CORRELATION MINIMIZING FRAMES

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Doctor of Philosophy

By

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CORRELATION MINIMIZING FRAMES

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Abstract

In this dissertation, we study the structure of correlation minimizing frames. A correlation minimizing (N,d)-frame is any uniform Parseval frame of N vectors in dimension, d, such that the largest absolute value of the inner products of any pair of vectors is as small as possible. We call this value the correlation constant. These frames are important as they are optimal for the 2-erasures problem.

We produce the actual correlation minimizing frames. To further study the structure of correlation minimizing frames, we obtain upper bounds on the correlation constant. In the real case, we find an upper bound on the correlation constant of a correlation minimizing (N,d)-frame. As a result, we prove the correlation constant goes to zero for fixed redundancy as the dimension and number of vectors increases proportionally by 2^k. When addressing the correlation constant for complex correlation minimizing (N,d)-frames, we consider circulant matrices which are also projections as the Grammian matrix of a uniform Parseval frame. We derive a relationship between these Grammian matrices and the Dirichelet kernel as well as the structure of quadratic residue. Utilizing these relationships, we obtain two upper bounds on the correlation constant. Furthermore, we investigate how the correlation constant behaves asymptotically in comparison to the Welch bound. In $L^2[0,1]$, the Laurent matrix is a projection defined by the Fourier transform of the characteristic function on an interval of fixed finite length in [0,1]. Considering the magnitude of the Fourier transform of the characteristic function on a set of sufficiently small size, we derive a bound on the correlation constant and construct a method to create a correlation constant that is arbitrarily small.

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CHAPTER 1

Background

1.1 History of Frames

1.1.1 Frames

A frame is a set of vectors in a Hilbert space that can be used to reconstruct each vector in the space from its inner products with the frame vectors. These inner products are generally called the *frame coefficients of the vector.* But unlike an orthonormal basis each vector may have infinitely many different representations in terms of its frame coefficients.

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [13] in 1952 to study some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [15] for signal processing. These ideas did not generate much interest outside of nonharmonic Fourier series and signal processing until the landmark paper of Daubechies, Grossmannn, and Meyer [11] in 1986, where they developed the class of tight frames for signal reconstruction. After this innovative work the theory of frames began to be widely studied. While orthonormal bases have been widely used for many applications, [12] [14] [27], it is the redundancy that makes frames useful in applications.

Today, frames play an important role in many applications in mathematics, science, and engineering. Some of these applications include internet coding [35], time-frequency analysis [19], speech and music processing [42][28], wireless communication [27], medical imaging [40][20], digital copy-write infringement [31], quantum computing [24], and many other areas. Applications often use Parseval or tight frames because these frames have the added advantage that each vector has one natural representation given by a simple formula involving its frame coefficients. If, in addition, the frame is equal-norm (or uniform) somewhat equal weight is given to each vector in the space. Examples of classes of uniform Parseval frames can be found in [7]. A special case, called a 2-uniform frame, is a uniform Parseval frame with the additional condition that the inner products between distinct pairs of frame vectors has equal modulus. These frames were discovered by Holmes and Paulsen [22] to be optimal for the 2 erasure problem.

1.1.2 Optimal frames for the 2-erasure problem

Suppose that we want to transmit the information contained in a d-tuple of numbers to a receiver over a noisy channel. Given a vector in a d-dimensional Hilbert space, if we use a frame with $N > d$ vectors then the information contained in this vector is now encoded in its N frame coefficients. When information is stored or encoded redundantly, so that it is spread among a large number of coefficients, then our chances of communicating the vector within some margin of error should be increased. The *erasure problem* assumes that the only types of errors that can occur is that each coefficient as it is transmitted is either lost entirely or received perfectly. If m of the coefficients are lost during transmission one wants the best possible reconstruction of the data, using "blind reconstruction". This means that one always reconstructs the same way, ignoring that m frame coefficients lost have been set to zero. The *erasures problem* asks: what is the best set of N vectors to encode the information so that when up to m coefficients are lost and the blind reconstruction formula is used, the most accurate reconstruction can occur, i.e., where the maximum over all input vectors of the error between the original and it are "best", or in some sense optimal, were studied in [22]. Frames that satisfied their optimality properties were called *m-erasure frames*.

In the case of the 1-erasure problem, only one coefficient is lost before reconstruction, it was proven by Cassaza-Kovacevic that uniform Parseval frames are the best, or optimal, choices of vectors to encode the data [6]. Thus, the 1-erasure frames are exactly the uniform Parseval frames.

The 2-erasure frames, introduced by Holmes and Paulsen [22], are the frames that are the optimal solution for the 2-erasure problem among all uniform Parseval frames, i.e., among all frames that are optimal for the 1-erasure problem. Holmes and Paulsen proved that the set of uniform Parseval frames for which the minimal angle between any pair of vectors is as large as possible are exactly the 2-erasure frames.

Furthermore, they identified that for some (N,d) pairs there exist frames that they called *2-uniform frames*, where the angle between each pair of distinct frame vectors is equal, and that when 2-uniform frames exist then they are exactly the set of 2-erasure frames. The 2-uniform frames are exactly the set of equiangular, uniform Parseval frames and this latter terminology is now much better known.

The problem with 2-uniform and equiangular tight frames is that for most pairs (N,d), they do not exist. For example, when $d = 3$ these are known to exist only for $N = 3, 4$, and 6. The lack of existence in most cases leads to the questions: For a given pair (N,d), what is the set of 2-erasure frames? Answering this question has also been addressed as minimizing the worst case coherence, finding Grassmannian equal-norm Parseval frames and finding frames with low coherence in [37] [38].

Heath and Strohmer in [35] studied frames from the aspect of coding theory. Consequently, they also considered the problem, for a given N and d, which unit norm tight frames satisfy the property that the largest magnitude of the inner products between two frame vectors is as small as possible. For given N and d, they called the unit norm frames that solved the problem Grassmannian frames. In the case when all of those magnitudes between frame vectors are equal, the magnitude equals the

Welch Bound [41] and the equiangular tight frame, was called optimal Grassmannian. The Grassmannian space $G(d, n)$ is the set of all n-dimensional subspaces of the space $\mathbb{R}^d$ or $\mathbb{C}^d$. The Grassmannian packing problem is the problem of finding the best packing of N n-dimensional subspaces in $\mathbb{R}^d$ or $\mathbb{C}^d$, such that the angle between any two of these subspaces becomes as large as possible. In the real case where n=1, the subspaces are real lines through the origin in $\mathbb{R}^d$ and the goal is to arrange N lines such that the angle between any two of the lines becomes as large as possible. Since maximizing the angle between lines is equivalent to minimizing the magnitude of the inner product of the unit vectors generating these lines, therefore finding optimal packings in $G(d, 1)$ is equivalent to finding finite Grassmannian frames, which is what motivated the name Grassmannian frames and optimal Grassmannian frames.

Applications of 2-uniform frames or equiangular tight frames occur in combinatorial design theory [35], digital fingerprinting codes [31], and many other areas. An illustration of one current application is in digital fingerprinting. Digital fingerprinting is a framework for marking media files, where user-specific signatures are used to deter illegal duplication and distribution. It is possible for multiple users to collude to produce a forgery that can potentially overcome a fingerprinting system. It was proposed, that an equiangular tight frame fingerprint design is robust to such attacks [31].

1.1.3 The Optimal Line Packing Problem

Optimal line packings were first researched by Hanntjes, in 1948, where he posed the problem of packing equiangular lines in real Euclidean space [21]. Then in 1973 this problem was analyzed by Lemmens and Seidel [26]. The optimal packings of N lines in $\mathbb{R}^3$ were studied by Conway, Hardin, and Sloan [9] for all values of $N \leq 55$. For some values of N, they were able to give closed form descriptions of these sets of lines, along with proofs that they were indeed optimal packings, while for many values of N, they were only able to give numerical approximations to these optimal packings.

Holmes and Paulsen [22] did numerical experiments that computed the approximate minimum angle between vectors for 2-erasure frames of N vectors in $\mathbb{R}^3$. Their computations showed that for some values of N the minimum angle between vectors appeared to be identical to the angle determined by [9] for optimal line packing results, up to the number of decimal places published. This lead them to conjecture that for these values of N, one could obtain a unit norm tight frame by choosing a unit vector from each line in the optimal line packing, and that, after scaling, the resulting uniform Parseval frame would be a 2-erasure frame.

1.2 Frame-Basics and Notation

Definition 1.2.1. Let $\mathcal{I}$ be a countable index set. A family $\mathcal{F} = \{f_i\}_{i\in\mathcal{I}}$ of elements in a (real or complex) Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there are constants

$0 < A \leq B < \infty$, called the *lower and upper frame bounds*, respectively so that for all $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{i \in \mathcal{I}} |\langle f, f_i \rangle|^2 \leq B\|f\|^2. \tag{1.1}$$

If A=B, then $\mathcal{F}$ is called a *tight frame* and when A=B=1, $\mathcal{F}$ is called a *Parseval frame.*

We denote the collection of all Parseval frames for a d dimensional Hilbert space consisting of N vectors as $\mathcal{F}(N, d)$. If $\mathcal{F} = \{f_1, ..., f_N\}$ is a Parseval frame for a d-dimensional Hilbert space, then we call $\mathcal{F}$ a (N, d)*-frame.*

We say $\mathcal{F} = \{f_i\}_{i \in I}$ is a *uniform (or equal-norm) frame* if its vectors are all the same length and *equiangular* if in addition there is a $c \geq 0$ where $|\langle f_j, f_k \rangle| = c$, for all $j \neq k$.

In general, a frame can have more vectors than the dimension of the Hilbert space and, in the case that the space is finite dimensional, we call

$$\frac{card(I)}{dim(\mathcal{H})}$$

the *redundancy* of the frame.

If $\mathcal{F} = \{f_i\}_{i \in I}$ is a frame for $\mathcal{H}$, the *analysis operator* is the bounded linear operator $V : \mathcal{H} \to \ell_2(I)$ given by $V(x)_i = \langle x, f_i \rangle$ for all $i \in I$. The *synthesis operator*, the adjoint of the analysis operator, $V^* : \ell_2(I) \to \mathcal{H}$ is defined by the formula, $V^*(e_i) = f_i$, where $\{e_i\}_{i \in \mathcal{I}}$ is the standard basis for $\mathcal{H}$. Therefore, the *frame operator* defined by,

$$Sx = V^*Vx = \sum_{i \in I} \langle x, f_i \rangle f_i,$$

is a positive, continuous invertible operator that satisfies $AI \leq S \leq BI$. The *canonical dual frame* is $H = \{h_i\}_{i \in \mathcal{I}}$, where $h_i = S^{-1} f_i$. If $\mathcal{F}$ has frame bounds A and B, the canonical dual frame bounds are $\frac{1}{B}$ and $\frac{1}{A}$.

In particular, $\mathcal{F}$ is a Parseval frame if and only if V is an isometry and this is if and only if $V^*V = I_{\mathcal{H}}$.

Thus, $\mathcal{F} = \{f_i\}_{i \in I}$ is a Parseval frame if and only if we have that

$$h = \sum_{i \in I} \langle h, f_i \rangle f_i, \, \forall h \in \mathcal{H}.$$

More generally, if $\mathcal{F} = \{f_i\}_{i \in I}$ is a tight frame for a Hilbert space $\mathcal{H}$ with constant A then

$$h = \frac{1}{A} \sum_{i \in I} \langle h, f_i \rangle f_i, \, \forall h \in \mathcal{H}.$$

This is known as the *sampling and reconstruction formula.*

On the other hand, the analysis operator V is an isometry if and only if the *Grammian matrix* $VV^* = (\langle f_j, f_i \rangle)$ is a projection with rank equal to $dim(\mathcal{H})$, and hence gives another characterization of Parseval frames.

If $\mathcal{F}$ is a uniform (N, d)-frame with analysis operator V, then $V^*V = I_d$ the $d \times d$ identity matrix and

$$d = rank(VV^*) = Tr(VV^*) = \sum_{i=1}^{N} \|f_i\|^2 = N\|f_k\|^2,$$

for any k. Thus, for a (N, d)-frame,

$$\frac{1}{\|f_k\|^2} = \frac{N}{d},$$

is the redundancy. For this reason, when $\mathcal{F} = \{f_i\}_{i \in I}$ is a uniform Parseval frame for an infinite dimensional Hilbert space, we still call $\frac{1}{\|f_k\|^2}$ the *frame redundancy.*

Furthermore, when $\mathcal{F}$ is a uniform (N,d)-frame, each of the diagonal entries of VV^* must be equal to d/N and therefore each frame vector must be of length $\sqrt{d/N}$.

Conversely, given an $N \times N$ self-adjoint projection P of rank d, we can always factor it as $P = VV^*$ for some $N \times d$ matrix V. It readily follows that $V^*V = I_d$ and hence V is the matrix of an isometry and so is the analysis operator of an (N,d)-frame. The vectors in this frame are the complex conjugates of the rows of V.

Moreover, if $P = WW^*$ is another factorization of P, then there exists a unitary U such that $W^* = UV^*$ and hence the two corresponding frames differ by multiplication by this unitary. Thus, P determines a unique unitary type I equivalence class of frames. A projection P corresponds to a uniform (N,d)-frame if and only if all of its diagonal entries are d/N.

Definition 1.2.2. Frames $\mathcal{F} = \{f_i\}_{i=1}^n$ and $\mathcal{G} = \{g_i\}_{i=1}^n$, are *type I equivalent* if there exists a unitary (orthogonal matrix, in the real case) U such that $g_i = Uf_i$ for all i.

Theorem 1.2.3. *[22] If V and W are the analysis operators for $\mathcal{F}$ and $\mathcal{G}$, respectively, then the following are equivalent*

1. $\mathcal{F}$ and $\mathcal{G}$ are type I equivalent

2. there exists a unitary(respectively, orthogonal matrix) U such that $V = WU$

3. $VV^* = WW^*$.

Definition 1.2.4. Frames $\mathcal{F} = \{f_i\}_{i=1}^n$ and $\mathcal{G} = \{g_i\}_{i=1}^n$ are *type II equivalent* if they are a permutation of the same set of vectors and they are *type III equivalent* if there exist numbers $\{\lambda_i\}_{i=1}^n$ of modulus such that $f_i = \lambda_i g_i$. Thus, in the real case if they differ by multiplication by ± 1. Two frames are *equivalent* if they belong to the same equivalence class in the equivalence relation generated by these three equivalence relations.

Theorem 1.2.5. *[22] If $\mathcal{F}$ and $\mathcal{G}$ are (N, d)-frames with analysis operators V and W, respectively, then they are equivalent if and only if $UVV^*U^* = WW^*$ for some $N \times N$ unitary U that is the product of a permutation matrix and a diagonal matrix with entries of modulus 1(± 1, in the real case).*

1.3 Outline

In Chapter 2, we define correlation minimizing frames and derive properties. Then we give examples for some of these correlation minimizing frames, some of which are also 2-uniform frames, in $\mathbb{R}^3$ for some values of N and prove that for some of the values of N where these two angles were shown to numerically agree, that one does indeed obtain tight frames. Additionally, we produce the actual correlation minimizing frames by using the geometric descriptions of the optimal line packings to find closed coordinates, and then choose unit vectors from those the optimal line packing coordinates. Also, we identify cases where the numerical estimates match but the line packing does not yield a correlation minimizing frame.

In Chapter 3, we begin by finding bounds on the correlation constant of a correlation minimizing (N,d)-frame. As a result, we show that the correlation constant goes to zero for fixed redundancy as the dimension and number of vectors increases proportionally by 2^k. Then, we consider the problem of finding bounds on the correlation constant in the complex case. We begin by laying out the structure of circulant matrices which are also projections. We view these circulant matrices as the Grammian matrix of a uniform (N,d)-frame. We obtain a relationships between these Grammian matrices and the Dirichelet kernel as well as the structure of quadratic residues in $\mathbb{Z}^N$. Finally, we utilize these relationship to derive bounds on the correlation constant. Furthermore, we investigate how the correlation constant behaves asymptotically in comparison to the Welch bound.

