TWO PROBLEMS IN GRAPH ALGEBRAS AND DYNAMICAL SYSTEMS

A Dissertation Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

James West

December 2017

TWO PROBLEMS IN GRAPH ALGEBRAS AND DYNAMICAL SYSTEMS

James West

APPROVED:

Dr. Mark Tomforde, Committee Chair
Department of Mathematics, University of Houston

Dr. William Ott
Department of Mathematics, University of Houston

Dr. Andrew Török
Department of Mathematics, University of Houston

Dr. Damon Hay
Department of Mathematics and Statistics,
Sam Houston State University

Dean, College of Natural Sciences and Mathematics

Acknowledgements

I never imagined I would be in the position to actually finish this dissertation. If I could have imagined it, I would have made sure to spend more time writing these acknowledgements. Thus, I will apologize in advance for this inadequate expression of my gratitude towards all of the people, from whom I received great support, during the unreasonable amount of time spent doubting my ability to finish this, and the short time it took to bite the bullet and finish. I wish I had the time to properly thank all the people who are not mentioned here.

This dissertation is, of course, the culmination of the hard work, support and most of all patience of many individuals. I waited six years to get serious about getting my undergraduate degree and it has taken me nearly eight years to finish this dissertation (three years working and nearly five years fretting). The lone advantage of such a circuitous path is the number of wonderful people I have been fortunate to meet during this time.

Many thanks to my advisors, Dr. Mark Tomforde and Dr. William Ott. They have both showed far more patience and understanding with me than was warranted. I benefitted greatly from their intelligence, wisdom, and guidance.

Thanks to Dr. Krešimir Josić for the time he spent introducing me to research and time spent meeting with me in later attempts to expand my research.

I am grateful to Dr. Andrew Török and Dr. Damon Hay for taking the time to be on my defense committee.

I am also fortunate for the pressure Dr. Ilya Timofeyev placed on me to stop worrying and finally finish this work.

Thank you to all of the great professors I have had in the UH math program, beginning with Dr. Matthew O'Malley, who was a wonderful teacher and always showed interest in me, even long after I was a student in his classes. I also received much aid and mentorship from Dr. Jutta Hausen, Dr. Mike Field, and Dr. Richard Sanders.

Thanks to Dillon Sexton, with whom it was always fun to work problems over beers. I also enjoyed working in graduate school with Becky Chen, who never allowed herself enough credit. These are merely two of the wonderful people I have been fortunate to meet and work with during my time as an undergraduate and as a graduate student. Despite the struggle, I often miss taking classes and the late nights working with my friends.

I want to acknowledge the support I have received from many of my colleagues here in the University of Houston math department during my time spent trying to finish this project. Just in the last two weeks, I have received invaluable help and encouragement from many of my colleagues in the math department including Sarah Chehade, Jennifer May, Moses Sosa, Andy Chang, and Mozahid Haque. This barely scratches the surface of listing all of the people in the math department who have supported and encouraged me. Thank you all so much!

Bekki George has always been incredibly supportive towards me. I am extremely lucky to have met her and been able to work with her during my time at UH.

I am grateful to know Tai McAlister. I have been seeking her aid with anything related to navigating school, work, and life since I was first introduced to her during my second semester at UH. Now that I work with her, Tai continues to be amazing. It is impossible to find an issue in the math department that would not be alleviated, and normally fixed outright, by speaking with her, a fact I have habitually abused.

I owe a great debt to Nara Thull, my boss during the four years I worked at Vitamin World after high school. She became a mother figure in my life and always treated me with love, while encouraging me to return to college. Without her guidance and support, I may have never finished an undergraduate degree, much less two graduate degrees. I will always be grateful to her for the role she played in me turning from misanthropic youth to something resembling an adult.

I want to thank my mother and father for their love and support during my years spent studying at University of Houston and for encouraging me to go back to college after the five years I spent out of school. I thank my siblings Whitney and Kevin who have not only loved and supported me, but also been protective of me, despite the fact that I am the oldest.

It saddens me to think about the fact that I should have finished this while my grandmother was still alive. I wish she could be with me now to be a part of this moment. She was always proud of me, even when I did not deserve it. Giving her a reason to be proud of me was a primary source of motivation throughout my time spent in school.

Finally, I want to thank my amazing fiancée Brittany. It is impossible to imagine anyone being luckier in love than I have been with you. You make me so happy every day and I am so lucky to spend the rest of my life with you.

TWO PROBLEMS IN GRAPH ALGEBRAS AND DYNAMICAL SYSTEMS

An Abstract of a Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

James West

December 2017

Abstract

There are two parts to this dissertation. The first topic comprises Chapter two of this document, where we consider classification of nonsimple graph C^*-algebras. There are many classes of nonsimple graph C^*-algebras that are classified by the six-term exact sequence in K-theory. In this paper we consider the range of this invariant and determine which cyclic six-term exact sequences can be obtained by various classes of graph C^*-algebras. To accomplish this, we establish a general method that allows us to form a graph with a given six-term exact sequence of K-groups by splicing together smaller graphs whose C^*-algebras realize portions of the six-term exact sequence. As rather immediate consequences, we obtain the first permanence results for extensions of graph C^*-algebras.

The second part considers a problem in dynamical systems. We prove that Lyapunov exponents of infinite-dimensional dynamical systems can be computed from observational data. Crucially, our hypotheses are placed on the observations, rather than on the underlying infinite-dimensional system. We formulate checkable conditions under which a Lyapunov exponent computed from experimental data is a Lyapunov exponent of the underlying infinite-dimensional dynamical system (provided that the observational scheme is typical in the sense of prevalence).

Contents

1 Introduction 1

2 Ranges of K-theoretic Invariants for Nonsimple Graph Algebras 3

2.1 Introduction 3

2.2 Preliminaries 9

2.2.1 Extension and K-theory preliminaries 10

2.2.2 Ordered K-theory preliminaries 13

2.2.3 Graph and graph C^*-algebra preliminaries 14

2.3 K-groups of AF and simple graph C^*-algebras 18

2.4 A method for realizing six-term exact sequences 31

2.5 Gluing graphs 45

2.5.1 Adhesive graphs 46

2.5.2 Order obstructions 52

2.6 Six-term exact sequences realized by graph C^*-algebras 54

2.6.1 Range of the Non-Unital Invariant 54

2.6.2 Range of the Unital Invariant 60
2.7 Permanence 65

3 Observing Lyapunov Exponents of Infinite-dimensional Systems 70
3.1 Introduction 70
3.1.1 Lyapunov exponents in finite dimensions 71
3.1.2 Lyapunov exponents in infinite dimensions 73
3.1.3 Observation of Lyapunov exponents 74
3.1.4 Embedding results via dimension characteristics 75
3.2 Linear prevalence 78
3.3 Projection of dynamics: the Hilbert space case 79
3.3.1 Continuous observables 80
3.3.2 Observing differentiable dynamics 81
3.4 Discussion 92

Bibliography 94

CHAPTER 1

Introduction

There are two parts to this dissertation. The first topic comprises chapter two of this document, where we consider nonsimple graph C^*-algebras. We are interested in studying the range of the invariant used to classify these objects, that is the six-term exact sequence in K-theory.

