end{pmatrix}\right\rangle + \left\langle (A_{22}-\delta I)\begin{pmatrix} f_1\\ \vdots \\ f_n \\ z \end{pmatrix}, \begin{pmatrix} f_1\\ \vdots \\ f_n \\ z \end{pmatrix}\right\rangle \right\}.$$

Next, we partition $\mathbb{N}\setminus\{0\}$ as $\mathbb{N}\setminus\{0\} = N_1\cup\cdots\cup N_n$, where

$$N_m = m+[n+(m-1)(n-1)]\frac{(n^k-1)}{n-1},\ldots,m+[n+m(n-1)]\frac{(n^k-1)}{n-1}, \quad k=0,1,2,\ldots.$$

Then, we write $A_{22}-\delta I = A_{22}^1+\cdots+A_{22}^n$, where

$$(A_{22}^m)_{ij} = \begin{cases} (A_{22}-\delta I)_{ij}, \text{ if } i,j\in N_m \\ 0, \text{else} \end{cases},$$

where $(A_{22}-\delta I)_{ij}$ denotes the i,j-th entry of $A_{22}-\delta I$, same for $(A_{22}^m)_{ij}$, $1\le m\le n$. It is not hard to see that A_{22}^m is unitarily equivalent to $A_{22}-\delta I$ for each m.

Decompose $\mathscr{H}^\infty = \oplus_{m=1}^n \mathscr{H}_m$ according to the partition $\mathbb{N}\setminus\{0\} = N_1\cup\cdots\cup N_n$, we have that

$$\langle Bh,h\rangle = \inf_{f_1\in\mathscr{H}}\cdots\inf_{f_n\in\mathscr{H}}\left\{\langle(1-\delta)h,h\rangle + \left\langle A_{12}\begin{pmatrix} f_1\\ \vdots\\ f_n\\ 0\\ \vdots\end{pmatrix},h\right\rangle\right.$$

$$+\left\langle A_{21}h,\begin{pmatrix} f_1\\ \vdots\\ f_n\\ 0\\ \vdots\end{pmatrix}\right\rangle + \inf_{z=z_1+\cdots z_n, z_i\in H_m}\left\{\left\langle A_{22}^1\begin{pmatrix} f_1\\ z\end{pmatrix},\begin{pmatrix} f_1\\ z\end{pmatrix}\right\rangle+\cdots\right.$$

$$\left.\left.+\left\langle A_{22}^n\begin{pmatrix} f_n\\ z\end{pmatrix},\begin{pmatrix} f_n\\ z\end{pmatrix}\right\rangle\right\}\right\}$$

$$= \inf_{f_1\in\mathscr{H}}\cdots\inf_{f_n\in\mathscr{H}}\left\{\langle(1-\delta)h,h\rangle + \left\langle A_{12}\begin{pmatrix} f_1\\ \vdots\\ f_n\\ 0\\ \vdots\end{pmatrix},h\right\rangle\right.$$

$$+\left\langle A_{21}h, \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ 0 \\ \vdots \end{pmatrix} \right\rangle + \inf_{z_1 \in H_1} \left\{ \left\langle A_{22}^1 \begin{pmatrix} f_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} f_1 \\ z_1 \end{pmatrix} \right\rangle \right\} + \cdots$$

$$+ \inf_{z_n \in H_n} \left\{ \left\langle A_{22}^n \begin{pmatrix} f_n \\ z_n \end{pmatrix}, \begin{pmatrix} f_n \\ z_n \end{pmatrix} \right\rangle \right\} \Bigg\}$$

$$= \inf_{f_1 \in \mathscr{H}} \cdots \inf_{f_n \in \mathscr{H}} \Bigg\{ \langle (1-\delta)h, h \rangle + \left\langle A_{12} \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ 0 \\ \vdots \end{pmatrix}, h \right\rangle$$

$$+\left\langle A_{21}h, \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ 0 \\ \vdots \end{pmatrix} \right\rangle + \inf_{z \in \mathscr{H}^\infty} \left\{ \left\langle (A-\delta I) \begin{pmatrix} f_1 \\ z \end{pmatrix}, \begin{pmatrix} f_1 \\ z \end{pmatrix} \right\rangle \right\} + \cdots$$