In Chapter 4, we consider when the Hilbert space is infinite dimensional. In this case, the Laurent matrix is a projection defined by the Fourier transform of the characteristic function on an interval of fixed finite length in [0,1]. We derive bounds on the magnitude of the Fourier transform of the characteristic function on a set of sufficiently small size. Additionally, we construct a method to create a correlation constant that is arbitrarily small.

CHAPTER 2

Correlation Minimizing Frames

Correlation Minimizing Frames were first defined by Holmes and Paulsen [22] as 2-erasure frames. Later Getzelman, Leonhard, and Paulsen renamed these correlation minimizing frames [16]. Chapters 2 includes an exposition of these results.

2.1 Optimal Frames for Erasures

To motivate the definition of *corrlation minimizing frames* we first consider the m-erasure problem and which frames are optimal, in a sense made precise by [22] for m=1 and m=2.

The idea behind treating frames as codes is that, given an original vector $x \in \mathbb{F}^d$, where $\mathbb{F}^d$ s a d dimensional field, and an (N, d)-frame with analysis operator V, one regards the vector $Vx \in \mathbb{F}^N$ as an encoded version of x, which then is transmitted, received, and finally decoded by applying V^*. If V is the analysis operator for the frame used to encode x, choose V^* to be the unique left inverse that minimizes both the operator norm and Hilbert-Schmidt norm.

Furthermore, suppose that during transmission some number, say m, of the frame coefficients, i.e components of the vector Vx, are lost prior to the reconstruction of x. In this case, we remove the components using the matrix E and represent the received vector as EVx, where E is a diagonal matrix of m 0's and N-m 1's corresponding to the entries of Vx that are, respectively, lost and received. The 0's in E can be thought of as the coordinates of Vx that have been erased.

There are two methods by which one could attempt to reconstruct x. Both require computation of a left inverse. For active reconstruction, the left inverse of EV is used. In the case of blind reconstruction, the left inverse V^* for analysis operator V can continue be used in which case x will only have been approximately reconstructed. If EV has a left inverse, then the left inverse of minimum norm is given by $T^{-1}W^*$ where $EV = WT$ is the polar decomposition and $T = |EV| = (V^*EV)^{\frac{1}{2}}$. Thus, the minimum norm of a left inverse is given by the inverse of the minimum eigenvalue of T, t_{min}^{-1}.

The 2-erasure problem was characterized by Holmes and Paulsen [22]. In their research, the norms of the error operators were considered, rather than those of the left inverses. This approach yields cleaner formulas. Their research described the frames for which the norms of these error operators are in some sense minimized, independent of which erasures occur. That is, for analysis operator V, V^* is used to reconstruct x.

The error in reconstructing x is given by

$$x - V^*EVx = V^*(I - E)Vx = (I - T^2)x = V^*DVx,$$

where D is a diagonal matrix of m 1's and N-m 0's. It follows that the norm of the error operator is $1 - t_{min}^2$.

Consequently, when a left inverse exists, the problem of minimizing the norm of a left inverse over all frames and all E with m-erasures is equivalent to minimizing the norm of the error operator over all frames. Moreover, they are both achieved by maximizing the minimal eigenvalue of T.

The first quantity defined, $d_m(V)$, represents the maximal norm of an error operator given that some set of m erasures occurs.

Definition 2.1.1. First, let $\mathcal{D}_m, 1 < m \leq N$ denote the set of $N \times N$ diagonal matrices with m 1's and $N - m$ 0's and for any isometry $V \in \mathcal{F}(N, d)$ set

$$d_m(V) = \max\{\|V^*DV\| : D \in \mathcal{D}_m\}.$$

Since $\mathcal{F}(N, d)$ is a compact set the value

$$e_1(N, d) = \inf\{d_1(V) : V \in \mathcal{F}(N, d)\}$$

is attained. We define the 1-*erasure frames* to be the nonempty compact set

$$\mathcal{E}_1(N, d) = \{V \in \mathcal{F}(N, d) : d_1(V) = e_1(N, d)\}$$

.

Proceeding inductively,for $1 \leq m \leq N$, set

$$e_m(N,d) = \inf\{d_m(V) : V \in \mathcal{E}_{m-1}(N,d)\} \tag{2.1}$$

and define the *m-erasure frames* to be the nonempty compact subset $\mathcal{E}_m(N,d)$ of $\mathcal{E}_{m-1}(N,d)$ where this infimum is attained.

As evident from the definition, we have decreasing sets of equivalent frames that will be the optimal solutions for fixed (N,d). Consequently, we examine the solution for the 1-erasure problem to rephrase and further motivate the solution for the 2-erasure problem. Casazza and Kovačević, in [6], proved that the optimal solution for the 1-erasure problem is the family of uniform (N, d)-frames. The theorem as stated in [22] is below.

Theorem 2.1.2. *[6] The set $\mathcal{E}_1(N,d)$ coincides with the family of uniform (N, d)-frames, and consequently, $e_1(N,d) = N/d$.*

Proof. Given an (N, d)-frame $\mathcal{F} = \{f_1, \ldots, f_N\}$, if we regard the frame vectors as column vectors, then the analysis operator V is the matrix whose j-th row is f_j^*. Given $D \in \mathcal{D}_1$ which is 1 in the $D_{j,j}$ entry, we have that

$$\|V^*DV\| = \|DV^*VD\| = \|f_j\|^2.$$

Hence,

$$d_1(V) = \max\{\|f_j\|^2 : 1 \leq j \leq N\}.$$

Since $\sum_{j=1}^{N} \|f_j\|^2 = tr(VV^*) = N$, we have that $\|f_j\|^2 \geq \frac{d}{N}$ for some $1 \leq j \leq N$. So $d_1(V)$ is clearly minimized when $\|f_j\|^2 = \frac{d}{N}$ independent of j. That is, when $\mathcal{F}$ is a uniform (N, d)-frame. □

As a special class of uniform (N,d)-frames which will be analyzed later, we use the correspondence between projection and the equivalence classes of these frames as proven in [22].

Theorem 2.1.3. *[22] There is a one-to-one correspondence between $N \times N$ rank d projections and type I equivalence classes of uniform (N,d)-frames.*

Proof. Given type I equivalent frames $\mathcal{F} = \{f_i\}_{i=1}^N$ and $\mathcal{G} = \{g_i\}_{i=1}^N$, with analysis operators V and W respectively. Since $\mathcal{F}$ and $\mathcal{G}$ are type I equivalent there exists a unitary (orthogonal matrix, in the real case) U such that $g_i = Uf_i$ for all $1 \leq i \leq N$. This is true if and only if V and W are the analysis operators for $V = WU$ or equivalently, if and only if $VV^* = WW^*$. Thus, there is a one-to-one correspondence between N by N rank d projections and type I equivalence classes of (N, d)-frames and consequently type I equivalence classes of uniform (N, d)-frame. □

By observing the relationship between $\mathcal{E}_1(N, d)$ and

$$\mathcal{E}_2(N, d) = \{V \in \mathcal{F}(N, d) : d_2(V) = e_2(N, d)\}$$

and uniform (N,d)-frames, we can consider the corresponding NxN projection to analyse the structure of equivalence classes of uniform (N,d)-frames. When considering the class of 2-erasure frames, there are some (N,d) pairs where 2-uniform frames occur.

Definition 2.1.4. (Definition 2.4, [22]) A (N,d)-frame, $\mathcal{F}$ is 2-uniform provided that $\mathcal{F}$ is a uniform (N,d)-frame and in addition $\|V^*DV\|$ is a constant for all $D \in \mathcal{D}_2$.

Theorem 2.1.5. *(Theorem 2.5, [22]) Let $\mathcal{F}$ be a uniform (N, d)-frame. Then F is 2-uniform if and only if $|\langle f_j, f_i\rangle| = c_{N,d}$ is constant for all $i \neq j$, where*

$$c_{N,d} = \sqrt{\frac{d(N-d)}{N^2(N-1)}}$$

.

Proof. Fix $i \neq j$, let V be the analysis operator for the uniform (N,d)-frame $\mathcal{F}$ and let D be the diagonal matrix that is 1 in the (i,i) and (j,j) entries and 0 elsewhere. Since $D^2 = D = D^*$, we have that

$$\|V^*DV\| = \|(DV)^*(DV)\| = \|DVV^*D\| = \| \begin{pmatrix} \frac{d}{N} & \langle f_i, f_j\rangle \\ \langle f_j, f_i\rangle & \frac{d}{N} \end{pmatrix} \|.$$

The norm of this 2x2 matrix is easily found to be $\frac{d}{N} + |\langle f_j, f_i\rangle|$ and thus $\mathcal{F}$ is 2-uniform if and only if $|\langle f_j, f_i\rangle|$ is constant, say c, for all $i \neq j$. To see the final claim, use the fact that $P = VV^*$ satisfies $P = P^2$. Equating diagonal entries of P and P^2, yields the equation

$$\frac{d}{N} = \left(\frac{d}{N}\right)^2 + (N-1)c^2.$$

This equation, when solved for c, yields the above formula for $c = c_{N,d} = \sqrt{\frac{d(N-d)}{N^2(N-1)}}$.

□

Previously, Holmes and Pauslen find the exact value of magnitudes of the inner product of a 2-uniform (N,d)-frame, i.e. the off diagonal entries of the Grammian matrix. Since these entries are equal, it follows that the angle between each pair of frame vectors will be the same.

Corollary 2.1.6. *(Corollary 2.6, [22]) Let $\mathcal{F}$ be a uniform (N,d)-frame. Then $\mathcal{F}$ is 2-uniform if and only if the angle between the lines spanned by every pair of frame vectors is equal to* $\cos^{-1}\left(\sqrt{\frac{N-d}{d(N-1)}}\right)$.

The families of frames satisfying the latter condition in the above proposition have also been studied independently in [35], where they are called equiangular frames and the corresponding unit norm frame optimal Grassmannain frames. For those (N,d) pairs where 2-uniform frames do not exist, a lower bound for the maximal norm of an error operator was identified.

Theorem 2.1.7. *(Proposition 2.7, [22]) Let natural numbers $d \leq N$ be given. If $\mathcal{F} = \{f_1, \ldots, f_n\}$ is a uniform (N,d)-frame, then for each i there exists $j \neq i$ such that $|\langle f_j, f_i \rangle| \geq c_{N,d}$. Consequently, if V denotes the analysis operator of F, then $d_2(V) \geq \frac{d}{N} + c_{N,d}$.*

Proof. Let $P = (p_{i,j}) = VV^*$ denote the Grammian matrix of $\mathcal{F}$. Using the fact that $P^2 = P$ and equating the (i,i)-th entry yields $\sum_{j=1}^{N} |p_{i,j}|^2 = \frac{d}{N}$ and hence,

$$\sum_{j=1, j\neq i}^{N} |p_{i,j}|^2 = \frac{d}{N} - |p_{i,i}|^2 = \frac{d}{N} - \left(\frac{d}{N}\right)^2 = \frac{d(N-d)}{N^2}$$

Since there are (N-1) terms in the above sum, at least one term must be larger than $\frac{N(N-d)}{(N-1)N^2} = c_{N,d}^2$ and the first result follows. The second claim follows from the formula for $\|V^*DV\|$ for any $D \in \mathcal{D}_2$ obtained in the proof of Proposition 2.1.5. $\square$

Consequently, the previous theorem can be used to determine the relationship 2-uniform frames have to the m-erasure problem and when a 2-uniform frame does not exist, a lower bound can be established.

Theorem 2.1.8. *(Theorem 2.8, [22]) Let natural numbers $d \leq N$ be given. If there exists a 2-uniform (N, d)-frame, then every frame in $\mathcal{E}_m(N,d)$ is 2-uniform for $2 \leq m$ and $e_2(N,d) = \frac{d}{N} + c_{N,d}$. If there does not exist a 2-uniform (N,d)-frame, then necessarily $e_2(N,d) > \frac{d}{N} + c_{N,d}$.*

Proof. The first statement follows from Proposition 2.1.5. To see the second statement, note that by compactness there must exist a uniform (N,d)-frame $\mathcal{F}$ with analysis operator V such that $e_2(N,d) = d_2(V)$. If $e_2(N,d) = \frac{N}{d} + c_{N,d}$, then the proof proposition 2.1.5 shows that for all $j \neq i$, we would have that $\langle f_j, f_i \rangle = |p_{i,j}| = c_{N,d}$, which implies that $\mathcal{F}$ is 2-uniform. Thus if $\mathcal{F}$ is not 2-uniform $e_2(N,d) > \frac{d}{N} + c_{N,d}$. □

2.2 Correlation Minimizing Frames

Now, we define *correlation minimizing frames* and relate them to *2-erasure frames* and, when they exist, *2-uniform frames* and their properties.

Definition 2.2.1. Let $\mathcal{F} = \{f_1, ... f_N\} \in \mathcal{E}_1(N,d)$. Then the *maximum correlation* is

$$M_\infty(\mathcal{F}) = \frac{N}{d} \cdot \max\{|\langle f_k, f_l \rangle| : k \neq l\}.$$

and

$$C(N,d) = \inf\{M_\infty(\mathcal{F}) : \mathcal{F} \in \mathcal{E}_1(N,d)\}. \tag{2.2}$$

is the *correlation constant for frames in* $\mathcal{E}_1(N,d)$. We call a uniform (N,d)-frame $\mathcal{F}$ *correlation minimizing* if $M_\infty(\mathcal{F}) = C(N,d)$.

Note that the factor $\frac{N}{d}$ must be included because $\mathcal{F} \in \mathcal{E}_1(N,d)$ implies that each vector in $\mathcal{F}$ has norm $\sqrt{d/N}$. Clearly, $0 \leq M_\infty(\mathcal{F}) \leq 1$ If $N = d$, then we take $\mathcal{F}$ to be an orthonormal basis for $\mathcal{H}$. For the case where $\mathcal{F}$ is "overcomplete", $N > d$, then $|\langle f_j, f_l\rangle|$ will depend the redundancy. For smaller redundancy $M_\infty(\mathcal{F})$ should be smaller. Smaller maximum correlation means a set is more nearly orthogonal. If $M_\infty(\mathcal{F}) = 1$ and the supremum is attained, then any two vectors where the supremum is attained are parallel. Therefore, larger maximum correlation indicates that the set contains vectors that are more nearly parallel.

These constants were introduced in [22] where it was proven that $C(N,d)$ was always attained, i.e., the infimum is actually a minimum. Frames called *2-uniform* in [22] or tight equiangular frames in [35] are correlation minimizing frames. The following result is essentially from [16].

Theorem 2.2.2. *A uniform (N,d)-frame is* correlation minimizing *if and only if it is in the set* $\mathcal{E}_2(N,d)$. *Consequently,*

$$\mathcal{E}_2(N,d) = \{\mathcal{F} \in \mathcal{E}_1(N,d) : M_\infty(\mathcal{F}) = C(N,d)\}$$

and a uniform (N,d)-frame is correlation minimizing if and only if it is a 2-erasure frame.

Then the relationship between correlation minimizing frames to 2-uniform and equiangular frames we have the result from [22]

Theorem 2.2.3. *([35][22]) Let $N \geq d$, and let $\mathcal{F} \in \mathcal{E}_1(N,d)$. Then*

$$M_\infty(\mathcal{F}) \geq \sqrt{\frac{N-d}{d(N-1)}}, \tag{2.3}$$

and equality holds iff $\mathcal{F}$ is equiangular.

If there exists an equiangular frame $\mathcal{F} \in \mathcal{E}_1(N,d)$, then it is 2-uniform, and necessarily a correlation minimizing (N,d)-frame. In this case every frame in $\mathcal{E}_2(N,d)$ is 2-uniform.

Furthermore, if $N > \frac{d(d+1)}{2}$ in the real case and $N > d^2$ in the complex case, then there is no equiangular frame in $\mathcal{E}_1(N,d)$ and equality cannot hold in the above equation.