We are able to create a general method that allows us to construct a graph with a given six-term exact sequence of K-groups by splicing together smaller graphs whose C^*-algebras realize portions of the six-term exact sequence.

It is our hope that our methods will contribute to future research into more general cases, such as situations where the C^*-algebras under investigations have more than one ideal. At present, there are no existing classification theories in this case.

The second part considers a problem in dynamical systems. We ask the question: Can Lyapunov exponents of infinite-dimensional dynamical systems be observed by projecting the dynamics into $\mathbb{R}^N$ using a 'typical' nonlinear projection map?

We answer this question affirmatively by developing embedding theorems for compact invariant sets associated with C^1 maps on Hilbert spaces.

Examples of such discrete-time dynamical systems include time-T maps and Poincaré return maps generated by the solution semigroups of evolution partial differential equations.

We make every effort to place hypotheses on the projected dynamics rather than on the underlying infinite-dimensional dynamical system.

In so doing, we adopt an empirical approach and formulate checkable conditions under which a Lyapunov exponent computed from experimental data will be a Lyapunov exponent of the infinite-dimensional dynamical system under study (provided the nonlinear projection map producing the data is typical in the sense of prevalence).

CHAPTER 2

Ranges of $\mathbf{K}$-theoretic Invariants for Nonsimple Graph Algebras

2.1 Introduction

The following paper was published in Transactions of the American Mathematical Society in 2016 [18]. The authors of this work are Søren Eilers, Takeshi Katsura, Mark Tomforde, and myself.

In any classification program for a given class of mathematical objects there are three goals one wishes to accomplish:

1. Associate an invariant to each object in the class in such a way that the invariant completely classifies objects in the class up to some notion of equivalence.

2. Describe a tractable method to compute the invariant for a given object.

3. Determine the range of the invariant; i.e., identify all invariants that can be realized from the objects in the class.

In this paper we accomplish goal (3) for certain classes of graph C^*-algebras that are classified up to stable isomorphism by a six-term exact sequence of abelian groups in K-theory.

For the class of C^*-algebras classification programs have been very successful in the past few decades, particularly the classification of C^*-algebras using K-theoretic data as the invariant. Two classification results were especially groundbreaking and opened several avenues for further research. The first classification is the seminal work of Elliott in the 1970's, where it was shown that the AF-algebras are classified up to stable isomorphism by their ordered K_0-group [24]. Later Effros, Handelman, and Shen showed that the range of this invariant is the class of all unperforated countable Riesz groups [17]. The second classification, occurring in the 1990's, showed that Kirchberg algebras (i.e., purely infinite, simple, separable, nuclear C^*-algebras) in the bootstrap class are classified up to stable isomorphism by the pair consisting of the K_0-group and the K_1-group [38, 56]. Moreover, it has been shown (cf. [61, 4.3.3]) that the range of this invariant is all pairs (G_0, G_1) of countable abelian groups.

In recent years more attention has been paid to the classification of nonsimple C^*-algebras [60, 49, 59], and in this paper we focus on the classification of graph

C^*-algebras. It is known that any simple graph C^*-algebra is either an AF-algebra or a Kirchberg algebra in the bootstrap class, and hence is classified by either Elliott's Theorem or the Kirchberg-Phillips Classification. In particular, for a simple graph C^*-algebra the pair $(K_0(C^*(E)), K_1(C^*(E)))$ is a complete stable isomorphism invariant, where we view $K_0(C^*(E))$ as a pre-ordered group.

A calculation for the K-theory without order was determined by Raeburn and Szymański [58, Theorem 3.2] and by Drinen and the third author [16, Theorem 3.1], and it is a consequence of these results that the K_1-group of a graph C^*-algebra must be a free abelian group. The order of the K_0-group was completely determined in [1] and [68]. The range of this invariant for simple graph C^*-algebras was calculated in independent work of Drinen and Szymański. Drinen showed that any AF-algebra is stably isomorphic to a graph C^*-algebra of a row-finite graph, and it follows from this that all simple Riesz groups are attained as the K_0-group of a simple AF graph C^*-algebra. Szymański [66] proved that for simple graph C^*-algebras that are Kirchberg algebras, all pairs of countable abelian groups (G_0, G_1) with G_1 free abelian may be attained, and moreover for any such pair one may choose a graph C^*-algebra associated with a graph that is row-finite, transitive, and has a countably infinite number of vertices.

In classification efforts for the nonsimple graph C^*-algebras, the first and third author have shown that if $C^*(E)$ is a graph C^*-algebra with a unique nontrivial ideal

$\mathcal{I}$, then the six-term exact sequence in K-theory

$$\begin{array}{ccccc} K_0(\mathcal{I}) & \xrightarrow{\iota_*} & K_0(C^*(E)) & \xrightarrow{\pi_*} & K_0(C^*(E)/\mathcal{I}) \\ \uparrow \partial_1 & & & & \downarrow \partial_0 \\ K_1(C^*(E)/\mathcal{I}) & \xleftarrow{\pi_*} & K_1(C^*(E)) & \xleftarrow{\iota_*} & K_1(\mathcal{I}) \end{array}$$

is a complete stable isomorphism invariant [23, Theorem 4.5]. Additionally, it was shown in [23, Theorem 4.7] that this six-term exact sequence is also a complete stable isomorphism invariant in the case when $\mathcal{I}$ is a largest ideal in the graph C^*-algebra and $\mathcal{I}$ is an AF-algebra. More recently [20], the first author, Restorff, and Ruiz have shown that the six-term exact sequence is also a complete stable isomorphism invariant in the case when $\mathcal{I}$ is a smallest ideal in the graph C^*-algebra $C^*(E)$, and $C^*(E)/\mathcal{I}$ is an AF-algebra. Thus there are several classes of graph C^*-algebras for which the six-term exact sequence arises as a complete stable isomorphism invariant. Consequently, these results accomplish goal (1) for several classes of nonsimple graph C^*-algebras, and even more results have been announced recently. With regards to goal (2), the computation of the invariant, it was shown by the first author, Carlsen, and the third author that the six-term exact sequence can be calculated from data provided by the vertex matrix of the graph [11, Theorem 4.1].

In this paper we turn our attention to goal (3) of classification: computing the range of the six-term exact sequence. Following [28] (cf. [21]) we focus our attention to *stenotic* extensions given by an ideal $\mathcal{I}$ that contains, or is contained in, any other ideal of the C^*-algebra $C^*(E)$ in question. We note that the classification results mentioned above cover all such extensions where ideal and quotient are either AF or simple. Basic results regarding the K-theory of extensions given by graph

C^*-algebras lead to the conditions that the K_1-groups must be free abelian groups, that the connecting map from $K_0(C^*(E)/\mathcal{I})$ to $K_1(\mathcal{I})$ must vanish, and when the extensions are stenotic with ideals and quotients that are either AF or simple we obtain certain further restrictions on the order of $K_0(C^*(E))$. We prove that these are the only restrictions, and combine our results with classification results to obtain the first permanence results for graph C^*-algebras, which allow us to determine from inspection of the K-theory when a given stable extension of AF or simple graph C^*-algebras is itself a graph C^*-algebra. As far as we can tell, this result is the first of its kind, even when restricted to the classical case of Cuntz-Krieger algebras where the necessary classification theory has been available since the work of [19, 30, 31]. We also provide complete results on the case when $C^*(E)$ is unital, determining the possible position of the order unit of $K_0(C^*(E))$ in this case.