$$+ \inf_{z \in \mathscr{H}^\infty} \left\{ \left\langle (A-\delta I) \begin{pmatrix} f_n \\ z \end{pmatrix}, \begin{pmatrix} f_n \\ z \end{pmatrix} \right\rangle \right\} \Bigg\}$$

$$= \inf_{f_1\in\mathscr{H}} \cdots \inf_{f_n\in\mathscr{H}} \left\{ \langle (1-\delta)h,h\rangle + \left\langle A_{12}\begin{pmatrix} f_1 \\ \vdots \\ f_n \\ 0 \\ \vdots \end{pmatrix}, h\right\rangle + \left\langle A_{21}h, \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ 0 \\ \vdots \end{pmatrix}\right\rangle \right.$$

$$\left. + \langle Bf_1,f_1\rangle + \cdots + \langle Bf_n,f_n\rangle \right\}.$$

Thus,

$$\begin{pmatrix} 1-B-\delta 1 & a_1 & \cdots & a_n \\ a_1^* & B & & \\ \vdots & & \ddots & \\ a_n^* & & & B \end{pmatrix} \geq 0.$$

Then, by adding $\frac{\delta}{2}I$ to the above matrix, we have that

$$\begin{pmatrix} 1-B-\frac{\delta}{2}1 & a_1 & \cdots & a_n \\ a_1^* & B+\frac{\delta}{2}1 & & \\ \vdots & & \ddots & \\ a_n^* & & & B+\frac{\delta}{2}1 \end{pmatrix} >> 0.$$

In addition, we have that $1-B-\frac{\delta}{2}1, B+\frac{\delta}{2}1 >> 0$. □

Having an alternative proof of the nuclearity of $\mathscr{O}_n$ ($2 \leq n < \infty$), we can give an alternative proof of the nuclearity of $\mathscr{O}_\infty$ with just a little effort. Here is the proof.

Theorem 3.3.3. *The Cuntz algebra $\mathscr{O}_\infty$ is nuclear.*

Proof. Using Lemma 3.1.2, we just need to show that $\mathscr{S}_\infty$ is C^*-nuclear.

It suffices to show that $(\mathscr{S}_\infty \otimes_{\min} \mathscr{A})^+ = (\mathscr{S}_\infty \otimes_{\max} \mathscr{A})^+$ for every unital C^*-algebra $\mathscr{A}$.

To see this, we choose $A \in (\mathscr{S}_\infty \otimes_{\min} \mathscr{A})^+$, then A has the form,

$$A = I \otimes X + \sum_{i \in F} S_i \otimes X_i + \sum_{i \in F} S_i^* \otimes X_i^*, \quad X_i \in \mathscr{A},$$

where F is a finite subset of $\mathbb{N}$. So there exists $N \in \mathbb{N}$, such that $F \subseteq \{1, \ldots, N\}$. This means, by the injectivity the min tensor product, we have that $A \in (\mathscr{S}_N \otimes_{\min} \mathscr{A})^+$.

But we have just shown that $\mathscr{S}_n$ is C^*-nuclear for n finite, so we have that

$$A \in (\mathscr{S}_N \otimes_{\min} \mathscr{A})^+ = (\mathscr{S}_N \otimes_{\max} \mathscr{A})^+ \subseteq (\mathscr{S}_\infty \otimes_{\max} \mathscr{A})^+.$$

Thus, we know that $\mathscr{S}_\infty \otimes_{\min} \mathscr{A} = \mathscr{S}_\infty \otimes_{\max} \mathscr{A}$. □

3.4 The Dual Operator System of $\mathscr{S}_n$

In this section, we prove some properties of the dual operator system of $\mathscr{S}_n$, denoted by $\mathscr{S}_n^d$, which is the operator system consisting of all (bounded) linear functionals on $\mathscr{S}_n$. First, we choose a basis for $\mathscr{S}_n^d$ as the following,

$$\{\delta_0, \delta_i, \delta_i^* : 1 \leq i \leq n\},$$

where

$$\delta_0(I) := 1, \delta_0(S_i) := \delta_0(S_i^*) = 0, \quad \text{for all } i;$$
$$\delta_i(I) := 0, \delta_i(S_j) := \delta_{ij}, \delta_i(S_k^*) := 0, \quad \text{for all } k;$$
$$\delta_i^*(I) := 0, \delta_i^*(S_j^*) := \delta_{ij}, \delta_i(S_k) := 0, \quad \text{for all } k,$$

where δ_{ij} is the Kronecker delta notation. So we have $\mathscr{S}_n^d = \text{span}\{\delta_0, \delta_i, \delta_i^* : 1 \leq i \leq n\}$.