Thus, we see that

$$C(N,d) \geq \sqrt{\frac{N-d}{d(N-1)}},$$

with equality if and only if there exists an equiangular frame. The quantity appearing on the right hand side of the above equation is known as the *Welch bound.*

Also, $\Theta(N,d) = arccos(C(N,d))$. Note that since *arccos* is a decreasing function,

$$\Theta(N,d) = \sup\{\Theta(\mathcal{F}) : \mathcal{F} \in \mathcal{E}_1(N,d)\}.$$

We shall call $C(N,d)$ the *correlation constant for frames in $\mathcal{E}_1(N,d)$* and so $\Theta(N,d)$ is the *maximum angle between vectors for frames in $\mathcal{E}_1(N,d)$.*

Example 2.2.4. (Proposition 2.3, [22]) For $m \geq 2$ and $N \geq 2$, every frame in $\mathcal{E}_m(N,2)$ is frame equivalent to the frame given by setting

$$f_j = \sqrt{\frac{2}{N}}\left(cos\left(\frac{\pi j}{N}\right), sin\left(\frac{\pi j}{N}\right)\right),$$

for $j = 1, \ldots, N$.

It is clear from the definition that $\mathcal{F}$ is a uniform (N,d)-frame. However, it is not a 2-uniform frame. To see it is correlation minimizing, we compute

$$|\langle f_i, f_j\rangle| = \frac{2}{N}|cos\left(\frac{i\pi}{N}\right) sin\left(\frac{j\pi}{N}\right) + cos\left(\frac{j\pi}{N}\right) sin\left(\frac{i\pi}{N}\right)| = \frac{2}{N}|cos(\left(\frac{\pi(i+j)}{N}\right))|.$$

Note $|cos(\theta)| = |cos(\theta-\pi)|$ and $cos(\theta)$ is decreasing on $\left(0, \frac{\pi}{2}\right)$. Therefore, $\frac{N}{2}M_\infty(\mathcal{F}) = cos\left(\frac{\pi}{N}\right)$. This is the lower bound for the (N,2)-Grassmannian frame, given in [2]. Hence, by scaling, $\sqrt{\frac{N}{2}}\mathcal{F}$ is the (N,2)-Grassmannian frame and we conclude $\mathcal{F}$ is correlation minimizing, with $C(N,2) = cos\left(\frac{\pi}{N}\right)$.

Since correlation minimizing frames are uniform (N,d)-frames, we know by compactness that the infimum is attained for all (N,d) pairs. So now, we look at the relationship Naimark's Theorem and the Naimark complement give when applied to correlation minimizing frames.

Theorem 2.2.5 (Naimark's Dilation Theorem)**.** *Let $\mathcal{H}_d$ be a d-dimensional Hilbert space and $\mathcal{H}_N$ be a N-dimensional Hilbert space. A family of vectors $\mathcal{F} = \{f_i\}_{i=1}^N$ is a Parseval frame for $\mathcal{H}_d$ if and only if the analysis operator V^* is an isometry satisfying $V^*f_i = Pe_i$ for all $i = 1, 2, \ldots, d$ where $\{e_i\}_{i=1}^N$ is an orthonormal basis for $\mathcal{H}_N$ and P is an orthogonal projection from H_N onto H_d. Moreover, $\{(I-P)e_i\}_{i=1}^N$ is a Parseval frame for an $(N-d)$-dimensional Hilbert space.*

$\{(I-P)e_i\}_{i=1}^N$ is called a *Naimark complement.* Since a Naimark complement is a Parseval frame for its span, the analysis operator is a projection of $\{Pe_i\}_{i=1}^N$ and therefore the Grammian for a $(N, N-d)$-frame. This leads us to the relationship found in [16] between a correlation minimizing (N,d)-frame and a correlation minimizing $(N, N-d)$-frame.

Theorem 2.2.6. *[16] Let $\mathcal{F} = \{f_i\}_{i=1}^N$ be a correlation minimizing (N,d)-frame with Grammian matrix G. Then $I_N - G$ is the Grammian matrix of a correlation minimizing $(N, N-d)$-frame where $C(N, N-d) = \frac{d}{N-d}C(N,d)$ Moreover, there is a one-to-one correspondence between equivalence classes of correlation minimizing (N,d)-frames and equivalence classes of correlation minimizing $(N, N-d)$-frames.*

Proof. Let $\mathcal{F} = \{f_i\}_{i=1}^N$ be any uniform (N,d)-frame with Grammian matrix G. Then G is a rank d projection all of whose diagonal entries are equal to $\frac{d}{N}$. Hence, $I_N - G$ is a rank $N-d$ projection all of whose diagonal entries are $\frac{N-d}{N}$. Hence, if we let $WW^* = I_N - G$ be any factorization, then by the results of the last section the rows of W(or their complex conjugates in the complex case) form a uniform $(N, N-d)$-frame whose Grammian is $I_N - G$. This frame is not uniquely determined by $I_N - G$, since many factorizations are possible, but it is unique up to type I equivalence by Theorem 1.2.3.

So choose one such uniform $(N, N-d)$-frame and denote it by $\mathcal{F}^\perp$. Now if $\mathcal{F}_i$ $i = 1, 2$ are any uniform (N,d)-frames with Grammians G_i $i = 1, 2$ and we let $\mathcal{F}_i^\perp$ $i = 1, 2$ be $(N, N-d)$-frames with Grammians $I_N - G_i$ obtained as above, then, by applying Theorem 1.2.5, we see that $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent if and only if $\mathcal{F}_1^\perp$ and $\mathcal{F}_2^\perp$ are equivalent.

Finally, since the maximum correlation of a frame is really just the maximum off-diagonal entry of its Grammian (appropriately scaled), we see that

$$M_\infty(\mathcal{F}^\perp) = \frac{d}{N-d} M_\infty(\mathcal{F}),$$

and so whenever $\mathcal{F}$ is a correlation minimizing uniform (N,d)-frame that $\mathcal{F}^\perp$ is a correlation minimizing uniform $(N, N-d)$-frame. This also shows

$$C(N, N-d) = \frac{d}{N-d} C(N, d).$$

□

Example 2.2.7. Referring back to Example 2.2.4 and using Theorem 2.2.6 we see that all (N,N-2) correlation minimizing frames are unique up to equivalence. Also, we obtain a formula for the correlation constant,

$$C(N, N-2) = \frac{2}{N-2} C(N, 2).$$

2.3 Grassmannian and Correlation Minimizing Frames in $\mathbb{R}^3$

The *line packing problem* is the problem of packing N lines in $\mathbb{R}^d$ so that the minimal angle between any two of them is as large as possible. Any solution to this problem is called a *Grassmannian line packing.* Given a Grassmannian line packing with $N \geq d$ if we choose one unit vector from each line, then this set of vectors always yields a frame for $\mathbb{R}^d$. Any frame obtained this way is called a *Grassmannian frame* by [35].

If a Grassmannian frame is a unit-norm tight frame, then after scaling the vectors by $\sqrt{d/N}$ we would obtain a uniform (N, d)-frame that is necessarily correlation minimizing.

A geometric approach to solving the line packing problem and list of best-known packings is posted on [32]. Conway, Hardin, and Sloane [9] find the Grassmannian line packings of N lines through the origin in $\mathbb{R}^3$, describe the packings geometrically and compute this minimal angle for $2 \leq N \leq 55$. For some values of N, they produce what the numerical calculations indicate to be the optimal packings and then are able to describe these packings, but do not provide proofs that these are the optimal line packings. In these cases, they refer to these explicit packings as the *putative* optimal line packings and we shall adopt their language.

A natural question that we shall study below is whether or not the Grassmannian frames arising from these Grassmannian line packings, putative and/or proven, are tight. The numerical experiments of [22] indicates that the answer should be "yes" for some values of N and "no" for other values.

In [22], the uniform $(N, 2)$-frames that are correlation minimizing were constructed, it was shown that these frames form a single equivalence class, and that these are Grassmannian.

First, note that whenever $N = d$ then any Parseval frame must be an orthonormal basis, since the frame operator V will be an isometry from $\mathbb{R}^d$ to $\mathbb{R}^d$ and hence will be an orthogonal matrix. Hence, the rows of V will be an orthonormal set. Moreover, every orthonormal basis is type I equivalent. Thus, there is a unique equivalence class of (d, d)-frames and an orthonormal basis is clearly correlation minimizing and Grassmannian.

So the first interesting case is the $(4, 3)$-frames. For this a corollary to Theorem 2.2.6 is useful. Let J_N denote the $N \times N$ matrix of all 1's.

Corollary 2.3.1. *[16] Up to equivalence there is a unique correlation minimizing $(N,1)$-frame and a unique correlation minimizing $(N,N-1)$-frame. These equivalence classes are represented by the uniform frames with Grammians $\frac{1}{N}J_N$ and $I_N - \frac{1}{N}J_N$, respectively. Moreover, both these frames are equiangular and so these frames are also Grassmannian.*

Proof. To obtain a uniform $(N,1)$-frame one must choose N numbers of modulus $1/\sqrt{N}$. But these are all equivalent to choosing the number $1/\sqrt{N}$ N-times. Thus, up to equivalence there is only one uniform $(N,1)$-frame and it has Grammian $\frac{1}{N}J_N$.

Hence, by the above theorem, up to equivalence there is only one $(N,N-1)$-frame and it has Grammian given by $I_N - \frac{1}{N}J_N$.

All these frames are equiangular since all the off-diagonal entries in their Grammians are of constant modulus $\frac{1}{N}$. □

We can now give a proof for the description of one representative of this equivalence class of frames in the case $N=4$, $d=N-1=3$.

Theorem 2.3.2. *[16] The lines generated by opposite vertices of the inscribed cube in the sphere centered at the origin is the optimal packing of 4 lines in 3-space. If we take the sphere of radius $\frac{\sqrt{3}}{2}$ centered at the origin and consider the 8 vectors determined by the vertices of this cube, then any set of 4 of these vectors that are not collinear yields a correlation minimizing, equiangular $(4,3)$-frame. In particular, one correlation minimizing, equiangular $(4,3)$-frame is given by*

$$(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2},+\frac{1}{2}),(-\frac{1}{2},+\frac{1}{2},-\frac{1}{2}),(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$$

and every other correlation minimizing (4,3)-frame is equivalent to this frame.

Proof. Let

$$V = \begin{pmatrix} {f_1}^* \\ {f_2}^* \\ {f_3}^* \\ {f_4}^* \end{pmatrix} = \begin{pmatrix} +\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Computing the Grammian yields $G = VV^* = I_4 - \frac{1}{4}J_4$ so that this is one representative of the unique correlation minimizing $(4,3)$-frame. The remaining claims are now straightforward to verify. $\square$

Now, we will look at the (5,3) correlation minimizing frame and the (5,3) Grassmannian frame and find that they are not the equivalent frames.

Theorem 2.3.3. *[16] A correlation minimizing* $(5,3)$*-frame is given by the vectors:*

$$c(1,0,0),\ c(\frac{-1-\sqrt{5}}{6}, \frac{15-\sqrt{5}}{a}, 0)$$
$$c(\frac{1-\sqrt{5}}{6}, \frac{-5-3\sqrt{5}}{a}, \frac{150-30\sqrt{5}}{ab}), c(\frac{-1+\sqrt{5}}{6}, \frac{5-3\sqrt{5}}{a}, \frac{-60\sqrt{5}}{ab})$$
$$c(\frac{1+\sqrt{5}}{6}, \frac{4\sqrt{5}}{a}, \frac{150-30\sqrt{5}}{ab}),$$

where $a = \sqrt{18\left(15-\sqrt{5}\right)}$, $b = \sqrt{150-30\sqrt{5}}$, *and* $c = \sqrt{3/5}$*. Every other correlation minimizing* $(5,3)$*-frame is equivalent to this frame.*

Proof. From [22] we have that the correlation minimizing $(5,2)$-frame is unique up to equivalence and one representative is given by the vectors

$$\{\left(cos\left(\frac{\pi k}{5}\right), sin\left(\frac{\pi k}{5}\right)\right) : k = 1,2,3,4,5\}.$$

Thus, by Theorem 2.2.6 the correlation minimizing $(5,3)$-frame will be unique up to equivalence and a representative Grammian will be given by $G = I_5 - G_{(5,2)}$, where $G_{(5,2)}$ is the Grammian of the above vectors.

Computing this Grammian yields,

$$G = \begin{pmatrix} 3/5 & \frac{2}{5}cos(\pi/5) & \frac{2}{5}cos(2\pi/5) & \frac{2}{5}cos(3\pi/5) & \frac{2}{5}cos(4\pi/5) \\ \frac{2}{5}cos(\pi/5) & 3/5 & \frac{2}{5}cos(\pi/5) & \frac{2}{5}cos(2\pi/5) & frac25cos(3\pi/5) \\ \frac{2}{5}cos(2\pi/5) & \frac{2}{5}cos(\pi/5) & 3/5 & \frac{2}{5}cos(\pi/5) & \frac{2}{5}cos(2\pi/5) \\ \frac{2}{5}cos(3\pi/5) & \frac{2}{5}cos(2\pi/5) & \frac{2}{5}cos(\pi/5) & 3/5 & \frac{2}{5}cos(\pi/5) \\ \frac{2}{5}cos(4\pi/5) & \frac{2}{5}cos(3\pi/5) & \frac{2}{5}cos(2\pi/5) & \frac{2}{5}cos(\pi/5) & 3/5 \end{pmatrix}.$$

This can be factored as $G = \frac{3}{5}UU^*$ where

$$U = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-1-\sqrt{5}}{6} & \frac{15-\sqrt{5}}{a} & 0 \\ \frac{1-\sqrt{5}}{6} & \frac{-5-3\sqrt{5}}{a} & \frac{150-30\sqrt{5}}{ab} \\ \frac{-1+\sqrt{5}}{6} & \frac{5-3\sqrt{5}}{a} & \frac{-60\sqrt{5}}{ab} \\ \frac{1+\sqrt{5}}{6} & \frac{4\sqrt{5}}{a} & \frac{150-30\sqrt{5}}{ab} \end{pmatrix},$$

with a and b as above. □

Corollary 2.3.4. *[16]The Grassmannian frame of 5 vectors in $\mathbb{R}^3$ is not a tight frame and hence is not a correlation minimizing frame.*

Proof. By inspection the largest off diagonal entry of the above G is $\frac{2}{5}cos(\pi/5)$, and the smallest angle produced by the vectors of this Grammian is equal to

$$\arccos(\frac{2}{3}cos(\pi/5))$$

which is approximately 57.361 degrees. This is not equal to the angle of the optimal packing of 5 lines found in [9]. Thus, if we take one unit vector from each of the 5 lines corresponding to the optimal packing of 5 lines through the origin in $\mathbb{R}^3$, then this set of vectors can not be a tight frame since its correlation is smaller. □

Thus, the correlation minimizing $(5,3)$-frame is an example that is not obtained via the optimal line packing. In the language of [35], the correlation minimizing $(5,3)$-frame is not a Grassmannian frame.

Holmes and Paulsen [22, Example 3.6] showed that the correlation minimizing $(6,3)$-frame is equiangular, that it is unique up to equivalence and a formula for obtaining its Grammian was given. Below we give a geometric description of the set of vectors for one representative of this equivalence class and give the vectors explicitly.

Theorem 2.3.5. *[16] The 6 vertices, that lie in the upper half plane of an icosahedron centered at the origin and symmetric about the xy-plane, form a correlation minimizing $(6,3)$-frame. Set $\alpha = \frac{1}{\sqrt{5}}$, then these are the vectors given by:*

$$f_1 = \tfrac{1}{\sqrt{2}}\,(0,0,1),$$
$$f_2 = \tfrac{1}{\sqrt{2}}\left(\sqrt{1-\alpha^2},0,\alpha\right),$$
$$f_3 = \tfrac{1}{\sqrt{2}}\left(\alpha\sqrt{\tfrac{1-\alpha}{1+\alpha}},\sqrt{\tfrac{(1+2\alpha)(1-\alpha)}{1+\alpha}},\alpha\right),$$
$$f_4 = \tfrac{1}{\sqrt{2}}\left(\alpha\sqrt{\tfrac{1-\alpha}{1+\alpha}},-\sqrt{\tfrac{(1+2\alpha)(1-\alpha)}{1+\alpha}},\alpha\right),$$
$$f_5 = \tfrac{1}{\sqrt{2}}\left(-\alpha\sqrt{\tfrac{1-\alpha}{1+\alpha}},\sqrt{\tfrac{(1-2\alpha)(1+\alpha)}{1-\alpha}},\alpha\right),$$
$$f_6 = \tfrac{1}{\sqrt{2}}\left(-\alpha\sqrt{\tfrac{1+\alpha}{1-\alpha}},-\sqrt{\tfrac{(1-2\alpha)(1+\alpha)}{1-\alpha}},\alpha\right).$$

Every other correlation minimizing $(6,3)$-frame is equivalent to this frame.