This paper is organized as follows. In Section 2.2 we establish some preliminaries and explain a key observation for our main results, namely, that for a graph C^*-algebra the six-term exact sequence from K-theory may be obtained by applying the Snake Lemma to a certain commutative diagram determined by the vertex matrix of the graph. In Section 2.3 we obtain some range results for the K-theory of simple graph C^*-algebras and AF graph C^*-algebras. In particular, we prove that there exist graph C^*-algebras from various classes (e.g., stable Kirchberg algebras, AF-algebras, simple Cuntz-Krieger algebras, unital Kirchberg algebras) that realize various K_0-groups and K_1-groups. We also need a slightly stronger version of Szymański's Theorem [66, Theorem 1.2]. We are able to give a new, shorter proof of Szymański's Theorem (see Theorem 2.3.3) that allows us to choose a graph with the additional properties that we need. In Section 2.4 we establish the main technical results of our paper. We prove a homological algebra result in Proposition 2.4.3 that allows us to produce various six-term exact sequences using the Snake Lemma and develop methods to arrange for positivity of the block matrices found there. Recasting our findings in the context of graphs in Section 2.5, we provide sufficient conditions on a pair of graphs E_1, E_3 with K-theory fitting in a given six-term exact sequence to ensure in Proposition 2.5.5 that a new graph E_2 may be created in a way realizing the given K-theory. This new graph is formed by taking the disjoint union of E_1 and E_3, and then drawing a number of edges from vertices in E_3 to vertices in E_1 as determined by Proposition 2.4.3. We also analyze the pre-order on the K_0-group of a graph C^*-algebra in terms of the pre-order on the K_0-groups of an ideal and its quotient. In Section 2.6 we present the main results of this paper. We apply all of our results to calculate the range of the six-term exact sequence in K-theory for various classes of graph C^*-algebras. We show that for graph C^*-algebras with a unique nontrivial ideal we are able to attain any six-term exact sequence satisfying the obvious obstructions mentioned earlier. We also determine exactly which six-term exact sequences are obtained in various other classes, including Cuntz-Krieger algebras with a unique nontrivial ideal, graph C^*-algebra extensions of unital Kirchberg algebras, graph C^*-algebras with a largest ideal that is AF, and graph C^*-algebras with a smallest ideal whose quotient is AF. In all these cases, the proof is obtained by using results from Section 2.3 to obtain graphs whose C^*-algebras realize portions of the six-term exact sequence, and then using Proposition 2.5.5 to splice these graphs together into a larger graph whose C^*-algebra has the required invariant. Finally, in Section 2.7 we show how to obtain permanence results from our results, giving a complete description in several cases of when an extension of two graph C^*-algebras is again a graph C^*-algebra.

The authors wish to thank Mike Boyle for inspiring discussions in the early phases of this work, and the authors would also like to thank Efren Ruiz for suggesting improvements in the later phases.

2.2 Preliminaries

For a countable set X, we let $\mathbb{Z}^X$ denote the free abelian group generated by the basis $\{\delta_x\}_{x\in X}$ indexed by X. For $n \in \mathbb{N}$, we denote $\mathbb{Z}^{\{1,2,\dots,n\}}$ by $\mathbb{Z}^n$ which is the direct sum of n copies of $\mathbb{Z}$, and denote $\mathbb{Z}^{\mathbb{N}}$ by $\mathbb{Z}^\infty$, which is the direct sum of countably infinite copies of $\mathbb{Z}$.

For two countable sets X and Y, a *column-finite* $Y \times X$ matrix with entries in $\mathbb{Z}$ is $A = (a_{y,x})_{(y,x)\in Y\times X}$ with $a_{y,x} \in \mathbb{Z}$ such that for each $x \in X$ there are only finitely many y with $a_{y,x} \neq 0$. The collection of all such matrices is denoted by $M_{Y,X}(\mathbb{Z})$. If $X = Y$, we denote $M_{X,X}(\mathbb{Z})$ by $M_X(\mathbb{Z})$. As above, we use notations like $M_{m,n}(\mathbb{Z})$ for $m, n \in \{1, 2, \dots, \infty\}$ which is the collection of all column-finite $m \times n$ matrices with entries in $\mathbb{Z}$.

Suppose X and Y are countable sets, and that $A = (a_{y,x})_{(y,x)\in Y\times X}$ and $B = (b_{y,x})_{(y,x)\in Y\times X}$ are matrices with $a_{y,x} \in \mathbb{Z}$ and $b_{y,x} \in \mathbb{Z}$ for all $x \in X$ and for all $y \in Y$. We write $A \leq B$ (respectively, $A < B$) to mean $a_{y,x} \leq b_{y,x}$ (respectively,

$a_{y,x} < b_{y,x}$) for all $x \in X$ and for all $y \in Y$. In this case, we say B dominates A (respectively, B strictly dominates A) or that A is subordinate to B (respectively, A is strictly subordinate to B).

For two countable sets X and Y, there is a one-to-one correspondence between elements of $M_{Y,X}(\mathbb{Z})$ and $\mathbb{Z}$-module maps from $\mathbb{Z}^X$ to $\mathbb{Z}^Y$: Each matrix $A = (a_{y,x})$ in $M_{Y,X}(\mathbb{Z})$ corresponds to a $\mathbb{Z}$-module map ϕ from $\mathbb{Z}^X$ to $\mathbb{Z}^Y$ by $\phi(\delta_x) = \sum_{y \in Y} a_{y,x}\delta_y$ which makes sense by the column-finite condition. When we have a matrix A we will often identify the matrix itself with the corresponding $\mathbb{Z}$-module map, using the notation A for both, and for $\xi \in \mathbb{Z}^X$ we will often write $A\xi$ in place of $A(\xi)$. If X is a subset of Y, we denote by $I \in M_{Y,X}(\mathbb{Z})$ the map defined by $I(\delta_x) = \delta_x$ for every $x \in X$, and by $P \in M_{X,Y}(\mathbb{Z})$ the map defined by $P(\delta_x) = \delta_x$ for every $x \in X$ and $P(\delta_y) = 0$ for every $y \in Y \setminus X$.

We note that the trivial abelian group $\{0\}$ is denoted by 0, and the unique $\mathbb{Z}$-module map from or to 0 is also denoted by 0. We have $\mathbb{Z}^{\emptyset} = \mathbb{Z}^0 = 0$ and for a countable set X the $X \times \emptyset$ matrix and the $\emptyset \times X$ matrix corresponding to the unique $\mathbb{Z}$-module maps 0 are denoted by $\emptyset$. Thus $M_{\emptyset,X}(\mathbb{Z}) = \{\emptyset\}$ and $M_{X,\emptyset}(\mathbb{Z}) = \{\emptyset\}$ by definition.