Then, we define an order structure on $\mathscr{S}_n^d$ by

$$(f_{ij}) \in M_p(\mathscr{S}_n^d)^+ \Longleftrightarrow (f_{ij}) : \mathscr{S}_n \to M_p \text{ is completely positive }.$$

It is a well-known result by Choi and Effros [5, Theorem 4.4] that with the order structure defined above, the dual space of a finite-dimensional operator system is again an operator system with an Archimedean order unit, and indeed, any strictly positive linear functional is an Archimedean order unit.

We claim that δ_0 is strictly positive. To see this suppose that $p \in \mathscr{S}_n^+$ with $\delta_0(p) = 0$. Then $p = \sum_{i=1}^n a_i S_i + \sum_{i=1}^n \overline{a_i} S_i^*$, Using the fact that, if S_i are Cuntz isometries, then $-S_i$ are also Cuntz isometries, we see that $-p \in \mathscr{S}_n^+$. Thus, $p = 0$.

Hence, $\mathscr{S}_n^d$ is an operator system with Archimedean order unit δ_0.

The following characterizes positive elements in $M_p(\mathscr{S}_n^d)$ of the form

$$I_p \otimes \delta_0 + \sum_{i=1}^n A_i \otimes \delta_i + \sum_{i=1}^n A_i^*.$$

Proposition 3.4.1. *An element $I_p \otimes \delta_0 + \sum_{i=1}^n A_i \otimes \delta_i + \sum_{i=1}^n A_i^* \otimes \delta_i^* \in M_p(\mathscr{S}_n^d)$ is positive if and only if $(A_1, \ldots, A_n)$ is a row contraction.*

Proof. Let $M = I_p \otimes \delta_0 + \sum_{i=1}^n A_i \otimes \delta_i + \sum_{i=1}^n A_i^* \otimes \delta_i^*$, and view M as a completely positive map from $\mathscr{S}_n$ to M_p, it satisfies $M(I) = I_p$, $M(S_i) = A_i$. Thus, since M us unitally completely positive, we have that $(A_1, \ldots, A_n)$ is a row contraction.
Conversely, if $(A_1, \ldots, A_n)$ is a row contraction, then there exists a unital completely positive map which sends S_i to A_i, S_i^* to A_i^*, by the universal property of $\mathscr{S}_n$. But this map is necessarily $M : \mathscr{S}_n \to M_p$, and this means $M \in M_p(\mathscr{S}_n^d)^+$. □

Proposition 3.4.2. *Let $\mathscr{A}$ be a unital C^*-algebra and $\phi : \mathscr{S}_n^d \to \mathscr{A}$ be a unital linear map. Then ϕ is completely positive if and only if ϕ is self-adjoint and*

$$w(A_1 \otimes \phi(\delta_1) + \cdots + A_n \otimes \phi(\delta_n)) \leq \frac{1}{2},$$

for each row contraction $(A_1, \ldots, A_n) \in M_p$, each $p \in \mathbb{N}$, where w denotes the numerical radius.

Proof. Suppose ϕ is unitally completely positive, then for

$$M = I_p \otimes \delta_0 + \sum_{i=1}^n A_i \otimes \delta_i + \sum_{i=1}^n A_i^* \otimes \delta_i^* \in M_p(\mathscr{S}_n^d)^+,$$

we must have

$$I_p \otimes \phi(M) = I_p \otimes I + \sum_{i=1}^n A_i \otimes \phi(\delta_i) + \sum_{i=1}^n A_i^* \otimes \phi(\delta_i^*)$$

$$= I_p \otimes I + \sum_{i=1}^{n} A_i \otimes \phi(\delta_i) + \sum_{i=1}^{n} A_i^* \otimes \phi(\delta_i)^* \geq 0.$$

By Proposition 3.4.1, M is positive if and only if $(A_1, \ldots, A_n)$ is a row contraction. Noting that $(zA_1, \ldots, zA_n)$ is also a row contraction, we then have that

$$I_p \otimes I + z\sum_{i=1}^{n} A_i \otimes \phi(\delta_i) + \bar{z}\sum_{i=1}^{n} A_i^* \otimes \phi(\delta_i)^* \geq 0,$$

which means that $w(\sum_{i=1}^{n} A_i \otimes \phi(\delta_i)) \leq \frac{1}{2}$, for each row contraction $(A_1, \ldots, A_n) \in M_p$ and each $p \in \mathbb{N}$.