Proof. From [2] we have the vectors defined above. For $k \neq l$, we compute $|\langle f_k, f_l\rangle| =$

$\frac{1}{\sqrt{5}}$. Thus, this set of vectors is equiangular and and each vector has norm $\frac{\sqrt{3}}{\sqrt{6}}$ so these must be a correlation minimizing $(6,3)$-frame.

In [9], it is observed that the 6 lines obtained by taking antipodal pairs of points on an icosahedron are equiangular. $\square$

To construct the (7,3) correlation minimizing frame we need the following proposition.

Proposition 2.3.6. *[16]If $\{f_1, \ldots, f_N\}$ is a uniform (N, d)-frame and $\{g_1, \ldots, g_M\}$ is a uniform (M, d)-frame, then $\{af_1, \ldots, af_N, bg_1, \ldots, bg_M\}$ is a uniform $(M+N, d)$-frame, where $a = \sqrt{N/(N+M)}$ and $b = \sqrt{M/(N+M)}$.*

Proof. Since $\|f_i\| = \sqrt{d/N}$ and $\|g_j\| = \sqrt{d/M}$ we have that $\|af_i\| = \|bg_j\| = \sqrt{d/(N+M)}$, so this set of vectors is uniform in norm. Finally, for any vector $x \in \mathbb{R}^d$, we have that

$$\sum_{i=1}^{N} |\langle x, af_i\rangle|^2 + \sum_{j=1}^{M} |\langle x, bg_j\rangle|^2 = a^2\|x\|^2 + b^2\|x\|^2 = \|x\|^2,$$

so the Parseval condition is met. $\square$

Theorem 2.3.7. *[16] Let $\{f_1, f_2, f_3, f_4\}$ be the correlation minimizing $(4,3)$-frame of Theorem 2.3.2 and let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis for $\mathbb{R}^3$, then $\{\sqrt{4/7}\, f_1, \sqrt{4/7}\, f_2, \sqrt{4/7}\, f_3, \sqrt{4/7}\, f_4, \sqrt{3/7}\, e_1, \sqrt{3/7}\, e_2, \sqrt{3/7}\, e_3\}$ is a correlation minimizing $(7,3)$-frame.*

Proof. By the above proposition, this set of vectors is a uniform $(7,3)$-frame. The inner products of pairs of these unequal vectors take on the values, $\{0, \pm\frac{\sqrt{3}}{7}, \pm\frac{1}{7}\}$, so that for this frame, $M_\infty(\mathcal{F}) = \frac{7}{3}\frac{\sqrt{3}}{7} = \frac{\sqrt{3}}{3}$. Since $cos^{-1}\left(\frac{\sqrt{3}}{3}\right)$ corresponds to the minimum angle for the Rhombic Dodecahedron [18], which is an optimal line packing angle for 7 lines in 3 space found by [9], this uniform Parseval frame must be correlation minimizing. $\square$

Since the correlation minimizing (7,3)-frame corresponds to an optimal line packing and $N \leq 7$, every correlation minimizing (7,3)-frame would yield an optimal line packing. But we do not know if every correlation minimizing (7,3)-frame is equivalent to this frame. In [9], they remark that the optimal packing of 7 lines in 3 space appears to be unique, but do not supply a proof. A related, and possibly easier, problem would be to decide if every optimal line packing of 7 lines in 3 space yields a tight frame.

Unfortunately, for $N \geq 8$ the results from Conway, Hardin, and Sloan in [9] come from running their optimization program 1500 times. In the cases of N=10, 12, and 16, the angle estimates match the estimates from Holmes and Paulsen in [22] and those estimated coordinates seem to correspond to geometric shapes. Since these are both simulation based and when the vectors are not equiangular there is no nice way to check to see if they indeed form a correlation minimizing frame. We also observe that in the N=10 case the vectors do not form a tight frame.

The optimal line packing for 10 lines in $\mathbb{R}^3$ is given numerically on Sloane's web site [32]. In [9], it was determined that there are infinitely many solutions to this optimal line packing problem. This occurs because the axial line can move freely over a small range of angles without affecting the minimum angle.

Theorem 2.3.8. *[16] The optimal line packing for* 10 *lines in* $\mathbb{R}^3$ *comprised of 2 axis vectors and the set of 8 vectors that are not collinear from the scaled hexakis bi-antiprism, given by*

$$\begin{gathered}
(1,0,0)\,,(0,-1,0)\,,\\
\left(\pm\frac{\sqrt{3}}{2},\frac{1}{2},0\right),\left(\beta,0,\pm\beta\sqrt{\sqrt{3}-1}\right)\\
\left(\frac{\beta}{2},\beta\frac{\sqrt{3}}{2},\beta\sqrt{\sqrt{3}-1}\right),\left(-\frac{\beta}{2},-\beta\frac{\sqrt{3}}{2},\beta\sqrt{\sqrt{3}-1}\right),\\
\left(\frac{\beta}{2},-\beta\frac{\sqrt{3}}{2},-\beta\sqrt{\sqrt{3}-1}\right),\left(-\frac{\beta}{2},\beta\frac{\sqrt{3}}{2},-\beta\sqrt{\sqrt{3}-1}\right).
\end{gathered}$$

where $\beta = 3^{-\frac{1}{4}}$*, is not a tight frame. Moreover, there does not exist a "rattle" of the axis that will yield a tight frame.*

Proof. First, we construct the hexakis bi-antiprim by taking two hexagonal antiprisms and joining them at the base. To create the first half, shift the coordinates for the hexigonal antiprism from [30]. For the second half, we use a shift and rotation of the same coordinates from [30]. Now, we join them at the base to complete the construction. From the set of 18 unique scaled vectors in the construction we consider 2 axis together with the set of 8 vectors that are not collinear. Set $\beta = 3^{-\frac{1}{4}}$ and define

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \beta & 0 & \beta\sqrt{\sqrt{3}-1} \\ \beta & 0 & -\beta\sqrt{\sqrt{3}-1} \\ \frac{\beta}{2} & \beta\frac{\sqrt{3}}{2} & \beta\sqrt{\sqrt{3}-1} \\ -\frac{\beta}{2} & -\beta\frac{\sqrt{3}}{2} & \beta\sqrt{\sqrt{3}-1} \\ \frac{\beta}{2} & -\beta\frac{\sqrt{3}}{2} & -\beta\sqrt{\sqrt{3}-1} \\ -\frac{\beta}{2} & \beta\frac{\sqrt{3}}{2} & -\beta\sqrt{\sqrt{3}-1} \end{pmatrix}$$

Recall that a set of vectors forms a uniform tight frame if and only if they are of equal norm and when they are entered as the rows of a matrix, then that matrix is a multiple of an isometry. Moreover, to be a multiple of an isometry, the columns of the matrix must be orthogonal and of equal norm.

The rows of V are unit norm. By inspection we see the columns are orthogonal. However, the columns of V do not have equal norm. Hence, no multiple of V is an isometry and so the rows are not a tight frame.

Now, we will consider the case where the axial lines ”rattle” to try to gain equality in the norm of the columns. Consider the first row as $v_1 = (a_1, b_1, c_1)$ and the second as $v_2 = (a_2, b_2, c_2)$. Since the vectors comprising the last eight entries of V are orthogonal, to keep the columns orthogonal, we will need the vectors $(a_1, a_2), (b_1, b_2)$ and (c_1, c_2) to be orthogonal. Since this is three vectors in $\mathbb{R}^2$, one of them must be zero. The norm of the first column is the largest so $a_1 = a_2 = 0$. The rows must be unit norm so $b_1^2 + c_1^2 = 1$ and $b_2^2 + c_2^2 = 1$. We still need the norms of the three columns to be equal. Thus, we get the system of equations.

$$\begin{cases} b_1^2 + c_1^2 = 1 \\ b_2^2 + c_2^2 = 1 \\ b_1^2 + b_2^2 = 1 \\ c_1^2 + c_2^2 = 3\sqrt{3} - \frac{9}{2} \end{cases}.$$

By subtracting the third equation from the first we see that $c_1^2 = b_2^2$. Plugging into equation 2 we get $c_1^2 + c_2^2 = 1$, which contradicts the fourth equation. Therefore, there is no choice of vectors that can make V a multiple of an isometry. $\square$

There are two more examples where the optimal angles in [22] and [9] for $N \geq 8$ match. In these cases, the simulations seem to show a relationship to a 3 dimensional shape. We know they are not 2-uniform, since of the number of vectors in dimension 3 violates Theorem 2.2. Taking into account that these angles in [9] are not proven to be optimal, these results are essentially from [16]. Unlike optimal (10,3) line packing, the closed form of the vectors do form a tight frame and give us a numerical upper bound for C(12,3) and C(16,3).

Theorem 2.3.9. *The putative* $(12, 3)$ *optimal line packing is obtained by considering the lines through the origin generated by opposite vertices of the rhombicubocathedron. Scaling the set of vertices of the rhombicuboctahedron to be vectors of length* $\frac{1}{2}$*, yields all permutations of the vectors* $\frac{1}{4}\left(\pm\frac{1}{\sqrt{2\sqrt{2}+5}}, \pm\frac{1}{\sqrt{2\sqrt{2}+5}}, \pm\frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}}\right)$. *Any set of 12 non-collinear vertices is a uniform (12,3)-frame. Moreover,* $\sqrt{\frac{3}{11}} < C(12, 3) \leq \frac{7+8\sqrt{2}}{17}$.

Proof. In [9], the putative optimal line packing of 12 lines in $\mathbb{R}^3$ is a rhombicuboctahedron. Define V such that the rows are the vectors of the rhombicuboctahedron in [18] . So,

$$V = \frac{1}{2}\begin{pmatrix}
\frac{1}{\sqrt{2\sqrt{2}+5}} & \frac{1}{\sqrt{2\sqrt{2}+5}} & \frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}} \\
\frac{1}{\sqrt{2\sqrt{2}+5}} & \frac{1}{\sqrt{2\sqrt{2}+5}} & \frac{-1-\sqrt{2}}{\sqrt{2\sqrt{2}+5}} \\
\frac{1}{\sqrt{2\sqrt{2}+5}} & -\frac{1}{\sqrt{2\sqrt{2}+5}} & \frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}} \\
\frac{1}{\sqrt{2\sqrt{2}+5}} & -\frac{1}{\sqrt{2\sqrt{2}+5}} & -\frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}} \\
\frac{1}{\sqrt{2\sqrt{2}+5}} & \frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}} & \frac{1}{\sqrt{2\sqrt{2}+5}} \\
\frac{1}{\sqrt{2\sqrt{2}+5}} & \frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}} & -\frac{1}{\sqrt{2\sqrt{2}+5}} \\
\frac{1}{\sqrt{2\sqrt{2}+5}} & \frac{-1-\sqrt[2]{2}}{\sqrt{2\sqrt{2}+5}} & \frac{1}{\sqrt{2\sqrt{2}+5}} \\
\frac{1}{\sqrt{2\sqrt{2}+5}} & -\frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}} & -\frac{1}{\sqrt{2\sqrt{2}+5}} \\
\frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}} & \frac{1}{\sqrt{2\sqrt{2}+5}} & \frac{1}{\sqrt{2\sqrt{2}+5}} \\
\frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}} & \frac{1}{\sqrt{2\sqrt{2}+5}} & -\frac{1}{\sqrt{2\sqrt{2}+5}} \\
\frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}} & -\frac{1}{\sqrt{2\sqrt{2}+5}} & \frac{1}{\sqrt{2\sqrt{2}+5}} \\
\frac{(1+\sqrt{2})}{\sqrt{2\sqrt{2}+5}} & -\frac{1}{\sqrt{2\sqrt{2}+5}} & -\frac{1}{\sqrt{2\sqrt{2}+5}}
\end{pmatrix}.$$

Each row V is of norm $\frac{1}{2}$. Additionally,we see the columns are orthogonal and of equal norm. Therefore, V is an isometry and we can conclude that the rows form a uniform Parseval frame that is a putative correlation minimizing frame. The scaled magnitude of the off diagonals of the Gram matrix are $\left|\frac{3-8\sqrt{2}}{17}\right| \approx .48904$, $\left|\frac{7+8\sqrt{2}}{17}\right| \approx .74452$, $\left|\frac{1}{2\sqrt{2}+5}\right| \approx .12774$. Since there is no equiangular (12,3)-frame, we can conclude that the Welch bound, $\sqrt{\frac{12-3}{3(12-1)}} = \sqrt{\frac{3}{11}} < C(12,3)$ while $C(12,3) \leq \frac{7+8\sqrt{2}}{17}$. $\square$

Theorem 2.3.10. *The putative* $(16,3)$ *optimal line packing is given by the lines generated by opposite vertices of the Biscribed Penatkis Dodecahedron. The set of vectors produced by scaling a set of opposite vertices to be unit norm is given by*

$$(0,c_0,\pm c_4)\,,(c_4,0,\pm c_0)\,,(c_0,\pm c_4,0)\,,(c_1,0,\pm c_3)\,,(c_3,\pm c_1,0)\,,(0,c_3,\pm c_1)\,,$$
$$(c_2,c_2,c_2)\,,(c_2,-c_2,-c_2)\,,(-c_2,c_2,-c_2)\,,(-c_2,-c_2,c_2)\,.$$

$c_0 = \frac{\sqrt{15}-\sqrt{3}}{6}$, $c_1 = \frac{\sqrt{10\left(5-\sqrt{5}\right)}}{10}$, $c_2 = \frac{\sqrt{3}}{3}$,$c_3 = \frac{\sqrt{10\left(5+\sqrt{5}\right)}}{10}$, *and* $c_4 = \frac{\sqrt{15}+\sqrt{3}}{6}$. *These vectors scaled by* $\frac{\sqrt{3}}{4}$ *yield a uniform (16,3)-frame. Moreover,* $\sqrt{\frac{13}{45}} < C(16,3) \leq \frac{\sqrt{10}}{60}(\sqrt{5+\sqrt{5}})(\sqrt{15}+\sqrt{3})$

Proof. Let the columns of W^* be the opposite vertices of the Biscribed Pentakis Dodecahedron with radius one centered at the origin in [30]. Set $c_0 = \frac{\sqrt{15}-\sqrt{3}}{6}$, $c_1 = \frac{\sqrt{10\left(5-\sqrt{5}\right)}}{10}$, $c_2 = \frac{\sqrt{3}}{3}$,$c_3 = \frac{\sqrt{10\left(5+\sqrt{5}\right)}}{10}$, and $c_4 = \frac{\sqrt{15}+\sqrt{3}}{6}$. It follows that,

$$W^* = \begin{pmatrix} 0 & 0 & c_4 & c_4 & c_0 & c_0 & c_1 & c_1 & c_3 & c_3 & 0 & 0 & c_2 & c_2 & -c_2 & -c_2 \\ c_0 & c_0 & 0 & 0 & c_4 & -c_4 & 0 & 0 & c_1 & -c_1 & c_3 & c_3 & c_2 & -c_2 & c_2 & -c_2 \\ c_4 & -c_4 & c_0 & -c_0 & 0 & 0 & c_3 & -c_3 & 0 & 0 & c_1 & -c_1 & c_2 & -c_2 & -c_2 & c_2 \end{pmatrix}.$$

By inspection we see that the columns of W^* are unit norm, the rows are equal norm and the rows are orthogonal. Hence, $V = \frac{\sqrt{3}}{4}W$ is an isometry and so its rows are a uniform $(16,3)$-frame. The scaled magnitude of the off diagonals of the Gram matrix are $\frac{1}{3}$, $\frac{\sqrt{5}}{3} \approx .74536$, $\frac{\sqrt{10}}{60}\sqrt{5+\sqrt{5}}(\sqrt{15}+\sqrt{3}) \approx .79465$, $\frac{\sqrt{10}}{60}\sqrt{5-\sqrt{5}}(\sqrt{15}-\sqrt{3}) \approx .18759$, and $\frac{\sqrt{10}}{10}\left(\sqrt{\sqrt{5}+3}-\sqrt{-\sqrt{5}+3}\right) \approx .44721$. Therefore, using the Welch bound, we can conclude $\sqrt{\frac{16-3}{3(16-1)}} = \sqrt{\frac{13}{45}} < C(16,3) \leq \frac{\sqrt{10}}{60}\sqrt{5+\sqrt{5}}(\sqrt{15}+\sqrt{3})$ $\square$

CHAPTER 3

Bounds on Correlation Minimizing frames

In this chapter, we study the asymptotic behavior of C(N,d). When $d = p^l$ is a power of a prime, then it is known that there exists a set of $p^l + 1$ mutually unbiased bases. The union of these vectors when appropriately scaled gives rise to a uniform Parseval frame $\mathcal{F}$ of $N = p^l(p^l + 1)$ vectors with $M_\infty(\mathcal{F}) = d^{-1/2}$, which is approximately $r^{-1/2}$ where $r = N/d$ is the frame redundancy. In this section we prove the stronger result that for fixed redundancy r, $C(N, d) \to 0$ as $N \to +\infty$.