2.2.1 Extension and K-theory preliminaries

In this paper, an *ideal* of a C^*-algebra will mean a closed two-sided ideal. Every nonzero C^*-algebra $\mathcal{A}$ has at least two ideals, 0 and $\mathcal{A}$, which are called *trivial ideals*. A *nontrivial ideal* is an ideal that is nonzero and proper, and a simple C^*-algebra has no nontrivial ideals).

Definition 2.2.1. An ideal $\mathcal{I}$ of $\mathcal{A}$ is said to be a *smallest ideal* (respectively, a *largest ideal*) if $\mathcal{I}$ is nontrivial and whenever $\mathcal{J}$ is a nontrivial ideal of $\mathcal{A}$, we have $\mathcal{I} \subseteq \mathcal{J}$ (respectively, $\mathcal{J} \subseteq \mathcal{I}$). An ideal $\mathcal{I}$ of $\mathcal{A}$ is called *stenotic* if for any ideal $\mathcal{J}$, either $\mathcal{J} \subseteq \mathcal{I}$ or $\mathcal{I} \subseteq \mathcal{J}$.

The notion of stenosis was introduced to C^*-algebras in [28]. Obviously any smallest or largest ideal is stenotic. Note that if $\mathcal{I}$ is a smallest (respectively, largest) ideal then $\mathcal{I}$ (respectively, $\mathcal{A}/\mathcal{I}$) is simple, but the converses are not true in general. A C^*-algebra $\mathcal{A}$ need not have a smallest ideal nor a largest ideal, but if it does, it will be unique. The uniqueness shows the following easy but useful observation.

Lemma 2.2.2. *Let $\mathcal{A}$ and $\mathcal{A}'$ be C^*-algebras. Suppose both $\mathcal{A}$ and $\mathcal{A}'$ have smallest (or largest) ideals $\mathcal{I}$ and $\mathcal{I}'$. Then $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic if and only if there exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}/\mathcal{I} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}' & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathcal{A}'/\mathcal{I}' & \longrightarrow & 0 \end{array}$$

in which the three vertical maps are isomorphisms.

If a C^*-algebra $\mathcal{A}$ has a unique nontrivial ideal $\mathcal{I}$, then $\mathcal{I}$ is smallest and largest (conversely a smallest and largest ideal is a unique nontrivial ideal). Thus we have an analogous result for C^*-algebras having unique nontrivial ideals.

One advantage of Lemma 2.2.2 is that a short exact sequence gives us a powerful invariant. From a short exact sequence

$$0 \longrightarrow \mathcal{I} \xrightarrow{\iota} \mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{I} \longrightarrow 0$$

of C^*-algebras, K-theory gives a cyclic six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{I}) & \xrightarrow{\iota_*} & K_0(\mathcal{A}) & \xrightarrow{\pi_*} & K_0(\mathcal{A}/\mathcal{I}) \\ {\scriptstyle \partial_1}\uparrow & & & & \downarrow{\scriptstyle \partial_0} \\ K_1(\mathcal{A}/\mathcal{I}) & \xleftarrow[\pi_*]{} & K_1(\mathcal{A}) & \xleftarrow[\iota_*]{} & K_1(\mathcal{I}) \end{array} \tag{2.1}$$

of abelian groups. For convenience of notation, we let $K_{\mathrm{six}}(\mathcal{A}, \mathcal{I})$ denote this cyclic six-term exact sequence of abelian groups.

If we have a cyclic six-term exact sequence $\mathcal{E}$ of the form

$$\begin{array}{ccccc} G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 \\ {\scriptstyle \delta_1}\uparrow & & & & \downarrow{\scriptstyle \delta_0} \\ F_3 & \xleftarrow{\gamma'} & F_2 & \xleftarrow{\epsilon'} & F_1 \end{array} \tag{2.2}$$

of abelian groups, then we say that $K_{\mathrm{six}}(\mathcal{A}, \mathcal{I})$ is *isomorphic* to $\mathcal{E}$ if there exist isomorphisms α_i, β_i for $i = 1, 2, 3$ making

$$\begin{array}{ccccccccc} K_0(\mathcal{I}) & & & \xrightarrow{\iota_*} & K_0(\mathcal{A}) & \xrightarrow{\pi_*} & & & K_0(\mathcal{A}/\mathcal{I}) \\ & \searrow^{\alpha_1} & & & \downarrow{\scriptstyle \alpha_2} & & & \swarrow^{\alpha_3} & \\ & & G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 & & \\ {\scriptstyle \partial_1}\uparrow & & {\scriptstyle \delta_1}\uparrow & & & & \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \partial_0} \\ & & F_3 & \xleftarrow{\gamma'} & F_2 & \xleftarrow{\epsilon'} & F_1 & & \\ & \nearrow^{\beta_3} & & & \uparrow{\scriptstyle \beta_2} & & & \nwarrow^{\beta_1} & \\ K_1(\mathcal{A}/\mathcal{I}) & & & \xleftarrow{\pi_*} & K_1(\mathcal{A}) & \xleftarrow{\iota_*} & & & K_1(\mathcal{I}) \end{array} \tag{2.3}$$

commute. We see from Lemma 2.2.2 and functoriality of K-theory that when two isomorphic C^*-algebras with smallest or largest ideals are given, the associated cyclic six-term exact sequences are isomorphic.

2.2.2 Ordered K-theory preliminaries

Lemma 2.2.2 shows that $K_{\text{six}}(\mathcal{A},\mathcal{I})$ is an invariant of a C^*-algebra $\mathcal{A}$ in the case the ideal $\mathcal{I}$ is either smallest or largest (or both). However to get a finer invariant we need to consider a pre-order on K_0-groups.

A *pre-ordered abelian group* is a pair (G, G^+), where G is an abelian group and G^+ is a subset of G satisfying $G^+ + G^+ \subseteq G^+$ and $0 \in G^+$. For $x, y \in G$ we write $x \leq y$ to mean $y - x \in G^+$. A group homomorphism $h : G_1 \to G_2$ between pre-ordered abelian groups is called an *order homomorphism* if $h(G_1^+) \subseteq G_2^+$. If h is also a group isomorphism with $h(G_1^+) = G_2^+$, then we call h an *order isomorphism.*

If $\mathcal{A}$ is a C^*-algebra, then $K_0(\mathcal{A})$ is a pre-ordered abelian group with $K_0(\mathcal{A})^+ := \{[p]_0 : p \in M_\infty(\mathcal{A})\}$. If $\mathcal{A}$ contains an approximate unit consisting of projections (which is the case for a graph C^*-algebra), then $K_0(\mathcal{A})^+ - K_0(\mathcal{A})^+ = K_0(\mathcal{A})$ (see [7, Proposition 5.5.5]).