Conversely, we suppose $w(\sum_{i=1}^{n} A_i \otimes \phi(\delta_i)) \leq \frac{1}{2}$, for each row contraction $(A_1, \ldots, A_n)$ in M_p and each $p \in \mathbb{N}$, and this implies that

$$I_p \otimes I + \sum_{i=1}^{n} A_i \otimes \phi(\delta_i) + \sum_{i=1}^{n} A_i^* \otimes \phi(\delta_i)^* \geq 0,$$

for each row contraction $(A_1, \ldots, A_n) \in M_p$ and each $p \in \mathbb{N}$.

Choose an arbitrary $N = B_0 \otimes \delta_0 + \sum_{i=1}^{n} B_i \otimes \delta_i + \sum_{i=1}^{n} B_i^* \otimes \delta_i^* \in M_p(\mathscr{S}_n^d)^+$, then for each $\varepsilon > 0$,

$$\varepsilon I_p \otimes \delta_0 + N = (\varepsilon I_p + B_0) \otimes \delta_0 + \sum_{i=1}^{n} B_i \otimes \delta_i + \sum_{i=1}^{n} B_i^* \otimes \delta_i^* \geq 0,$$

which implies

$$I_p \otimes \delta_0 + \sum_{i=1}^{n} (\varepsilon I_p + B_0)^{-\frac{1}{2}} B_i (\varepsilon I_p + B_0)^{-\frac{1}{2}} \otimes \delta_i + \sum_{i=1}^{n} (\varepsilon I_p + B_0)^{-\frac{1}{2}} B_i^* (\varepsilon I_p + B_0)^{-\frac{1}{2}} \otimes \delta_i^* \geq 0.$$

By Proposition 3.4.1, we have that $((\varepsilon I_p+B_0)^{-\frac{1}{2}}B_1(\varepsilon I_p+B_0)^{-\frac{1}{2}},\ldots,(\varepsilon I_p+B_0)^{-\frac{1}{2}}B_n(\varepsilon I_p+B_0)^{-\frac{1}{2}})$ is a row contraction, and therefore

$$I_p\otimes I+\sum_{i=1}^{n}(\varepsilon I_p+B_0)^{-\frac{1}{2}}B_i(\varepsilon I_p+B_0)^{-\frac{1}{2}}\otimes\phi(\delta_i) \\ +\sum_{i=1}^{n}(\varepsilon I_p+B_0)^{-\frac{1}{2}}B_i^*(\varepsilon I_p+B_0)^{-\frac{1}{2}}\otimes\phi(\delta_i)^*\geq 0.$$

Thus,

$$\phi(\varepsilon I_p\otimes\delta_0+N)\geq 0,\qquad\text{for each }\varepsilon>0.$$

So we have that $\phi(N)\geq 0$, and this completes the proof. □

Remark 3.4.3. Since compressions of row contractions are still row contractions, it follows that if

$$w(A_1\otimes\phi(\delta_1)+\cdots+A_n\otimes\phi(\delta_n))\leq\frac{1}{2},$$

for each row contraction $(A_1,\ldots,A_n)\in M_p$, each $p\in\mathbb{N}$, then for Cuntz isometries $S_1,\ldots,S_n$,

$$w(S_1\otimes\phi(\delta_1)+\cdots+S_n\otimes\phi(\delta_n))\leq\frac{1}{2},$$

where S_i's are Cuntz isometries and the tensor product is the minimal one so that $S_1\otimes a_1^*+\cdots+S_n\otimes a_n^*\in\mathscr{O}_n\otimes_{\min}\mathscr{A}$. Conversely, using the universal property of $\mathscr{S}_n$, we have that for each row contraction $(A_1,\ldots,A_n)\in M_p$, the map sending S_i to A_i, S_i^* to A_i^*, I to I_p is completely positive and hence

$$w(S_1\otimes\phi(\delta_1)+\cdots+S_n\otimes\phi(\delta_n))\leq\frac{1}{2}$$

implies that

$$w(A_1 \otimes \phi(\delta_1) + \cdots + A_n \otimes \phi(\delta_n)) \leq \frac{1}{2}.$$