3.0.1 Direct Sums and Tensor Products

Definition 3.0.11. Let A be an $m \times n$ matrix and B be a $p \times q$ matrix. Then the *direct sum* of A and B is denoted $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. The *Kronecker or tensor product* of A and B is the block matrix whose i,j block is $(a_{i,j}B)$ and is denoted $A \otimes B = (a_{i,j}B)_{i,j}$. If C is a $n \times r$ matrix and D is a $q \times p$ matrix, then $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.

The 2x2 rotation matrix $R_j = \begin{pmatrix} cos(\theta_j) & -sin(\theta_j) \\ sin(\theta_j) & cos(\theta_j) \end{pmatrix}$ rotates vectors in the xy-Cartesian plane counter clockwise by the angle θ_j. Given a vector $v \in \mathbb{R}^2$, one computes the rotated vector, by the matrix multiplication $R_j v$. The inverse, R_j^*, rotates the vector counter clockwise by $-\theta_j$ or clockwise by θ_j. Below we consider the rotation matrix $R_i^* R_j$, which rotates a vector in the counter clockwise direction by $\theta_i - \theta_j$. We observe that applying the rotation matrix k times to a vector simply rotates the vector counter clockwise by the angle $k(\theta_i - \theta_j)$.

Lemma 3.0.12. *Let R_j be the rotation matrix $R_j = \begin{pmatrix} cos(\theta_j) & -sin(\theta_j) \\ sin(\theta_j) & cos(\theta_j) \end{pmatrix}$ then*

$$(R_i^* R_j)^k = \begin{pmatrix} cos(k(\theta_i - \theta_j)) & -sin(k(\theta_i - \theta_j)) \\ sin(k(\theta_i - \theta_j)) & cos(k(\theta_i - \theta_j)) \end{pmatrix}$$

Proof. Let$R_j = \begin{pmatrix} cos(\theta_j) & -sin(\theta_j) \\ sin(\theta_j) & cos(\theta_j) \end{pmatrix}$. Then

$$\begin{aligned}
R_i^* R_j &= \begin{pmatrix} cos(\theta_i) & sin(\theta_i) \\ -sin(\theta_i) & cos(\theta_i) \end{pmatrix} \begin{pmatrix} cos(\theta_j) & -sin(\theta_j) \\ sin(\theta_j) & cos(\theta_j) \end{pmatrix} \\
&= \begin{pmatrix} cos(\theta_i)cos(\theta_j) + sin(\theta_i)sin(\theta_j) & cos(\theta_j)sin(\theta_i) - cos(\theta_i)sin(\theta_j) \\ -cos(\theta_j)sin(\theta_i) + cos(\theta_i)sin(\theta_j) & cos(\theta_i)cos(\theta_j) + sin(\theta_i)sin(\theta_j) \end{pmatrix} \\
&= \begin{pmatrix} cos(\theta_i - \theta_j) & -sin(\theta_i - \theta_j) \\ sin(\theta_i - \theta_j) & cos(\theta_i - \theta_j) \end{pmatrix}.
\end{aligned}$$

Now, assume $(R_i^* R_j)^k = \begin{pmatrix} cos(k(\theta_i - \theta_j)) & -sin(k(\theta_i - \theta_j)) \\ sin(k(\theta_i - \theta_j)) & cos(k(\theta_i - \theta_j)) \end{pmatrix}$. Then

$$\begin{aligned}
(R_i^* R_j)^{k+1} &= (R_i^* R_j)^k (R_i^* R_j) \\
&= \begin{pmatrix} cos(\theta_i)cos(\theta_j) + sin(\theta_i)sin(\theta_j) & cos(\theta_j)sin(\theta_i) - cos(\theta_i)sin(\theta_j) \\ -cos(\theta_j)sin(\theta_i) + cos(\theta_i)sin(\theta_j) & cos(\theta_i)cos(\theta_j) + sin(\theta_i)sin(\theta_j) \end{pmatrix} \\
&= \begin{pmatrix} cos((k+1)(\theta_i - \theta_j)) & -sin((k+1)(\theta_i - \theta_j)) \\ sin((k+1)(\theta_i - \theta_j)) & cos((k+1)(\theta_i - \theta_j)) \end{pmatrix}
\end{aligned}$$

□

Definition 3.0.13. Let A$\in M_n(\mathbb{R})$ then we let $\max\{A\}$ denote the maximum of the absolute values of the entries in A. This is also known as the ℓ_∞ norm of the matrix A, denoted as $\|A\|_\infty$.

Lemma 3.0.14. *Let $A, B \in M_n(\mathbb{R})$, then* $\max\{A \otimes B\} = \max A \cdot \max B$.

Proof. Let A and B $\in M_n(\mathbb{R})$ and $A = (a_{i,j})$. Then $A \otimes B = (a_{i,j}B)$. If $|a_{ij}| = \max A$ is the maximum entry in absolute values then the maximum element of $A \otimes B$ is in block of $(\max A)\, B$. Furthermore, if $|b_{kl}| = \max B$ is the maximum entry in B then $\max A \max B$ is the maximum element of $A \otimes B$. □

The direct sum of projection matrices is a projection matrix. Also, the Kronecker tensor of projection matrices is also a projection matrix.

3.0.2 Correlation Constant Bounds

Lemma 3.0.15. *Let k be a natural number greater than 1, fix N to be a natural number greater than 1. Then there exist unitary matrices, $U_1, \ldots, U_N \in M_{2^k}(\mathbb{R})$, such that*

$$\max\{U_i^* U_j\} \leq cos^k(\frac{\pi}{2N})$$

for all $i \neq j$, $1 \leq i, j \leq N$.

Proof. Let j be a natural number between 1 and N and R_j be the rotation matrix through the angle θ_j and set $\theta_j = \frac{j\pi}{2N}$. Define $U_j = \otimes_{l=1}^k (R_j)_l$.

Since R_j is unitary matrix and U_j is the tensor product of k unitary matrices it is also unitary. Lemma 3.0.12 gives us

$$U_i^* U_j = (\otimes_{l=1}^k (R_i)_l)^* (\otimes_{l=1}^k (R_j)_l) = (\otimes_{l=1}^k (R_i)^* (R_j))_l$$

is the tensor product of rotations.

Since N is greater than one, cosine is decreasing and is greater than sine values in $[0, \frac{\pi}{4}]$ the max of $R_i^* R_j$ is $cos(\frac{(i-j)\pi}{2N})$, which is less than or equal to $cos(\frac{\pi}{2N})$.

We now will proceed by induction on k.

For k=2,

$$\begin{aligned}\max\{U_i^*U_j\} &= \max\{\otimes_{l=1}^2 (R_i^*R_j)_l\}\\ &= cos(\frac{(i-j)\pi}{2N})cos(\frac{(i-j)\pi}{2N})\\ &\leq cos^2(\frac{\pi}{2N})\end{aligned}$$

Assume, for k $\max\{U_i^*U_j\} \leq cos^k(\frac{\pi}{2N})$. So for k+1 $U_i^*U_j = \otimes_{l=1}^{k+1}(R_i^*R_j)_l$. Therefore,

$$\begin{aligned}\max\{U_i^*U_j\} &= \max\{\otimes_{l=1}^k (R_i^*R_j)_l \otimes (R_i^*R_j)\}\\ &= cos^k(\frac{(i-j)\pi}{2N})cos(\frac{(i-j)\pi}{2N})\\ &\leq cos^{k+1}(\frac{\pi}{2N})\end{aligned}$$

□

Theorem 3.0.16. *For k a natural number greater than or equal to 1,*

$$C(2^kN, 2^kd) \leq cos^k(\frac{\pi}{2N})C(N,d)$$

Proof. Let P_F=$(p_{i,j})$ be the NxN projection matrix that is the Grammian of uniform (N,d)-frame $\mathcal{F}$ with correlation constant C(N,d). Define P=$P_F \otimes I_{2^k}$=$(p_{i,j}I_{2^k})$. Since the tensor product of projections is a projection, P is a projection. So P is the Grammian matrix of a $(2^kN, 2^kd)$ Parseval frame with correlation constant $C(2^kN, 2^kd)$. By Theorem 3.0.15 there exists $U_1, \ldots, U_N$ unitary matrices in $M_{2^k}(\mathbb{R})$ where for $i \neq j$ $\max\{U_i^*U_j\} \leq cos^k(\frac{\pi}{2N})$.

Define $U{=}\oplus_{l=1}^{N}(U_i)_l$. So $U^*PU{=}(p_{ij}U_i^*U_j)_l$ is unitary equivalent to P and therefore has equal rank. Then, $C(2^kN, 2^kd) = C(N, d)\max\{U_i^*U_j\}$ with Lemma 3.0.15 applied yields $C(2^kN, 2^kd) \leq C(N, d)cos^k(\frac{\pi}{2N})$.

□

Corollary 3.0.17. *For fixed redundancy, $r = N/d$ the correlation constant $C(2^kN, 2^kd)$ goes to zero as k goes to infinity.*

Proof. For k in the natural numbers we have

$$0 \leq C(2^kN, 2^kd) \leq cos^k(\frac{\pi}{2N})C(N, d).$$

Since $cos(\frac{\pi}{2N}){<}1$ and C(N,d) is a constant, $cos^k(\frac{\pi}{2N})C(N, d)$ goes to zero as k goes to infinity. So $C(2^kN, 2^kd)$ goes to zero as k goes to infinity. □

3.1 Correlation Minimization for Circulant Matrices

Circulant matrices have many applications and are a well known family of matrices. Their basic properties and applications in pure mathematics use linear algebra, abstract algebra, geometry, and the discrete Fourier transform and can be found in [12]. Real world applications range from wireless communications in [27] to hardware complexity [14]. For our purposes, we will consider only a subset of circulant matrices with the property that they are also projection matrices. Since these projection matrices are also the grammian of a Parseval frame, we can establish bounds on the correlation constant of correlation minimizing frame, which we will denote $C_{circ}(N,d)$. Using this upper bound, we then study the asymptotic behavior of $C_{circ}(N,d)$ and the Welch bound.

3.2 Circulant Matrices as Projections

We begin with the definition and useful properties of the subset of circulant matrices that are projections.

Definition 3.2.1. A circulant matrix is defined as

$$C = \begin{bmatrix} c_0 & c_{N-1} & c_{N-2} & \cdots & \cdots & c_1 \\ c_1 & c_0 & c_{N-1} & \cdots & \cdots & c_2 \\ \vdots & \vdots & \ddots & \ddots & & c_3 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ c_{N-1} & c_{N-2} & c_{N-3} & \cdots & \ddots & c_0 \end{bmatrix} = circ(c_0, \ldots, c_{N-1})$$

To analyze the structure of these circulant matrices that are projections, we will look at the relationship between matrices in NxN circulant matrices and the discrete Fourier transform matrix.

Definition 3.2.2. Given $N \in \mathbb{N}$, the discrete Fourier transform (DFT) matrix is defined by $F = \frac{1}{\sqrt{N}}(e^{\frac{2\pi ijl}{N}})_{j,l=0}^{N-1}$.

Proposition 3.2.3. *If C is an $N \times N$ circulant matrix, then C can be written as $C = F\Sigma F^*$, where F is the discrete Fourier transform matrix and Σ is a diagonal matrix with the diagonal containing the eigenvalues of C. Conversely, every matrix of the form $F\Sigma F^*$ for some diagonal matrix Σ is circulant.*

$F\Sigma F^*$ is called the *spectral decomposition* of the circulant matrix C. We will denote the set of NxN rank d circulant matrices by $C^{(N,d)}$. Since $C \in C^{(N,d)}$ is uniquely determined by its spectral decomposition, and more specifically Σ, we will identify C by its spectral decomposition. That is, given an ordered set $\Lambda = \{\lambda_0, \dots, \lambda_{N-1}\}$, let $\Sigma = diag\,\{\lambda_0, \dots, \lambda_{N-1}\}$ and set $C = F^*\Sigma F$. Now, consider the specific subset of circulant matrices that are projections, with notation $C_P^{(N,d)}$. We identify a matrix $C = circ(c_0, \dots, c_{N-1}) = F^*\Sigma F \in C^{(N,d)}$ by its spectral decomposition with $\Sigma = diag\,\{\lambda_0, \dots, \lambda_{N-1}\}$. However, for $C \in C_P^{(N,d)}$ the ordered set $\{\lambda_0, \dots, \lambda_{N-1}\}$ that become the diagonals of Σ, are a combination of zeros and ones that will be assigned as follows. Given a subset $S \in \{0, \dots, N-1\}$, $|S| = d$, define $\lambda_j = \begin{cases} 1 & j \in S \\ 0 & \text{otherwise} \end{cases}$, for $j = 0, \dots, N-1$. From Proposition 3.2.3 we see that there is a one to one correspondence between $C \in C_P^{(N,d)}$ and subsets of $\{0, \dots, N-1\}$ with cardinality d given by $S \to F\Sigma_S F^*$. Now, using well known properties of diagonalized matrices, matrix algebra, and the previous lemma, we find an equation for the entries in a circulant matrix that is a projection.

Definition 3.2.4. Given $N, d \in \mathbb{N}$, $N > d$, let $C = circ(c_0, \dots, c_{N-1}) \in C_P^{(N,d)}$. Then the constant $C_{circ}(N,d) = \min\limits_{C \in C_P^{(N,d)}} \max\limits_{0<j\leq N-1} |c_j|$.

Remark 3.2.5. When computing or bounding the correlation constant, C(N,d), for complex (N,d)-frames we have $C(N,d) \leq C_{circ}(N,d)$

3.2.1 An upper bound for $C_{circ}(N, d)$

In this section we will consider the circulant projection matrix

$$C = F^*\Sigma F = circ(c_0, \dots, c_{N-1}) \in C_P^{(N,d)}$$

uniquely determined by $\Sigma = diag(\lambda_0, \dots, \lambda_{N-1})$ with $\lambda_j = \begin{cases} 1 & 0 \leq j \leq d-1 \\ 0 & \text{otherwise} \end{cases}$.

To begin, we will need some basic definitions and properties or the Fejer and Dirichlet kernel.