For a C^*-algebra $\mathcal{A}$ and its ideal $\mathcal{I}$, we let $K_{\text{six}}^+(\mathcal{A},\mathcal{I})$ denote the same sequence as (2.1) but the three K_0-groups are considered as pre-ordered abelian groups. If we have a cyclic six-term exact sequence $\mathcal{E}^+$ of the same form as (2.2) but G_1, G_2, and G_3 are pre-ordered abelian groups, then we say that $K_{\text{six}}^+(\mathcal{A},\mathcal{I})$ is *order isomorphic* to $\mathcal{E}^+$ if α_i is an order isomorphism of pre-ordered groups for $i = 1, 2, 3$.

A *Riesz group* is an ordered abelian group G that is unperforated (i.e., if $g \in G$, $n \in \mathbb{N}$, and $ng \in G^+$ then $g \in G^+$) and has the Riesz interpolation property (i.e., for all $g_1, g_2, h_1, h_2 \in G$ with $g_i \leq h_j$ for $i, j = 1, 2$ there is an element $z \in G$ such that $g_i \leq z \leq h_j$ for $i, j = 1, 2$). If $\mathcal{A}$ is an AF-algebra, then $K_0(\mathcal{A})$ is a countable Riesz group. Moreover, there is a lattice bijection between the ideals of $\mathcal{A}$ and the ideals of $K_0(\mathcal{A})$ given by $\mathcal{I} \mapsto \iota_*(K_0(\mathcal{I}))$, where $\iota \colon \mathcal{I} \to \mathcal{A}$ is the inclusion map. If G is an ordered abelian group, then an *ideal* in G is a subgroup H of G with $H^+ = H \cap G^+$, $H = H^+ - H^+$, and whenever $x, y \in G$ with $0 \leq x \leq y$ and $y \in H^+$, then $x \in H$. An ordered abelian group is *simple* if it has no nontrivial ideals.

A pre-ordered abelian group G is called *trivially pre-ordered* if $G^+ = G$. If $\mathcal{A}$ is a simple purely infinite C^*-algebra, then $K_0(\mathcal{A})$ is trivially pre-ordered (see [62, Exercise 4.6 and Exercise 5.7]).

2.2.3 Graph and graph C^*-algebra preliminaries

A (directed) graph $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. A vertex $v \in E^0$ is called a *sink* if $|s^{-1}(v)| = 0$, and v is called an *infinite emitter* if $|s^{-1}(v)| = \infty$. A graph E is said to be *row-finite* if it has no infinite emitters. If v is either a sink or an infinite emitter, then we call v a *singular vertex*. We write E^0_{sing} for the set of singular vertices. Vertices that are not singular vertices are called *regular vertices* and we write E^0_{reg} for the set of regular vertices. A *cycle* is a sequence of edges $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ with $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < n$ and $r(\alpha_n) = s(\alpha_1)$. We call the vertex $r(\alpha_n) = s(\alpha_1)$ the *base point* of the cycle α. A *loop* is a cycle of length 1.

If E is a graph, a *Cuntz-Krieger E-family* is a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ with orthogonal ranges which satisfy the *Cuntz-Krieger relations*:

1. $s_e^* s_e = p_{r(e)}$ for every $e \in E^1$;

2. $s_e s_e^* \leq p_{s(e)}$ for every $e \in E^1$;

3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every $v \in E^0$ that is not a singular vertex.

The *graph C^*-algebra* $C^*(E)$ is defined to be the C^*-algebra generated by a universal Cuntz-Krieger E-family. The graph C^*-algebra is unital if and only if E^0 is a finite set in which case $1_{C^*(E)} = \sum_{v \in E^0} p_v$.

For any graph E, the *regular vertex matrix* is the $E^0 \times E^0_{\text{reg}}$ matrix R_E with

$$R_E(v, w) := |\{e \in E^1 : r(e) = v \text{ and } s(e) = w\}|.$$

Since $w \in E^0_{\text{reg}}$, all entries of R_E are finite, and R_E is column-finite. Hence we get $R_E \in M_{E^0, E^0_{\text{reg}}}(\mathbb{Z})$. We note that by the definition of regular vertices each column contains at least one nonzero entry. If E has no regular vertices then $R_E = \emptyset \in M_{E^0, \emptyset}(\mathbb{Z})$. Recall that $I \in M_{E^0, E^0_{\text{reg}}}(\mathbb{Z})$ is defined by $I(\delta_v) = \delta_v$ for $v \in E^0_{\text{reg}}$.

Proposition 2.2.3 ([58, Theorem 3.2], [16, Theorem 3.1])**.** *Let E be a graph. Then we have*

$$K_0(C^*(E)) \cong \operatorname{coker}(R_E - I) \quad \textit{and} \quad K_1(C^*(E)) \cong \ker(R_E - I).$$

From this proposition, we see that $K_0(C^*(E))$ and $K_1(C^*(E))$ are countable abelian groups, and in addition, $K_1(C^*(E))$ is a free abelian group because any subgroup of a free abelian group is free. We also have

$$\operatorname{rank} K_0(C^*(E)) + |E^0_{\text{reg}}| = \operatorname{rank} K_1(C^*(E)) + |E^0|. \tag{2.4}$$

Let E be a graph. A subset $H \subseteq E^0$ is *hereditary* if whenever $e \in E^1$ and $s(e) \in H$, then $r(e) \in H$. A hereditary subset H is *saturated* if whenever $v \in E^0_{\mathrm{reg}}$ with $r(s^{-1}(v)) \subseteq H$, then $v \in H$. For a hereditary subset H, we can define two graphs E_1 and E_3 by

$$E_1 = (H, s^{-1}(H), r, s), \qquad E_3 = (E^0 \setminus H, E^1 \setminus r^{-1}(H), r, s) \tag{2.5}$$

where r and s are restrictions of those for E. The set H is saturated if and only if we have $(E_3)^0_{\mathrm{reg}} \supseteq E^0_{\mathrm{reg}} \setminus H$. If a saturated hereditary subset H satisfies $(E_3)^0_{\mathrm{reg}} = E^0_{\mathrm{reg}} \setminus H$ then we say that H has *no breaking vertices*. Note that if E is row-finite, then every saturated hereditary subset has no breaking vertices.

For a saturated hereditary subset H, we denote by $\mathcal{I}_H$ the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$.