On the other hand, we have that

$$w(S_1 \otimes \phi(\delta_1) + \cdots + S_n \otimes \phi(\delta_n)) \leq \frac{1}{2}$$

if and only if

$$I \otimes 1 + \sum_{j=1}^{n} S_j \otimes a_j^* + \sum_{j=1}^{n} S_j^* \otimes a_j \geq 0.$$

So we have proved the following corollary.

Corollary 3.4.4. *A unital linear map $\phi : \mathscr{S}_n^d \to \mathscr{A}$ is completely positive if and only if ϕ is self-adjoint and*

$$w(S_1 \otimes \phi(\delta_1) + \cdots + S_n \otimes \phi(\delta_n)) \leq \frac{1}{2},$$

where $S_1, \ldots, S_n$ are Cuntz isometries if and only if $(\phi(\delta_1)^*, \ldots, \phi(\delta_n)^*)$ is a dual row contraction.

In [17], the joint numerical radius for $n-$tuple of operators $(T_1, \ldots, T_n) \in B(\mathscr{H})$ is defined as:

$$w(T_1, \ldots, T_n) := \sup \left| \sum_{\alpha \in F_n^+} \sum_{j=1}^{n} \langle h_\alpha, T_j h_{g_j \alpha} \rangle \right|,$$

where F_n is the free group on n generators $g_1, \ldots, g_n$, and the supremum is taken over all families of vectors $\{h_\alpha\}_{\alpha \in F_n^+} \subseteq \mathscr{H}$ with $\sum_{\alpha \in F_n^+} \|h_\alpha\|^2 = 1$.

It was shown in the same paper that $w(T_1, \ldots, T_n) = w(S_1 \otimes T_1^* + \cdots + S_n \otimes T_n^*)$ [17, Corollary 1.2], where w on the right hand side is the numerical radius of an operator on $\mathscr{H}$

defined in the usual way. Thus, it is natural to extend the notion of joint numerical radii of n-tuples to the category of C^*-algebras.

Definition 3.4.5. Let $\mathscr{A}$ be a C^*-algebra. The **joint numerical radius** of $(a_1,\dots,a_n)$—an n-tuple in $\mathscr{A}$, is defined as:

$$w(a_1,\dots,a_n) := w(S_1\otimes a_1^* + \cdots + S_n \otimes a_n^*),$$

where S_i's are Cuntz isometries.

Remark 3.4.6. Let $\mathscr{A}$ be a C^*-algebra. Then the n-tuple $(a_1,\dots,a_n)\in\mathscr{A}$ is a dual row contraction if and only if

$$w(a_1,\dots,a_n) \le \frac{1}{2}.$$

Theorem 3.4.7. *Let $\mathscr{E}_n' = \operatorname{span}\{I_{n+1}, E_{i0}, E_{0i} : 1 \le i \le n\} \subseteq M_{n+1}$, then $\mathscr{S}_n^d$ is completely order isomorphic to $\mathscr{E}_n'$ via the map $\theta : \mathscr{S}_n^d \to \mathscr{E}_n'$, with $\theta(\delta_0) = I_{n+1}$, $\theta(\delta_i) = E_{0i}$, $\theta(\delta_i^*) = E_{i0}$, for $1 \le i \le n$.*

Proof. We first show that θ is completely positive. By Corollary 3.4.4 and Remark 3.4.6, we just need to show that for n Cuntz isometries $S_1,\dots,S_n$,

$$w\left(\begin{pmatrix} 0 & S_1 & \cdots & S_n \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}\right) \le \frac{1}{2},$$

which is equivalent to

$$\begin{pmatrix} I & zS_1 & \cdots & zS_n \\ \bar{z}S_1^* & I & & \\ \vdots & & \ddots & \\ \bar{z}S_n^* & & & I \end{pmatrix} \geq 0 \quad \text{for all } z \in \mathbb{T}$$

which clearly holds since $(zS_1, \ldots, zS_n)$ is row contraction.