Definition 3.2.6. Let n be a natural number. Then the *Dirichlet kernel* is

$$D_m(x) = \sum_{j=-m}^{m} e^{xij}$$

and the *Fejer Kernel* is

$$F_M(x) = \sum_{|j|=M} (1 - \frac{|j|}{M}) e^{xij} = \sum_{n=0}^{M} D_m(x)$$

Lemma 3.2.7. *Let n and N be natural numbers. Then the Dirichlet Kernel has closed form*

$$D_m(x) = \frac{\sin((2m+1)\frac{x}{2})}{sin(\frac{x}{2})}$$

and the Fejer Kernel has closed form

$$F_M(x) = \frac{1}{M} \frac{\sin^2(\frac{Mx}{2})}{sin^2(\frac{x}{2})}.$$

Furthermore, for $M > 1$ odd,

$$F_M(x) = \frac{1}{M} (D_{\frac{M-1}{2}}(x))^2$$

Lemma 3.2.8. *Let N and d be natural numbers with $N > d$ and d odd. Let $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$ with $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ defined by $\lambda_j = \begin{cases} 1 & 0 \leq j \leq d-1 \\ 0 & otherwise \end{cases}$. Then $|c_k| = \frac{\sqrt{d}}{N}|D_{\frac{d-1}{2}}\left(\frac{2\pi k}{N}\right)|$.*

Proof. Let N and d be natural numbers with $N > d$ and d odd. Set $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$ with $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ defined by $\lambda_j = \begin{cases} 1 & 0 \leq j \leq d-1 \\ 0 & \text{otherwise} \end{cases}$ and $\omega_k = e^{\frac{2\pi ik}{N}}$. Then,

$$\begin{aligned}
|c_k|^2 &= \frac{1}{N^2}\sum_{l,j=0}^{d-1} e^{\frac{2\pi ik(j-l)}{N}} \\
&= \frac{1}{N^2}\left(d + (d-1)\omega_k + (d-2)\omega_k^2 + \ldots + \omega_k^{d-1} + (d-1)\omega_k + (d-2)\omega_k^2 + \ldots + \omega_k^{d-1}\right) \\
&= \frac{d}{N^2}\sum_{|j|\leq d}(1 - \frac{|j|}{d})\omega_k^j = d^2\frac{1}{Nd}\sum_{|j|=d}(1 - \frac{|j|}{d})\omega_k^j \\
&= \frac{d^2}{N^2}\frac{1}{d}\sum_{|j|=d}(1 - \frac{|j|}{d})(e^{\frac{2\pi ik}{N}})^j \\
&= \frac{d^2}{N^2}F_d(\frac{2\pi k}{N}) \\
&= d^2\frac{1}{N^2 d}(D_{\frac{d-1}{2}}(\frac{2\pi k}{N}))^2.
\end{aligned}$$

Therefore, $|c_k| = \frac{\sqrt{d}}{N}|D_{\frac{d-1}{2}}(\frac{2\pi k}{N})|$. $\square$

Remark 3.2.9. Before continuing, we review a few key facts about the maximum and minimum of the Dirichlet Kernel. While these facts are not new, in our case they are not specifically addressed in the literature.

3.2.2 The Dirichlet kernel

The Dirichlet kernel has zeros every $\frac{2j\pi}{2m+1}$ for $j = 1, \ldots, 2m$. Between consecutive zeros, $D_m(x)$ must have at least one local extrema, making a total of 2m local extrema. However, $D'_m(x)$ is a trigonometric polynomial of degree m, so it can have at most 2m roots. Therefore, we can conclude that there is exactly one local extrema between each pair of consecutive zeros. For our purposes we are only interested in the unique maximum and minimum and where they occur. The unique maximum of $D_m(x)$ is in $\left[\frac{-2\pi}{2m+1}, \frac{2\pi}{2m+1}\right]$ at x=0. Let $x_0 = 0$ and x_j denote the unique critical point of $D_m(x)$ in the interval $\left[\frac{2j\pi}{2m+1}, \frac{2(j+1)\pi}{2m+1}\right]$, for $j = 1, \ldots, 2m-1$. The Dirichlet kernel lies between the envelopes $\frac{1}{sin(x/2)}$ and $-\frac{1}{sin(x/2)}$ and is tangent to one or the other at the points $\frac{\pi(2j-1)}{2m+1}$ for $j = 1, \ldots, 2m+1$, the second point produces the minimum on the interval $\left[\frac{2\pi}{2m+1}, \frac{4\pi}{2m+1}\right]$ and the first giving the absolute value of the minimum. Also, on the interval $0 < x \leq \pi$, $1/sin(x/2)$ is decreasing. Hence, it follows that for $1 \leq j \leq m$

$$|D_m(x_{j-1})| > |D_m(x_{j-1} - \frac{2\pi}{2m+1})| > |D_m(x_j)|. \tag{3.1}$$

Given the relationship established between $|c_k|$ and the Dirichlet kernel for d odd and the nice properties of the maximum and minimum and critical points of $D_m(x)$, we will give an upper bound on C(N,d) by looking at NxN circulant matrices that are projections with odd rank d.

In addition to these facts, we need the following lemma.

Lemma 3.2.10. *Let d be an odd natural number. For* $0 \leq y \leq \frac{\pi}{d}$*,*

$$D_{\frac{d-1}{2}}(y) > \left| D_{\frac{d-1}{2}}\left(\frac{3\pi}{d}\right) \right|.$$

Proof. Let d be an odd natural number. Using the definition above, we know $D_{\frac{d-1}{2}}(x)$ has a maximum at x=0 and is strictly decreasing on the interval $\left[0, \frac{2\pi}{d}\right]$, where $x = \frac{2\pi}{d}$ is the first positive zero. Clearly, the interval $\left[0, \frac{\pi}{d}\right] \subset \left[0, \frac{2\pi}{d}\right]$. Hence $D_{\frac{d-1}{2}}(x)$ is strictly decreasing on $\left[0, \frac{\pi}{d}\right]$. Furthermore, we know from the above properties that the minimum of $D_{\frac{d-1}{2}}(x)$ is at $x = \frac{3\pi}{d}$ and $|D_{\frac{d-1}{2}}(x)|$ is strictly increasing on $\left[\frac{2\pi}{d}, \frac{3\pi}{d}\right]$.

Since $|sin\left(\frac{\pi}{2d}\right)| < |sin\left(\frac{3\pi}{2d}\right)|$, we have

$$\left|D_{\frac{d-1}{2}}\left(\frac{\pi}{d}\right))\right| = \frac{1}{sin(\frac{\pi}{2d})} > \frac{1}{|sin(\frac{3\pi}{2d})|} = \left|D_{\frac{d-1}{2}}\left(\frac{3\pi}{d}\right)\right|.$$

Therefore, we can conclude for $0 \leq y \leq \frac{\pi}{d}$, $D_{\frac{d-1}{2}}(y) > \left|D_{\frac{d-1}{2}}\left(\frac{3\pi}{d}\right)\right|$.

□

3.2.3 Application of Dirichlet Kernel in bounding the Correlation Constant

Proposition 3.2.11. *Let N and d be natural numbers with $N > d$ and d odd, where $\frac{N}{d} > 2$. Let $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$ with $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ defined by* $\lambda_j = \begin{cases} 1 & 0 \leq j \leq d-1 \\ 0 & otherwise \end{cases}$. *Then $|c_1| \geq |c_k|$, for $1 < k \leq N-1$.*

Proof. Let N and d be natural numbers, d odd, where $\frac{N}{d} > 2$. Define $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$ by $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ where $\lambda_j = \begin{cases} 1 & 0 \leq j \leq d-1 \\ 0 & \text{otherwise} \end{cases}$.
From Lemma 3.3.1, for $0 \leq k \leq N-1$, $|c_k| = \frac{\sqrt{d}}{N}|D_{\frac{d-1}{2}}(\frac{2\pi k}{N})|$. To apply Lemma 3.2.10 we need $0 \leq \frac{2\pi k}{N} < \frac{\pi}{d}$. On this interval k=1 gives the smallest x value and therefore

largest value of $D_{\frac{d-1}{2}}(\frac{2\pi k}{N})$. Hence, we need $\frac{2\pi}{N} < \frac{\pi}{d}$, so $2 < \frac{N}{d}$. We have, for $2 < \frac{N}{d}$, $\sqrt{d}|D_{\frac{d-1}{2}}(\frac{2\pi k}{N})|$ is largest at k=1. Given the decreasing local extrema property of the absolute value of the Dirichlet kernel, for $1 < k \leq N-1$, $|c_1| \geq |c_k|$.

$\square$

Corollary 3.2.12. *Let N and d be natural numbers, d odd, where $\frac{N}{d} > 2$. Then*

$$\frac{1}{\sqrt{d}}|D_{\frac{d-1}{2}}(\frac{2\pi}{N})| \geq C_{circ}(N,d).$$

Proof. Let N and d be natural numbers, d odd, where $\frac{N}{d} > 2$. If $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$ with $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ defined by $\lambda_j = \begin{cases} 1 & 0 \leq j \leq d-1 \\ 0 & \text{otherwise} \end{cases}$, then $\frac{N}{d}\max_k |c_k| \geq C_{circ}(N,d)$. From the previous proposition we have $|c_1| \geq |c_k|$, for $1 < k \leq N-1$. Therefore, we can conclude $\frac{1}{\sqrt{d}}|D_{\frac{d-1}{2}}(\frac{2\pi}{N})| \geq C_{circ}(N,d)$ $\square$

Corollary 3.2.12 provides an bound on $C_{circ}(N,d)$. Now, we determine how asymptotically close $C_{circ}(N,d)$ will get to the Welch bound. Going forward, we consider the case of where d is a fixed odd natural number and N is a natural number that will vary with the condition that $\frac{N}{d} > 2$.

Proposition 3.2.13. *Let N and d be natural numbers, d > 1 odd, where $\frac{N}{d} > 2$. Then*

$$\lim_{N\to\infty} \frac{C_{circ}(N,d)}{W(N,d)} = d.$$

Proof. Let N and d be natural numbers, d odd, where $\frac{N}{d} > 2$. Proposition 3.2.12

gives a bound on $C_{circ}(N,d)$. Hence,

$$\frac{C_{circ}(N,d)}{W(N,d)} \leq \frac{\frac{1}{\sqrt{d}}|D_{\frac{d-1}{2}}(\frac{2\pi}{N})|}{\sqrt{\frac{N-d}{d(N-1)}}} = \sqrt{\frac{N-1}{N-d}}\frac{sin\left(\frac{d\pi}{N}\right)}{sin\left(\frac{\pi}{N}\right)}. \tag{3.2}$$

Then, using l'Hospital's rule to take the limit we have, the limit of the right hand side of 3.2 goes to d. □

Now, we consider the ratio

$$\frac{C_{circ}(N,d)}{W(N,d)} \leq \sqrt{\frac{N-1}{N-d}}\frac{sin\left(\frac{d\pi}{N}\right)}{sin\left(\frac{\pi}{N}\right)}$$

as a function of x. Define $N = \frac{1}{x}$. Then as N goes to infinity x will go to zero. Consequently, we consider the function $f(x) = \sqrt{\frac{1-x}{1-dx}}\frac{sin(d\pi x)}{sin(\pi x)}$, where $0 < x \leq \frac{1}{d+1}$.

To determine the behavior of $f(x)$ on the interval $\left(0, \frac{1}{d+1}\right]$, we start by looking the first derivative. This will determine when f(x) is increasing and when f(x) is decreasing on $\left(0, \frac{1}{d+1}\right]$. Using the chain rule, we obtain

$$f'(x) = \frac{1}{2}\left(\frac{1-x}{1-dx}\right)^{-\frac{1}{2}} \frac{d-1}{(1-dx)^2}\frac{\sin(d\pi x)}{\sin(\pi x)} + \left(\frac{1-x}{1-dx}\right)^{\frac{1}{2}} \frac{\pi d\cos(d\pi x)\sin(\pi x) - \pi\sin(d\pi x)\cos(\pi x)}{\sin^2(\pi x)}.$$

First, we define

$$h(x) = \pi d\cos(d\pi x)\sin(\pi x) - \pi\sin(d\pi x)\cos(\pi x).$$

Notice $h(0) = 0$ and

$$h'(x) = -(d^2-1)\sin(d\pi x)\sin(\pi x)$$

is negative for all $x \in \left(0, \frac{1}{d+1}\right]$. Thus, h(x) is negative on $\left(0, \frac{1}{d+1}\right]$. Given this fact, it will be useful to rewrite $f'(x)$.

$$f'(x) = \left(\frac{1-x}{1-dx}\right)^{\frac{1}{2}} \left(\frac{(d-1)\sin(d\pi x)\sin(\pi x) + 2(1-x)(1-dx)h(x)}{2(1-dx)(1-x)\sin^2(\pi x)}\right).$$

Now, we see that both $\left(\frac{1-x}{1-dx}\right)^{\frac{1}{2}}$ and the denominator are positive on $\left(0, \frac{1}{d+1}\right]$. Therefore, we turn our attention to the sign of the numerator. To begin, we observe that the numerator,

$$g(x) = (d-1)\sin(d\pi x)\sin(\pi x) + 2(1-x)(1-dx)h(x)$$

is zero at x=0. As before, we take the derivative of $g(x)$. That is,

$$g'(x) = (d-1)h(x) - 2(1-dx)h(x) - 2d(1-x)h(x) + h'(x).$$

We know that $h'(x)$ is negative. Hence, only the first term is what needs to be considered. Now, rewrite the first term of the numerator as,

$$(d-1)h(x) - 2(1-dx)h(x) - 2d(1-x)h(x) = h(x)(-3-d+4dx).$$

Recall that h(x) is negative on $\left(0, \frac{1}{d+1}\right]$. Hence the second term is negative on $\left(\frac{d+3}{4d}, \frac{1}{d+1}\right]$ and positive on $\left(0, \frac{d+3}{4d}\right)$. Moreover, we now know the numerator is negative on the interval $\left(\frac{d+3}{4d}, \frac{1}{d+1}\right)$ and may not be negative on $\left(0, \frac{d+3}{4d}\right)$. This leads to the following properties of f(x).

Lemma 3.2.14. *Let* $f(x) = \sqrt{\frac{1-x}{1-dx}}\frac{sin(d\pi x)}{sin(\pi x)}$, *with* $0 < x \le \frac{1}{d+1}$. *Then there exists a unique* $x^* \in \left(\frac{1}{4d}, \frac{1}{3d}\right)$ *such that f(x) is increasing on* $(0, x^*)$ *and decreasing on* $\left(x^*, \frac{1}{d+1}\right]$

Proof. Let $f(x) = \sqrt{\frac{1-x}{1-dx}}\frac{sin(d\pi x)}{sin(\pi x)}$ and $x \in \left(0, \frac{1}{d+1}\right]$. Since $x = \frac{1}{3d} \in \left(\frac{d+3}{4d}, \frac{1}{d+1}\right)$, we know $f'\left(\frac{1}{3d}\right) < 0$. Hence f(x) is decreasing on $\left(\frac{1}{3d}, \frac{1}{d+1}\right]$. The case where $x = \frac{1}{4d}$ will need to be addressed more precisely. We proceed by computing the numerator of $f'(x)$. Recall the numerator of f(x),

$$g\,(x) = (-3 - d + 4dx)h(x) + h'(x).$$

Plugging in $x = \frac{1}{4d}$ we have,

$$\begin{aligned} g\left(\frac{1}{4d}\right) &= \left(-3 - d + 4d\frac{1}{4d}\right) h\left(\frac{1}{4d}\right) + h'\left(\frac{1}{4d}\right) \\ &= \frac{-\pi\sqrt{2}}{2}\left((d+2)\left(dsin\left(\frac{\pi}{4d}\right) - cos\left(\frac{\pi}{4d}\right)\right) - (d^2-1)\pi sin\left(\frac{\pi}{4d}\right)\right) \\ &> 0. \end{aligned}$$

Therefore, $f'(\frac{1}{4d}) > 0$. Moreover, since $f'(\frac{1}{3d}) < 0$ and $f'(\frac{1}{4d}) > 0$, there exists at least one $x^* \in \left(\frac{1}{4d}, \frac{1}{3d}\right)$ such that $f'(x^*) = 0$. To get the uniqueness, we need only look at the second derivative on the interval on $\left[\frac{1}{4d}, \frac{1}{3d}\right]$.

$$\begin{aligned} f''(x) = &-\frac{(d-1)^2 sin(d\pi x)}{4(1-x)^{\frac{3}{2}}(1-dx)^{\frac{5}{2}}sin(\pi x)} + \frac{(d-1)d\pi cos(d\pi x)}{(1-x)^{\frac{1}{2}}(1-dx)^{\frac{3}{2}}sin(\pi x)} \\ &- \frac{(d-1)\pi sin(d\pi x)cos(\pi x)}{(1-x)^{\frac{1}{2}}(1-dx)^{\frac{3}{2}}sin^2(\pi x)} + \frac{d(d-1)sin(d\pi x)}{(1-x)^{\frac{1}{2}}(1-dx)^{\frac{5}{2}}sin(\pi x)} \\ &- \left(\frac{1-x}{1-dx}\right)^{\frac{1}{2}}\frac{d^2\pi^2 sin(d\pi x)}{sin(\pi x)} - \left(\frac{1-x}{1-dx}\right)^{\frac{1}{2}}\frac{2d\pi^2 cos(d\pi x)cos(\pi x)}{sin^2(\pi x)} \\ &+ \left(\frac{1-x}{1-dx}\right)^{\frac{1}{2}}\frac{2\pi^2 sin(d\pi x)cos^2(\pi x)}{sin^3(\pi x)} \\ &+ \left(\frac{1-x}{1-dx}\right)^{\frac{1}{2}}\frac{\pi^2 sin(d\pi x)}{sin(\pi x)}. \end{aligned}$$

Combining all the terms, the denominator becomes

$$4(1-x)^{\frac{3}{2}}(1-dx)^{\frac{5}{2}}sin^3(\pi x).$$

For $x \in \left[\frac{1}{4d}, \frac{1}{3d}\right]$, πx is less than $\frac{\pi}{2}$, thus $sin^3(\pi x) > 0$. Since $x \in \left[\frac{1}{4d}, \frac{1}{3d}\right]$ is always less than 1 and $\frac{1}{4} \leq dx \leq \frac{1}{3}$, we can conclude

$$4(1-x)^{\frac{3}{2}}(1-dx)^{\frac{5}{2}}sin^3(\pi x) > 0.$$

Now, we turn our attention to the numerator. To determine that the numerator is negative, we will need a couple of facts. When $x \in \left[\frac{1}{4d}, \frac{1}{3d}\right]$ with $d > 1$ and odd, πx will be less than $\frac{\pi}{4}$. Hence $cos(\pi x) > sin(\pi x)$. Furthermore, $d\pi x \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$. So, it follows that $sin(d\pi x) > cos(d\pi x)$.