Proposition 2.2.4. *Let $E = (E^0, E^1, r, s)$ be a graph, and let H be a saturated hereditary subset of E^0 such that H has no breaking vertices. Let E_1 and E_3 be the two graphs as in* (2.5) *for H. Then we have the following:*

1. *There is a natural embedding from $C^*(E_1)$ onto a full corner of $\mathcal{I}_H$.*
2. *We have $C^*(E)/\mathcal{I}_H \cong C^*(E_3)$.*
3. *We have $E^0 = E^0_1 \sqcup E^0_3$ and $E^0_{\mathrm{reg}} = (E_1)^0_{\mathrm{reg}} \sqcup (E_3)^0_{\mathrm{reg}}$.*
4. *There exists a row-finite matrix $X \in M_{E^0_1,(E_3)^0_{\mathrm{reg}}}(\mathbb{Z}^+)$ such that under the decomposition in (3), we get $R_E = \begin{pmatrix} R_{E_1} & X \\ 0 & R_{E_3} \end{pmatrix}$.*

5. *$K_{\text{six}}(C^*(E), \mathcal{I}_H)$ is isomorphic to*

$$\begin{array}{ccccc}
\operatorname{coker}(R_{E_1}-I) & \longrightarrow & \operatorname{coker}\begin{pmatrix} R_{E_1}-I & X \\ 0 & R_{E_3}-I \end{pmatrix} & \longrightarrow & \operatorname{coker}(R_{E_3}-I) \\
{\scriptstyle [X]}\uparrow & & & & \downarrow{\scriptstyle 0} \\
\ker(R_{E_3}-I) & \longleftarrow & \ker\begin{pmatrix} R_{E_1}-I & X \\ 0 & R_{E_3}-I \end{pmatrix} & \longleftarrow & \ker(R_{E_1}-I)
\end{array}$$

where the horizontal maps are the obvious inclusions or projections, and $[X]$ is the map implemented by multiplication by X.

Proof. Statements (1) and (2) are standard (see [11]). It is straightforward to check (3) and (4). Finally (5) follows from [11, Remark 4.2] and [11, Theorem 1]. □

Remark 2.2.5. One can show that the sequence (5) above is nothing but the long exact sequence obtained by applying the Snake Lemma from homological algebra (see [47]) to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{(E_1)^0_{\text{reg}}} & \longrightarrow & \mathbb{Z}^{E^0_{\text{reg}}} & \longrightarrow & \mathbb{Z}^{(E_3)^0_{\text{reg}}} & \longrightarrow & 0 \\
 & & {\scriptstyle R_{E_1}-I}\downarrow & & {\scriptstyle R_E-I}\downarrow & & {\scriptstyle R_{E_3}-I}\downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^{E_1^0} & \longrightarrow & \mathbb{Z}^{E^0} & \longrightarrow & \mathbb{Z}^{E_3^0} & \longrightarrow & 0
\end{array}$$

with exact rows. See Proposition 2.4.1.

Remark 2.2.6. For a graph E and a gauge-invariant ideal $\mathcal{I}$ of $C^*(E)$, there is a computation of $K_{\text{six}}(C^*(E), \mathcal{I})$ in [11, 4.1] similar to (5) of Proposition 2.2.4. In particular, the index map from $K_0(C^*(E)/\mathcal{I})$ to $K_1(\mathcal{I})$ is always 0 in this case.

Remark 2.2.7. If $C^*(E)$ has a unique nontrivial ideal $\mathcal{I}$, then $\mathcal{I} = \mathcal{I}_H$ for a saturated hereditary subset H of E^0 such that H has no breaking vertices [23, Lemma 3.1].

2.3 K-groups of AF graph C^*-algebras and simple graph C^*-algebras

It has been shown by the work of many hands that K-theoretic invariants completely classify, up to stable isomorphism, the class of AF graph C^*-algebras and the class of simple graph C^*-algebras. The range of the invariants has also been computed. We review and reprove some of these results in this section to get sharper results regarding realization of graphs.

The following is a refined realization of AF graph C^*-algebras.

Proposition 2.3.1. *If (G, G^+) is a countable Riesz group, then there exists a row-finite graph E such that*

(1) E has no sinks, no sources, and a countably infinite number of vertices,

(2) E has no cycles (so that, in particular, $C^*(E)$ is an AF-algebra),

(3) $(K_0(C^*(E)), K_0(C^*(E))^+) \cong (G, G^+)$, and

(4) $C^*(E)$ is stable

Proof. By the Effros-Handelman-Shen Theorem [17], there exists an AF-algebra A whose K_0-group is order isomorphic to (G, G^+). By Drinen's Theorem [14, Theorem 1] there exists a row-finite graph F such that F has no cycles and $C^*(F)$ is stably isomorphic to A. Let E be the graph obtained by adding a tail (cf. [15]) to every sink of F and a head to every source of F (cf. [15]). Then E has no sinks or sources, E has a countably infinite number of vertices, E has no cycles, and $C^*(E)$ is isomorphic to $C^*(F) \otimes \mathbb{K}$ so that $(K_0(C^*(E)), K_0(C^*(E))^+) \cong (G, G^+)$. □

We next consider simple graph C^*-algebras. The "Dichotomy for simple graph C^*-algebras" [15, Remark 2.16] states that any simple graph C^*-algebra $C^*(E)$ is either purely infinite (if E contains a cycle) or AF (if E contains no cycles). Consequently, the Kirchberg-Phillips Classification Theorem and Elliott's Theorem imply that any simple graph C^*-algebra is classified up to stable isomorphism by the pair $(K_0(C^*(E)), K_1(C^*(E)))$, where we consider $K_0(C^*(E))$ as a pre-ordered abelian group. If $C^*(E)$ is purely infinite, then $K_0(C^*(E))$ is trivially pre-ordered; i.e. $K_0(C^*(E))^+ = K_0(C^*(E))$. If $C^*(E)$ is AF, then $K_0(C^*(E))$ is an ordered group (in fact, a Riesz group) and $K_0(C^*(E))^+$ is a proper subset of $K_0(C^*(E))$. Thus the ordering on the K_0-group can distinguish whether the simple graph C^*-algebra is purely infinite or AF. We have already seen in Proposition 2.3.1 that all simple Riesz groups are realized as the K_0-group of an AF graph C^*-algebra (and, moreover, that the graph may be chosen to have certain properties). Since an AF-algebra whose K_0-group is a simple Riesz group is simple, AF graph C^*-algebras given in Proposition 2.3.1 for simple Riesz groups are necessarily simple. So in order to complete the description of the range of K-theoretic invariants for simple graph C^*-algebras, we only need to consider simple purely infinite graph C^*-algebras. We know that the pre-order of the K_0-group has to be trivial. We also know that the K_1-group has to be free by Proposition 2.2.3. Szymański proved that these are the only restrictions on the K-groups, and we prove a sharper version of his result below (see Proposition 2.3.3) that gives us extra control of the choice of a graph.