Next, we show that θ^{-1} is also completely positive. Let $p \in \mathbb{N}$ and note that $M_p(\mathscr{E}_n') = \mathscr{E}_n'(M_p)$, we can write a positive element $A \in M_p(\mathscr{E}_n')$ as

$$\begin{pmatrix} A_0 & A_1 & \cdots & A_n \\ A_1^* & A_0 & & \\ \vdots & & \ddots & \\ A_n^* & & & A_0 \end{pmatrix},$$

where $A_i \in M_p$. Consider $\varepsilon I_p \otimes I_n + A$, where I_p denotes the identity matrix in M_p, for $\varepsilon > 0$, and let $B = (\varepsilon I_p + A_0)^{-\frac{1}{2}}$, we have that

$$\begin{pmatrix} I_p & BA_1B & \cdots & BA_nB \\ BA_1^*B & I_p & & \\ \vdots & & \ddots & \\ BA_n^*B & & & I_p \end{pmatrix} \geq 0.$$

This implies that $(BA_1B, \ldots, BA_nB)$ is a row contraction, and hence

$$(\theta^{-1})^{(p)}\left(\begin{pmatrix} I_p & BA_1B & \cdots & BA_nB \\ BA_1^*B & I_p & & \\ \vdots & & \ddots & \\ BA_n^*B & & & I_p \end{pmatrix}\right) = I_p \otimes \delta_0 + \sum_{i=1}^n BA_iB \otimes \delta_i$$

$$+ \sum_{i=1}^n BA_i^*B \otimes \delta_i^* \geq 0,$$

by Proposition 3.4.1. Thus,

$$(\varepsilon I_p + A_0) \otimes \delta_0 + \sum_{i=1}^n A_i \otimes \delta_i + \sum_{i=1}^n A_i^* \otimes \delta_i^* \geq 0, \quad \text{for all } \varepsilon > 0,$$

which is the same as $(\theta^{-1})^{(p)}(\varepsilon I_p \otimes I_n + A) \geq 0$, for all $\varepsilon > 0$. Since θ is unital, we know that $(\theta^{-1})^{(p)}(A) \geq 0$. Hence, θ^{-1} is also completely positive. □

We recall the following result of Kavruk:

Theorem 3.4.8. *[11, Theorem 4.1] Let $\mathscr{S}$ be a finite-dimensional operator system. Then $\mathscr{S}$ is C*-nuclear if and only if $\mathscr{S}^d$ is C*-nuclear.*

Corollary 3.4.9. *We have that $\mathscr{E}_n'$ is a C*-nuclear operator system.*

Proof. Since $\mathscr{S}_n$ is C*-nuclear, $\mathscr{S}_n^d$ is C*-nuclear by the above theorem. But $\mathscr{E}_n' = \mathscr{S}_n^d$ up to complete order isomorphism. □

Remark 3.4.10. The operator system $\mathscr{E}_n'$ seems more elementary to deal with and if we could show directly that $\mathscr{E}_n'$ is C*-nuclear, then that would imply by Kavruk's result that

$\mathscr{S}_n$ is C*-nuclear, which in turn would give another proof of the nuclearity of the Cuntz algebras. However, we have been unable to prove directly that $\mathscr{E}_n'$ is C^*-nuclear.

3.5 A Lifting Theorem for Joint Numerical Radius

The local lifting property of an operator system $\mathscr{S}$ is defined in [13]:

Definition 3.5.1. Let $\mathscr{S}$ be an operator system, $\mathscr{A}$ be a unital C^*-algebra, $I \lhd \mathscr{A}$ be an ideal, $q : \mathscr{A} \to \mathscr{A}/I$ be the quotient map and $\phi : \mathscr{S} \to \mathscr{A}/I$ be a unital completely positive map. We say ϕ **lifts locally**, if for every finite-dimensional operator system $\mathscr{S}_0 \subseteq \mathscr{S}$, there exists a completely positive map $\psi : \mathscr{S}_0 \to \mathscr{A}$ such that $q \circ \psi = \phi$. We say that $\mathscr{S}$ has the **operator system locally lifting property** (OSLLP) if for every C^*-algebra $\mathscr{A}$ and every ideal $I \subseteq \mathscr{A}$, every unital completely positive map $\phi : \mathscr{S} \to \mathscr{A}/I$ lifts locally.