The numerator is,

$$\begin{gathered}
-(d-1)^2 sin(d\pi x)sin^2(\pi x) + 4\pi d(d-1)(1-x)(1-dx)sin^2(\pi x)cos(d\pi x) \\
-4\pi(d-1)(1-x)(1-dx)sin(d\pi x)sin(\pi x)cos(\pi x) + 4d(d-1)(1-x)sin(d\pi x)sin^2(\pi x) \\
-4\pi^2 d^2(1-x)^2(1-dx)^2 sin(d\pi x)sin^2(\pi x) - 8\pi^2 d(1-x)^2(1-dx)^2 cos(d\pi x)cos^2(\pi x) \\
+8\pi^2(1-x)^2(1-dx)^2 cos^2(\pi x)sin(d\pi x) + 4\pi^2(1-x)^2(1-dx)^2 sin(d\pi x)sin^2(\pi x), \\
= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8.
\end{gathered}$$

We will proceed by grouping the terms and determine that each grouping is negative.

First, we consider $T_2 + T_3$.

$$\begin{aligned} T_2 + T_3 &= 4\pi d(d-1)(1-x)(1-dx)sin^2(\pi x)cos(d\pi x) \\ &\qquad - 4\pi(d-1)(1-x)(1-dx)sin(d\pi x)sin(\pi x)cos(\pi x) \\ &= 4\pi(d-1)(1-x)(1-dx)sin(\pi x)(dcos(d\pi x)sin(\pi x) - sin(d\pi x)cos(\pi x)) \\ &= 4\pi(d-1)(1-x)(1-dx)sin(\pi x)\frac{h(x)}{\pi} \\ &= 4(d-1)(1-x)(1-dx)sin(\pi x)h(x) \end{aligned}$$

Clearly, the first 5 factors are positive. From the proof of Proposition 3.2.13, h(x) is negative on $\left(0, \frac{1}{d+1}\right]$. Hence, we conclude that $T_2 + T_3 < 0$.

Now, we consider $T_6 + T_7$.

$$\begin{aligned} T_6 + T_7 &= -8\pi^2 d(1-x)^2(1-dx)^2 cos(d\pi x)cos^2(\pi x) + \\ &\qquad 8\pi^2(1-x)^2(1-dx)^2 cos^2(\pi x)sin(d\pi x) \\ &= 8\pi^2 d(1-x)^2(1-dx)^2 cos^2(\pi x)(-dcos(d\pi x) + sin(d\pi x)) \end{aligned}$$

The first 6 factors are positive. Hence we need to show $-dcos(d\pi x) + sin(d\pi x)$ is negative. Plugging in $x = \frac{1}{4d}$, we have $-d\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} < 0$. At $x = \frac{1}{3d}$, we get $\frac{-d}{2} + \frac{\sqrt{3}}{2}$. Since $d > 1$ and odd $\frac{-d}{2} + \frac{\sqrt{3}}{2} < 0$. Furthermore, the first derivative, $d^2\pi sin(d\pi x) + d\pi cos(d\pi x)$ is strictly positive. So we can conclude $-dcos(d\pi x) + sin(d\pi x)$ is negative on $\left[\frac{1}{4d}, \frac{1}{3d}\right]$,as well as $T_6 + T_7 < 0$.

Finally, we look at $T_1 + T_4 + T_5 + T_8$.

$$\begin{aligned}
T_1 + T_4 + T_5 + T_8 = &-(d-1)^2 sin(d\pi x) sin^2(\pi x) + 4d(d-1)(1-x) sin(d\pi x) sin^2(\pi x) \\
&- 4\pi^2 d^2 (1-x)^2 (1-dx)^2 sin(d\pi x) sin^2(\pi x) \\
&+ 4\pi^2 (1-x)^2 (1-dx)^2 sin(d\pi x) sin^2(\pi x) \\
= &(d-1) sin(d\pi x) sin^2(\pi x)(-(d-1) + 4d(1-x)) \\
&+ 4\pi^2 (1-x)^2 (1-dx)^2 sin(d\pi x) sin^2(\pi x)(d^2-1)
\end{aligned}$$

Hence,

$$T_1+T_4+T_5+T_8 = (d-1) sin(d\pi x) sin^2(\pi x)(-4\pi^2(1-x)^2(1-dx)^2(d+1)+3d+1-4dx)$$

The first 3 factors are positive. Which leaves the sign of

$$k(x) = -4\pi^2(1-x)^2(1-dx)^2(d+1) + 3d + 1 - 4dx$$

to be determined. To conclude this is less than zero, we begin by looking at the values at $x = \frac{1}{4d}$ and $\frac{1}{3d}$. At $x = \frac{1}{4d}$ we have

$$\begin{aligned}
k\left(\frac{1}{4d}\right) &= -4\pi^2\left(1 - \frac{1}{4d}\right)^2\left(1 - d\frac{1}{4d}\right)^2(d+1) + 3d + 1 - 4d\frac{1}{4d} \\
&= \frac{(-144\pi^2 + 192)d^3 - 72\pi^2 d^2 + 62\pi^2 d - 9\pi^2}{64d^2} \\
&< 0.
\end{aligned}$$

At $x = \frac{1}{3d}$ we have

$$\begin{aligned}
k\left(\frac{1}{3d}\right) &= -4\pi^2\left(1 - \frac{1}{3d}\right)^2\left(1 - d\frac{1}{3d}\right)^2(d+1) + 3d + 1 - 4d\frac{1}{3d} \\
&= \frac{(-144\pi^2 + 243)d^3 - 75\pi^2 d^2 + 80\pi^2 d - 16\pi^2}{81d^2} \\
&< 0
\end{aligned}$$

Also,

$$k'(x) = 8\pi^2(d+1)(1-x)(1-dx)^2 + 8\pi^2 d(d+1)(1-x)(1-dx)^2 - 4d > 0.$$

So $k(x) < 0$ on $\left[\frac{1}{4d}, \frac{1}{3d}\right]$., which means $T_1 + T_4 + T_5 + T_8 < 0$.

Therefore the numerator is less than zero for all $x \in \left[\frac{1}{4d}, \frac{1}{3d}\right]$. Therefore, $f'(x)$ has exactly one zero, $x^* \in \left[\frac{1}{4d}, \frac{1}{3d}\right]$, which proves the claim.

□

The previous proof also yields that $f(x)$ is concave down on $\left(0, \frac{1}{d+1}\right]$. This prompts the following lemma.

Lemma 3.2.15. *Let d be an odd natural number greater than 1. Then* $f(x) = \sqrt{\frac{1-x}{1-dx}\frac{sin(d\pi x)}{sin(\pi x)}}$ *is strictly increasing near zero on* $\left(0, \frac{1}{d+1}\right]$.

Proof. Let d be an odd natural number greater than one and $f(x) = \sqrt{\frac{1-x}{1-dx}\frac{sin(d\pi x)}{sin(\pi x)}}$. Taking the derivative of f(x), we have

$$f'(x) = \frac{d-1}{2(1-x)^{\frac{1}{2}}(1-dx)^{\frac{3}{2}}}\frac{\sin(d\pi x)}{\sin(\pi x)} + \left(\frac{1-x}{1-dx}\right)^{\frac{1}{2}} \frac{\pi d\cos(d\pi x)\sin(\pi x) - \pi\sin(d\pi x)\cos(\pi x)}{\sin^2(\pi x)}.$$

Then, using L'Hopsital's rule two times, we have $f'(0) = \lim_{x\to 0}\frac{d(d-1)}{2} > 0$. Hence, f(x) is increasing near zero. □

This lemma allows us to get a better approximation of $\frac{C_{circ}(N,d)}{W(N,d)}$, for N sufficiently large.

Corollary 3.2.16. *Let N and d be natural numbers, d odd, where $\frac{N}{d} > 2$. Then, for N large enough $\frac{C_{circ}(N,d)}{W(N,d)} \leq d + \frac{d(d-1)}{2N}$.*

Proof. Let N and d be natural numbers, d odd, where $\frac{N}{d} > 2$. Using the previous lemma, for large enough N, $f\left(\frac{1}{N}\right) \geq d$. Furthermore, since $f'(x)$ is decreasing, the maximum value of $f'(x)$ will be as x approaches 0, which is $\frac{d(d-1)}{2}$. Thus we can write, for sufficiently large N,

$$d \leq f\left(\frac{1}{N}\right) = f(x) \leq f(0) + \frac{d(d-1)}{2N} = d + \frac{d(d-1)}{2N}.$$

□

Now, we consider the case where N is small.

Corollary 3.2.17. *Let $f(x) = \sqrt{\frac{1-x}{1-dx}\frac{sin(d\pi x)}{sin(\pi x)}}$. Then as $x^- \to \frac{1}{d+1}$, f(x) decreases to $\sqrt{d}$.*

Proof. Let $f(x) = \sqrt{\frac{1-x}{1-dx}\frac{sin(d\pi x)}{sin(\pi x)}}$. From Lemma 3.2.14, f(x) decreases on $\left(\frac{1}{3d}, \frac{1}{d+1}\right]$. Furthermore, since $sin\left(\frac{d\pi}{d+1}\right) - sin\left(\frac{\pi}{d+1}\right) = 2cos\left(\frac{\frac{d\pi}{d+1}+\frac{\pi}{d+1}}{2}\right) sin\left(\frac{\frac{d\pi}{d+1}-\frac{\pi}{d+1}}{2}\right) = 0$, we get that $sin\left(\frac{d\pi}{d+1}\right) = sin\left(\frac{\pi}{d+1}\right)$. Therefore,

$$\lim_{x \to \frac{1}{d+1}} \sqrt{\frac{1-x}{1-dx}\frac{sin(d\pi x)}{sin(\pi x)}} = \sqrt{\frac{1-\frac{d\pi}{d+1}}{1-\frac{\pi}{d+1}}\frac{sin(d\pi x)}{sin(\pi x)}} = \sqrt{d}$$

□

Initially, we defined $N = \frac{1}{x}$, so $x = \frac{1}{N}$. Then using the previous Corollary, we can conclude that as $\frac{N}{d} \to 1$,

$$\frac{C(N,d)}{W(N,d)} \leq \frac{C_{circ}(N,d)}{W(N,d)} \leq \frac{1}{\sqrt{d}}|D_{\frac{d-1}{2}}(\frac{2\pi}{N})| \to \sqrt{d}.$$

3.3 Correlation Constant upper bound via Difference sets

Lemma 3.3.1. *Let* $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$ *, with* $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ *defined by the ordered set* $\lambda_0, \ldots, \lambda_{N-1}$*. Then*

$$c_k = \frac{1}{N}\sum_{j=0}^{N-1} \lambda_j e^{\frac{2\pi ijk}{N}}.$$

Proof. Let $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$ with $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ and $F_j = [\frac{1}{\sqrt{N}} e^{\frac{-2\pi ijn}{N}}]$,$n = 0, \ldots, N-1$, is the associated jth column from F. Define $k = j - l$. Thus for each k,

$$c_k = F_j \Sigma F_l^* = \sum_{j-l=0}^{N-1} \frac{1}{\sqrt{N}} e^{\frac{2\pi inj}{N}} \lambda_{j-l} \frac{1}{\sqrt{N}} e^{\frac{-2\pi iln}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} \lambda_k e^{\frac{2\pi ink}{N}}.$$

□

Lemma 3.2.8 precisely defines the optimization problem for finding C(N,d), for $C \in C_P^{(N,d)}$.

3.3.1 Shift invariant projections and cyclic equiangular frames

Kalra, in [23], defined the relationship between cyclic equiangular frames and difference sets. To begin, we will look at the relationship between cyclic equiangular frames and circulant matrices.

Definition 3.3.2. A subset D of a finite (*additive*) Abelian group G is said to be a (N, d, λ)-difference set of G if for some fixed natural number λ, every nonzero element of G can be written as the difference of two elements in D in exactly λ ways, where $|G| = N$ and $|H| = d$.

Given a (N, d, λ)-difference set, we define the circulant matrix $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$, with $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$, by $\lambda_j = \begin{cases} 1 & j \in D \\ 0 & \text{otherwise} \end{cases}$, for $j = 0, \ldots, N-1$.

Defining circulant matrices in this manner, we have the following Lemma.

Lemma 3.3.3. *Let D be a (N, d, λ)-difference set. Then, for the corresponding $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$, $|c_k|^2 = \frac{1}{N^2}(|D| - \lambda)$.*

Proof. Let D be a (N, d, λ)-difference set and c_k, be an entry in the circulant matrix $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$, with $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ defined by $\lambda_j = \begin{cases} 1 & j \in D \\ 0 & \text{otherwise} \end{cases}$. From Lemma 3.3.1 we know $c_k = \frac{1}{N}\sum_{j=0}^{N-1} \lambda_j e^{\frac{2\pi ikj}{N}}$. So,

$$
\begin{aligned}
|c_k|^2 &= c_k \bar{c_k} \\
&= \frac{1}{N}\sum_{j=0}^{N-1} \lambda_j e^{\frac{2\pi ikj}{N}} \frac{1}{N}\sum_{l=0}^{N-1} \lambda_l e^{\frac{-2\pi ikl}{N}} \\
&= \frac{1}{N^2}\sum_{l,j\in D} e^{\frac{2\pi ik(j-l)}{N}} \\
&= \frac{1}{N^2}\left(\sum_{l=j\in D} \underbrace{e^{\frac{2\pi ik(j-l)}{N}}}_{=1} + \sum_{l\neq j\in D} e^{\frac{2\pi ik(j-l)}{N}}\right) \\
&= \frac{1}{N^2}\left(|D| - \lambda \sum_{j-l\in \mathbb{Z}_N\setminus\{0\}} e^{\frac{2\pi ik(j-l)}{N}}\right) \\
&= \frac{1}{N^2}\left(|D| - \lambda\right).
\end{aligned}
$$

□

Since every nonzero element of $\mathbb{Z}_N$ must be a difference in exactly λ distinct ways and there are d(d-1) total differences, $\lambda = \frac{d(d-1)}{N-1}$. Now, it follows that

$$|c_k|^2 = \frac{1}{N^2}\left(|D| - \lambda\right) = \frac{1}{N^2}\left(d - \frac{d(d-1)}{N-1}\right) = \frac{N-d}{N^2 d(N-1)}.$$

So, we have $|c_k| = \sqrt{\frac{N-d}{N^2 d(N-1)}}$, which is the Welch Bound scaled for uniform (N,d)-frames.

By increasing the rank by one, we don't necessarily have an equiangular frame. Since a difference set yields a equiangular frame, we will consider the construction of a $(N, d+1)$-frame by taking a one element of $\mathbb{Z}_N \backslash D$ to how close it will be to the Welch bound.