Lemma 2.3.2. *Let G be a countable abelian group and let F be a countable free abelian group. Then there exists an exact sequence*

$$0 \longrightarrow F \stackrel{\iota}{\longrightarrow} \mathbb{Z}^\infty \stackrel{\phi}{\longrightarrow} \mathbb{Z}^\infty \stackrel{\pi}{\longrightarrow} G \longrightarrow 0.$$

Proof. Let X be a generating set for G, which must be countable since G is countable. Let $\mathcal{F}(X)$ be the free abelian group generated by X. Then $\mathcal{F}(X) \cong \mathbb{Z}^m$, where $m = |X|$, and by the universal property of free abelian groups there exists a surjective homomorphism $\pi_0 : \mathbb{Z}^m \to G$. Define $\pi : \mathbb{Z}^\infty \oplus \mathbb{Z}^m \to G$ by $\pi(x, y) := \pi_0(y)$. Since $\ker \pi$ is a subgroup of a countable free abelian group, $\ker \pi$ is a countable free abelian group. Moreover, since $\mathbb{Z}^\infty \oplus 0 \subseteq \ker \pi$, it follows that there is an isomorphism $\psi : \mathbb{Z}^\infty \to \ker \pi$. In addition, since F is a countable free abelian group there exists an isomorphism $i_0 : F \to \mathbb{Z}^n$ for some $n \in \{0, 1, 2, \dots, \infty\}$. Define $\phi : \mathbb{Z}^\infty \oplus \mathbb{Z}^n \to \mathbb{Z}^\infty \oplus \mathbb{Z}^m$ by $\phi(x, y) = \psi(x)$, and define $i : F \to \mathbb{Z}^\infty \oplus \mathbb{Z}^n$ by $i(x) := (0, i_0(x))$. One can verify that

$$0 \longrightarrow F \stackrel{i}{\longrightarrow} \mathbb{Z}^\infty \oplus \mathbb{Z}^n \stackrel{\phi}{\longrightarrow} \mathbb{Z}^\infty \oplus \mathbb{Z}^m \stackrel{\pi}{\longrightarrow} G \longrightarrow 0$$

is exact. Since $Z^\infty \oplus \mathbb{Z}^n \cong \mathbb{Z}^\infty$ and $Z^\infty \oplus \mathbb{Z}^m \cong \mathbb{Z}^\infty$, the result follows. □

Proposition 2.3.3 (Szymański's Theorem)**.** *Let G be a countable abelian group and let F be a countable free abelian group. Then there exists a row-finite graph E with countably infinite E^0 that satisfies the following properties:*

(1) Every vertex in E is the base point of at least two loops,

(2) E is transitive (so that, in particular, $C^*(E)$ is simple and purely infinite),

(3) $K_0(C^*(E)) \cong G$ *and* $K_1(C^*(E)) \cong F$*, and*

(4) $C^*(E)$ *is stable.*

Proof. By Lemma 2.3.2 there exists a group homomorphism $\phi : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ with $\operatorname{coker}\phi \cong G$ and $\ker\phi \cong F$. Let $A_0 \in M_\infty(\mathbb{Z})$ be the matrix representation of ϕ. Then A_0 is a column-finite matrix. Define $|A_0| \in M_\infty(\mathbb{Z})$ to be the entry-wise absolute value of A_0; i.e., $|A_0|(i,j) := |A_0(i,j)|$. Also define

$$A_1 := |A_0| + \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 1 & \\ 0 & 0 & 1 & 1 & \\ \vdots & \vdots & & & \ddots \end{pmatrix}.$$

Let $\mathcal{I}$ denote the identity matrix in $M_\infty(\mathbb{Z})$, and define

$$A := \begin{pmatrix} A_0 + A_1 + I & A_1 \\ I & 2I \end{pmatrix}.$$

We observe that A is a column finite square matrix with non-negative entries, and also observe that the diagonal is everywhere greater or equal to 2. Hence we can find a graph E with no singular vertices such that $R_E = A$. Since A is indexed by a countably infinite set, the set E^0 of vertices is countable and infinite. Since every vertex in E is regular, E is row-finite. In addition, the fact that every diagonal entry of A is two or larger shows that every vertex in E is the base point of at least two loops. Furthermore, one can see from the definition of A that E is transitive. Finally, we have

$$A - I = \begin{pmatrix} A_0 + A_1 & A_1 \\ I & I \end{pmatrix} = \begin{pmatrix} I & A_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}.$$

Since the matrices $\left(\begin{smallmatrix} I & A_1 \\ 0 & I \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} I & 0 \\ I & I \end{smallmatrix}\right)$ are invertible (with the inverses $\left(\begin{smallmatrix} I & -A_1 \\ 0 & I \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} I & 0 \\ -I & I \end{smallmatrix}\right)$), $A - I$ has a kernel and cokernel which is isomorphic to those of the matrix $\left(\begin{smallmatrix} A_0 & 0 \\ 0 & I \end{smallmatrix}\right)$, and hence to those of A_0. Thus

$$K_0(C^*(E)) \cong \operatorname{coker}(A - I) \cong \operatorname{coker} A_0 \cong G$$

and

$$K_1(C^*(E)) \cong \ker(A - I) \cong \ker A_0 \cong F.$$

Finally, since $C^*(E)$ is a nonunital Kirchberg algebra, it is stable by [72]. □

Remark 2.3.4. Proposition 2.3.3, together with Lemma 2.3.2, gives a shorter proof of Szymański's Theorem [66, Theorem 1.2] with the added conclusion (1).

The following proposition summarizes the arguments above.

Proposition 2.3.5. *The class of graph C^*-algebras that are either AF or simple is classified up to stable isomorphism by K_0-groups as pre-ordered abelian groups and K_1-groups as abelian groups. A pair (G, F) of a countable pre-ordered abelian group G and a countable abelian group F is in the range of invariants in this class if and only if either*

- *G is a Riesz group and $F = 0$, or*
- *G is trivially preordered and F is free.*

Moreover, one can realize the above invariants by a stable graph C^**-algebra* $C^*(E)$ *for a row-finite graph* E *with no sinks and no sources.*

In order to get a finer invariant for the classification up to isomorphism, one needs to consider scales of K_0-groups in the AF case, and positions of units in K_0-groups in unital simple purely infinite case. With this extra data, one can classify, up to isomorphism, all graph C^*-algebras that are either AF or simple. However, the computation of the range of this finer invariant is not as straightforward as above. In fact, the (nonunital, but nonstable) case of AF graph C^*-algebras is very complicated (see [37]). For the nonunital simple purely infinite case, K_0-groups and K_1-groups are already a complete invariant up to isomorphism since they are stable. In what follows, we complete the computation of the range of the invariant for the unital simple purely infinite case.

For a unital simple purely infinite C^*-algebra A, the extra information we need for classification up to isomorphism is the element $[1_A]_0 \in K_0(A)$ defined by the unit 1_A of A. For a group G and an element g_0, we write $(K_0(A), [1_A]_0) \cong (G, g_0)$ if there exists an isomorphism from $K_0(A)$ to G sending $[1_A]_0$ to g_0.

Recall that a graph C^*-algebra $C^*(E)$ is unital if and only if the set E^0 of vertices is finite, and in this case the unit $1_{C^*(E)}$ is $\sum_{v\in E^0} p_v$. Under the isomorphism $K_0(C^*(E)) \cong \operatorname{coker}(R_E - I)$ in Proposition 2.2.3 the element $[1_{C^*(E)}]_0$ corresponds to the equivalence class $[\mathbf{1}]$ where $\mathbf{1} := \sum_{v\in E^0} \delta_v \in \mathbb{Z}^{E^0}$ is the vector all of whose entries are 1.

Note that if E^0 is finite, then Proposition 2.2.3 and (2.4) shows that the two groups $G := K_0(C^*(E))$ and $F := K_1(C^*(E))$ satisfy

- G is a finitely generated abelian group,
- F is a finitely generated free abelian group with $\operatorname{rank} F \leq \operatorname{rank} G$.