Theorem 3.5.2. *[13] Let $\mathscr{S}$ be an operator system, then the following are equivalent:*

1. *$\mathscr{S}$ has the OSLLP;*
2. *$\mathscr{S} \otimes_{\min} B(\mathscr{H}) = \mathscr{S} \otimes_{\max} B(\mathscr{H})$.*

We have seen that the operator system $\mathscr{S}_n^d$ is C^*-nuclear (Theorem 3.4.8). In particular, we have that for a Hilbert space $\mathscr{H}$,

$$\mathscr{S}_n^d \otimes_{\min} B(\mathscr{H}) = \mathscr{S}_n^d \otimes_{\max} B(\mathscr{H}).$$

Thus, the operator system $\mathscr{S}_n^d$ has the lifting property (LP).

By using the LP of $\mathscr{S}_n^d$, we are able to derive the following result concerning the joint numerical radius.

Theorem 3.5.3. *Let $\mathscr{A}$ be a unital C^*-algebra and $J \lhd \mathscr{A}$ be an ideal. Suppose $T_1 + J, \dots, T_n + J \in \mathscr{A}/J$, then there exist $W_1, \dots, W_n \in \mathscr{A}$ with $W_i + J = T_i + J$ for each $1 \le i \le n$, such that $w(W_1, \dots, W_n) = w(T_1 + J, \dots, T_n + J)$.*

Proof. Suppose $w(T_1 + J, \dots, T_n + J) = K$. If $K = 0$, then clearly $T_i + J = 0$ for each $1 \le i \le n$. So we can choose $W_i = 0$ for every $1 \le i \le n$.

So we consider the case when $K > 0$. A little scaling shows that

$$w(\frac{T_1}{2K} + J, \dots, \frac{T_n}{2K} + J) = \frac{1}{2}.$$

So the linear map $\phi : \mathscr{S}_n^d \to \mathscr{A}/J$ defined by

$$\phi(\delta_0) = I + J, \quad \phi(\delta_i) = \frac{T_i^*}{2K} + J, \quad \phi(\delta_i^*) = \frac{T_i}{2K} + J$$

is unitally completely positive.

We have known that $\mathscr{S}_n^d$ has the LP, so there exists a unitally completely positive map $\hat{\phi} : \mathscr{S}_n^d \to \mathscr{A}$ such that $\pi \circ \hat{\phi} = \phi$, where π denotes the canonical map from $\mathscr{A}$ onto $\mathscr{A}/J$. Let $W_i^* = 2K\hat{\phi}(\delta_i)$, we have that $W_i^* + J = T_i + J$. Moreover, by proposition....., we know that $(\frac{W_1}{2K}, \dots, \frac{W_n}{2K})$ is a co-row contraction. Hence, we have that

$$w(W_1, \dots, W_n) \le K.$$

Now, to complete the proof, we need to show that $w(W_1,\dots,W_n)=K$. Suppose that

$$w(\frac{W_1}{2K},\dots,\frac{W_n}{2K})<\frac{1}{2}.$$

Then there exists an $\varepsilon>0$, such that

$$w(\frac{(1+\varepsilon)W_1}{2K},\dots,\frac{(1+\varepsilon)W_n}{2K})<\frac{1}{2}.$$

However, this implies that

$$I\otimes 1+\sum_{i=1}^{n}S_i\otimes\frac{(1+\varepsilon)W_i^*}{2K}+\sum_{i=1}^{n}S_i^*\otimes\frac{(1+\varepsilon)W_i}{2K}\geq 0,$$

in $\mathscr{S}_n\otimes_{\min}\mathscr{A}$. Since $\mathrm{id}\otimes\pi$ is completely positive, we further have that

$$I\otimes 1+J+\sum_{i=1}^{n}S_i\otimes\frac{(1+\varepsilon)T_i^*+J}{2K}+\sum_{i=1}^{n}S_i^*\otimes\frac{(1+\varepsilon)T_i+J}{2K}\geq 0.$$

It now follows that

$$w(T_1+J,\dots,T_n+J)\leq\frac{K}{1+\varepsilon},$$

which is a contradiction. □

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