Theorem 3.3.4. *Let D be a (N, d, λ)-difference set and $p \in \mathbb{Z}_N \backslash D$. For* $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$, *with* $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ *defined by* $\lambda_j = \begin{cases} 1 & j \in D \cup \{p\} \\ 0 & otherwise \end{cases}$,

$$|c_k|^2 = \frac{1}{N^2}\left(|D| - \lambda + 2\sum_{j\in D}\cos(\frac{2\pi k(j-p)}{N})\right).$$

Proof. Let D be a (N, d, λ)-difference set and fix $p \in \mathbb{Z}_N \backslash D$ and $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$, with $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ defined by

$\lambda_j = \begin{cases} 1 & j \in D \cup \{p\} \\ 0 & otherwise \end{cases}$. Then,

$$
\begin{aligned}
|c_k|^2 = c_k \bar{c_k} &= \frac{1}{N}\sum_{j=0}^{N-1} \lambda_j e^{\frac{2\pi ikj}{N}} \frac{1}{N}\sum_{l=0}^{N-1} \lambda_l e^{\frac{-2\pi ikl}{N}} \\
&= \frac{1}{N^2}\left(\sum_{l,j\in D\cup\{p\}} e^{\frac{2\pi ik(j-l)}{N}}\right) \\
&= \frac{1}{N^2}\left(|D| - \lambda + \sum_{\substack{j\in D \\ l=p}} e^{\frac{2\pi ik(j-p)}{N}} + \sum_{\substack{l\in D \\ j=p}} e^{\frac{2\pi ik(p-l)}{N}}\right) \\
&= \frac{1}{N^2}\left(|D| - \lambda + 2\sum_{j\in D} Re(e^{\frac{2\pi ik(j-p)}{N}})\right) \\
&= \frac{1}{N^2}\left(|D| - \lambda + 2\sum_{j\in D} \cos(\frac{2\pi k(j-p)}{N})\right).
\end{aligned}
$$

□

3.3.2 Quadratic Residues

We begin with the $(2d+1, d+1, \lambda)$-difference set, where d is odd, that is the set of quadratic residues including zero. Then we increase the rank of $C = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d+1)}$ by one. Now we chose a specific case of $C = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d+2)}$ and determine how close we get to the correlation constant, $C(2d+1, d+2)$.

Definition 3.3.5. $a \in \mathbb{Z}_N$ such that $\gcd(a, N) = 1$, is called a quadratic residue of an odd prime N if and only if $x^2 \equiv a \mod N$ has a solution in $\mathbb{Z}_N$. If a is not a quadratic residue, then it is called a quadratic nonresidue of N.

Due to the multiplicative properties of Z_N, we know that the product of two quadratic residues is a residue, the product of two quadratic nonresidues is a quadratic residue, and the product of a quadratic nonresidue and a quadratic residue is a quadratic nonresidue. Additionally, if a is a quadratic residue then its additive inverse, $-a$ will be a quadratic nonresidue.

Lemma 3.3.6. *Let $N = 2d+1$ be an odd prime and d be prime. If D is the set of quadratic residues mod N, then D is a multiplicative subgroup of $\mathbb{Z}_N^*$ and the set of quadratic nonresidues is $-D$. Moreover, $\mathbb{Z}_N = D \cup -D \cup \{0\}$.*

Lemma 3.3.7. *Let N be a prime integer such that $N = 2d+1$, where d is odd. Then the set of quadratic residues form a difference set. Moreover, the set of residues together with $\{0\}$ forms a $(N, d+1, \lambda')$-difference set, where $\lambda' = \frac{d(d+1)}{N-1} = \frac{d+1}{2}$.*

Since d is odd λ' will be a natural number. For the corresponding $C \in C_P^{(N,d)}$ we then have $|c_k| = \frac{1}{2d+1}\sqrt{\frac{d+1}{2}}$. We will denote the set of quadratic residues together with zero as D_0 and the set of quadratic nonresidues with zero as $(-D)_0$, which is $-D_0$. Applying the previous lemma, $Z_N = D_0 \cup (-D)$. Going forward, we define $D_p = D_0 \cup \{p\}$ and the corresponding circulant matrix $C = F^*\Sigma F = circ(c_0, \ldots, c_{N-1}) \in C_P^{(N,d)}$ by $\Sigma = diag(\lambda_0, \ldots, \lambda_{N-1})$ with $\lambda_j = \begin{cases} 1 & j \in D_p \\ 0 & \text{otherwise} \end{cases}$.

Proposition 3.3.8. *Let N be an prime integer such that $N = 2d+1$, where d is odd. If $p \in -D$, then $\left\{\sum_{j \in D_0} \cos(\frac{2\pi k(j-p)}{N})\right\}_{k=0}^{N-1}$ is independent of p.*

Proof. Let $p \in -D$ and $k \in \mathbb{Z}_N^*$. Since p is a quadratic nonresidue, $-p$ is a quadratic residue. Then for $j \in D_0$, consider the set of values $k(j-p) = k(-p) + kj$ for

$1 \leq k \leq N-1$. This set can be written as $k(-p) + kD_0$. Using the multiplication properties of quadratic residues,

$$k(-p+D_0) = \begin{cases} k(-p) + D_0 & k \in D \\ k(-p) + (-D)_0 & k \in -D \end{cases}.$$

Furthermore, since $0 \leq k \leq N-1$ can be thought of as the additive subgroup generated by one, which is Z_N. Z_N is a field. Therefore, for any choice of $p \in -D$, $(-p)\langle 1 \rangle$ is Z_N. Since the only place zero appears is when $k = 0$, we can restrict k to $1 \leq k \leq N-1$. So, for $1 \leq k \leq N-1$, $k(-p+D_0)$ is the same set for any choice of p. Hence, $\left\{ \sum_{j \in D_0} \cos(\frac{2\pi k(j-p)}{N}) \right\}_{k=0}^{N-1}$ is independent of p. □

Corollary 3.3.9. *Let N be an prime integer such that $N = 2d+1$, where d is odd. If $p \in -(D_0)$, then $\min\limits_{D_p} \max\limits_{1 \leq k \leq N-1} |c_k|$ is independent of p.*

Proof. Let $p \in -(D_0)$. From Theorem 3.3.4 and Lemma 3.3.7,

$$|c_k|^2 = \frac{|D_0| - \lambda'}{N^2} + \frac{2}{N^2} \sum_{j \in D_0} \cos(\frac{2\pi k(j-p)}{N}).$$

The first term is independent of $p \in -D$. Proposition 3.3.8 gives us that the third term is independent of p. Therefore, $\min\limits_{D_p} \max\limits_{1 \leq k \leq N-1} |c_k|$ is independent of p. □

CHAPTER 4

Correlation Minimization for Laurent Matrices

In this section we will consider the problem of finding a measurable set in $[0,1]$ where the correlation constant of the associated Laurent matrix is as small as possible. To do this, we will examine the behavior of the Fourier coefficients of the characteristic function on a measurable set $A \subset [0,1]$. To begin, we examine the relationship between a Laurent matrix and the Fourier coefficients of a bounded measurable function.

4.0.1 Laurent Matrices

Definition 4.0.10. Given a sequence of complex numbers $a = (a_p)_{p\in\mathbb{Z}}$, the *Laurent matrix* of the sequence is the matrix

$$L_a = [a_{p-q}]_{p,q\in\mathbb{Z}} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \\ \ddots & a_0 & a_{-1} & a_{-2} & \ddots \\ \ddots & a_1 & a_0 & a_{-1} & \ddots \\ \ddots & a_2 & a_1 & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \end{bmatrix}.$$

By a classical theorem of Toeplitz [39], the Laurent matrix L_a is a bounded linear operator on $\ell^2(\mathbb{Z})$ if and only if the entries are the Fourier coefficients of some function $f_a \in L^\infty([0,1])$. If such a function exists, then it is unique almost everywhere. For $L^2([0,1])$ with orthonormal basis $\{e^{2\pi inx}\}_{n=-\infty}^{\infty}$ and $f \in L^\infty([0,1])$ define the multiplication operator $M_f : L^2([0,1]) \to L^2([0,1])$ by $\phi \to f\phi$. If $U : L^2([0,1]) \to \ell^2(\mathbb{Z})$ is the unitary transformation defined by $Ue_n = e^{2\pi inx}$, then $UM_{f_a}U^* = L_a$. So M_{f_a} and L_a are unitarily equivalent. Also, notice that this equivalence gives us that $L_aL_b = L_c$, with multiplication defined as $f_c = f_af_b$. Now, we will examine the special case of the infinite matrix generated by the multiplication operator with $f(x) = \chi_A$. If A is a measurable subset of [0,1], then $f = \chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ is an $L^\infty([0,1])$ function.

Defined by $a_{p-q} = \langle M_{\chi_A} e^{2\pi ipt}, e^{2\pi iqt} \rangle$. For $p \neq q$

$$\begin{aligned}\left\langle M_{\chi_A} e^{2\pi ipt}, e^{2\pi iqt} \right\rangle &= \int_0^1 M_{\chi_A} e^{2\pi ipt} \cdot e^{2\bar{\pi} iqt} dt \\ &= \int_A e^{2\pi i(p-q)t} dt \\ &= \frac{1}{2\pi i\,(p-q)} e^{2\pi i(p-q)t}|_A \\ &= \hat{\chi_A}(p-q).\end{aligned}$$

Since $\langle M_{\chi_A} e^{2\pi ipt}, e^{2\pi iqt} \rangle$ is the $(p-q)$th Fourier coefficient of χ_A, we can define the values of a particular set of Laurent matrices by $\langle M_{\chi_A} e^{2\pi ipt}, e^{2\pi iqt} \rangle$. Also, we have that the set of Laurent matrices constructed in this manner are bounded.

Remark 4.0.11. In particular, we see that L_a is a projection if and only if f_a is equal almost everywhere to the characteristic function of some measurable subset of $[0, 1]$. Then, by equivalence, M_f is a projection if and only if there exists A is a measurable subset of $[0, 1]$ and $f = \chi_A$ almost everywhere.

4.0.2 Minimizing entries of the Laurent matrices

This construction for Laurent matrices yield projections, and therefore produces Grammian matrices of Parseval frames.

Going forward, we will only consider those Laurent matrices of the form above with the goal of fixing the measure of A and determining how small $|\hat{\chi_A}(p-q)|$ can be for $p \neq q$. To begin, we need the following lemmas.

Lemma 4.0.12. *Let $\delta > 0$ and $p, k \in \mathbb{N}$. If $A = \cup_{j=0}^{p-1}[\frac{j}{p}, \frac{j}{p} + \delta] \subset [0,1]$ and $n \in \mathbb{Z}$, then*

$$\hat{\chi}_A(-n) = \begin{cases} 0 & p \nmid n \\ \frac{e^{2\pi i k p \delta} - 1}{2\pi i k} & n = kp, \end{cases} . \tag{4.1}$$

Proof. Choose $\delta > 0$. Let $A = \cup_{j=0}^{p-1}[\frac{j}{p}, \frac{j}{p} + \delta] \subset [0,1]$ with $p, k \in \mathbb{N}$. Then for $n \in \mathbb{Z}$,

$$\begin{aligned} \hat{\chi}_A(-n) &= \sum_{j=0}^{p-1} \int_{\frac{j}{p}}^{\frac{j}{p}+\delta} e^{2\pi i n t} dt \\ &= \frac{1}{2\pi i n} \sum_{j=0}^{p-1} e^{2\pi i n(\frac{j}{p}+\delta)} - e^{2\pi i n \frac{j}{p}} \\ &= \frac{1}{2\pi i n} \sum_{j=0}^{p-1} e^{2\pi i n \frac{j}{p}} (e^{2\pi i n \delta} - 1) \\ &= \frac{e^{2\pi i n \delta} - 1}{2\pi i n} \sum_{j=0}^{p-1} e^{2\pi i n \frac{j}{p}} \end{aligned}$$

Now, if n is not divisible by p

$$\hat{\chi}_A(-n) = \frac{e^{2\pi i n \delta} - 1}{2\pi i n} \left(\frac{1 - 1}{1 - e^{2\pi i n \frac{n}{p}}} \right) = 0$$

Otherwise, n can be written as $n = kp$. Then

$$\begin{aligned}\hat{\chi}_A(-n) &= \frac{e^{2\pi i n\delta} - 1}{2\pi i k p} \sum_{j=0}^{p-1} \underbrace{e^{2\pi i n a_j}}_{=1} \\ &= \frac{e^{2\pi i n\delta} - 1}{2\pi i k p} p \\ &= \frac{e^{2\pi i n\delta} - 1}{2\pi i k}.\end{aligned}$$

□

For constructed set $A = \cup_{j=0}^{p-1} \left[\frac{j}{p}, \frac{j}{p} + \delta\right] \subset [0,1]$, we can now derive a bound on $|\hat{\chi}_A(n)|$.

Lemma 4.0.13. *Let $\delta > 0$ and $p \in \mathbb{N}$. If $A = \cup_{j=0}^{p-1}[\frac{j}{p}, \frac{j}{p} + \delta] \subset [0,1]$ and $n, k \in \mathbb{Z}$, then*

$$|\hat{\chi}_A(n)| \leq \frac{1}{\pi k}$$

Proof. Choose $\delta > 0$. Let $p \in \mathbb{N}$ and define $A = \cup_{j=0}^{p-1}[\frac{j}{p}, \frac{j}{p} + \delta] \subset [0,1]$. Then, for $n, k \in \mathbb{Z}$, using lemma 4.0.12 we have

$$\hat{\chi}_A(-n) = \begin{cases} 0 & p \nmid n \\ \frac{e^{2\pi i k p\delta} - 1}{2\pi i k} & n = kp, \end{cases}.$$

So, for n not divisible by p we trivially have it. Otherwise, write n as $n = kp$. Now,

$$\begin{aligned}|\hat{\chi}_A(n)| &= |\frac{e^{2\pi i k p\delta} - 1}{2\pi i k}| \\ &\leq \frac{|e^{2\pi i k p\delta}| + 1}{2\pi k} \\ &\leq \frac{1}{\pi k},\end{aligned}$$

which proves the claim. □

Theorem 4.0.14. *Given $\epsilon > 0$, by choosing $p \in \mathbb{N}$ and $\delta > 0$ so that $p\delta < 1$ and $p\delta < \epsilon$ there exists a set $A \subset [0,1]$, with $|A| = p\delta$, where $|\hat{\chi}_A(n)| < \epsilon$ is achieved for all $n \neq 0$*

Proof. Let $\epsilon > 0$, choose $p \in \mathbb{N}$ and $\delta > 0$ so that $p\delta < 1$ and $p\delta < \epsilon$. Define the set $A = \cup_{j=0}^{p-1}[\frac{j}{p}, \frac{j}{p} + \delta] \subset [0,1] =$, so $|A| = p\delta$. Now, fix $k \in \mathbb{N}$ so that $\frac{1}{\pi k} < \epsilon$. Then, from Lemma 4.0.13, we have for $k > \frac{1}{\pi\epsilon}$ that $|\hat{\chi}_A| < \epsilon$

For $1 \leq k \leq \frac{1}{\pi\epsilon}$, from Lemma 4.0.13, we have that

$$|\hat{\chi}_A(n)| = |\frac{e^{2\pi i k p\delta} - 1}{2\pi k}| \leq \frac{2\pi k p\delta}{2\pi k} \leq p\delta < \epsilon.$$

So, given $\epsilon > 0$, we have built a set A with measure $|A| < \epsilon$ such that $|\hat{\chi}_A(n)| < \epsilon$ for $n \neq 0$.

□

Remark 4.0.15. Furthermore, since $|\hat{\chi}_A| = 1 - |\hat{\chi}_{A^c}|$ for all $n \neq 0$ we also have a set where $|A^c| > 1 - \epsilon$

In general, for any A

$$\begin{aligned} |\hat{\chi}_A(n)| &= | < \chi_A, e^{2\pi i n t} > | \\ &\leq \|\chi_A\| \|e^{2\pi i n t}\| \\ &= \sqrt{|A|} \\ &< \epsilon \end{aligned}$$

when $|A| < \epsilon^2$. So we have improved the bound by constructing this special case of the set A.

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