The following proposition shows that these are the only restrictions for the K-groups, and there is no restriction on the position of $[1_{C^*(E)}]$ in $K_0(C^*(E))$. In this case, as opposed to what we saw above, not every vertex may be chosen regular.

Proposition 2.3.6. *Let G be a finitely generated abelian group, and let F be a free abelian group with* $\operatorname{rank} F \leq \operatorname{rank} G$*. Let g_0 be an element of G.*

Then there exists a graph E with finite E^0 that satisfies (1)–(2) of Proposition 2.3.3 as well as

(3') $(K_0(C^*(E)), [1_{C^*(E)}]_0) \cong (G, g_0)$ and $K_1(C^*(E)) \cong F$.

Proof. By the fundamental theorem of finitely generated abelian groups, there exist unique integers $k, n \geq 0$ and $m_1, m_2, \ldots, m_k \geq 2$ with $m_1 \mid m_2 \mid \cdots \mid m_k$ such that

$$G \cong (\mathbb{Z}/m_1\mathbb{Z}) \oplus (\mathbb{Z}/m_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/m_k\mathbb{Z}) \oplus \mathbb{Z}^n. \tag{2.6}$$

We can then find a generating set $\{\gamma_i\}_{i=1}^{k+n}$ of G with the relations $m_i\gamma_i = 0$ for $i = 1, \ldots, k$. There exist integers $(a_i)_{i=1}^{k+n}$ such that $g_0 = \sum_{i=1}^{k+n} a_i\gamma_i$. By subtracting m_i from a_i for $i \leq k$ many times and replacing γ_i with $-\gamma_i$ for $i > k$ if necessary, we may assume that $a_i \leq 0$ for all i. Let $n' = \operatorname{rank} F$, which is at most $\operatorname{rank} G = n$ by the assumption.

We denote the bases of $\mathbb{Z}^{1+k+n}$ and $\mathbb{Z}^{1+k+n'}$ by $\{\delta_i\}_{i=0}^{k+n}$ and $\{\delta_i\}_{i=0}^{k+n'}$ (the indices have been shifted compared to the convention in Section 2.2). We define a surjective map $\pi_0\colon \mathbb{Z}^{1+k+n} \to G$ by $\pi_0(\delta_0) = 0$ and $\pi_0(\delta_i) = \gamma_i$ for $i = 1, 2, \ldots, k+n$, and we define $A_0 \in M_{1+k+n,1+k+n'}(\mathbb{Z})$ by $A_0(\delta_0) = \delta_0$, $A_0(\delta_i) = m_i\delta_i$, for $i = 1, \ldots, k$ and $A_0(\delta_i) = 0$ for $i = k+1, \ldots, k+n'$. Then we have $\operatorname{im} A_0 = \ker \pi_0$. In matrix form, we have

$$A_0 = \begin{pmatrix} 1 & & & & & & \\ & m_1 & & & & & \\ & & m_2 & & & & \\ & & & \ddots & & & \\ & & & & m_k & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{pmatrix}.$$

We set $b_i = 1 - a_i \geq 1$ for $i = 1, 2, \ldots, k+n$. Consider two square matrices $P \in M_{1+k+n,1+k+n}(\mathbb{Z})$ and $Q \in M_{1+k+n',1+k+n'}(\mathbb{Z})$ by

$$P = \begin{pmatrix} 1 & & & & & \\ b_1 & 1 & & & & \\ b_2 & & 1 & & & \\ \vdots & & & \ddots & & \\ b_k & & & & 1 & \\ \vdots & & & & & \ddots \\ b_{k+n} & & & & & & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Note that P and Q are invertible. We set $A \in M_{1+k+n,1+k+n'}(\mathbb{Z})$ by

$$A = PA_0Q = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & \ldots & 1 \\ b_1 & b_1 + m_1 & b_1 & \ldots & b_1 & \ldots & b_1 \\ b_2 & b_2 & b_2 + m_2 & & b_2 & \ldots & b_2 \\ \vdots & \vdots & & \ddots & & & \vdots \\ b_k & b_k & b_k & & b_k + m_k & \vdots & b_k \\ \vdots & \vdots & \vdots & & \ldots & & \vdots \\ b_{k+n} & b_{k+n} & b_{k+n} & \cdots & b_{k+n} & \cdots & b_{k+n} \end{pmatrix}.$$

Since $b_i \geq 1$ and $m_i \geq 2$, all entries of A are positive. We define $\pi\colon \mathbb{Z}^{1+k+n} \to G$ by

$\pi = \pi_0 \circ P^{-1}$. Then π is a surjection satisfying

$$\ker \pi = P(\ker \pi_0) = P(\operatorname{im} A_0) = \operatorname{im}(PA_0) = \operatorname{im} A,$$

and hence π induces an isomorphism

$$\bar{\pi}\colon \operatorname{coker} A \ni [x] \mapsto \pi(x) \in G,$$

and therefore $\ker A$ is a free abelian group whose rank is $n+(1+k+n')-(1+k+n) = n'$.

We set $\mathbf{1} = \sum_{i=0}^{k+n} \delta_i \in \mathbb{Z}^{1+k+n}$. We are going to show $\bar{\pi}([\mathbf{1}]) = g_0$. Since $b_i + a_i = 1$ for $i = 1, 2, \ldots, k+n$, we have

$$P\Big(\delta_0 + \sum_{i=1}^{k+n} a_i \delta_i\Big) = \sum_{i=0}^{k+n} \delta_i = \mathbf{1}, \quad \begin{pmatrix} 1 & & & \\ b_1 & 1 & & \\ \vdots & & \ddots & \\ b_{k+n} & & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ \vdots \\ a_{k+n} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Hence we have

$$\bar{\pi}([\mathbf{1}]) = \pi(\mathbf{1}) = \pi_0(P^{-1}(\mathbf{1})) = \pi_0\Big(\delta_0 + \sum_{i=1}^{k+n} a_i \delta_i\Big) = \sum_{i=1}^{k+n} a_i \gamma_i = g_0.$$

Let E be the graph such that

- $E^0 = \{0, 1, \ldots, k+n\}$,
- there are infinitely many edges from $i \in E^0$ with $k + n' < i \leq k+n$ to every vertex, and
- $0, 1, \ldots, k+n'$ are regular, and the regular vertex matrix R_E of E is $A + I$.

Then E is a graph with $1 + k + n$ vertices, each vertex is the base point of at least two loops, and E is transitive. The computation of K-theory follows from Proposition 2.2.3 and the first part of this proof. □

Remark 2.3.7. Let G, F, and $g_0 \in G$ be as in Proposition 2.3.6, and let $k, n, m_1, \dots, m_k$ be as in (2.6). Below we will show that if a graph E satisfies the condition (3') in Proposition 2.3.6 then the number $|E^0|$ of vertices of E is at least $k + n$. We also show that in many cases including the case that G and F are arbitrary, but $g_0 = 0$, it is necessary that $|E^0|$ is strictly greater than $k+n$. Thus the number $1+k+n$ of vertices of the graph E in the proof above is the smallest possible in these cases.

Observe that if E satisfies condition (3) then we have a surjective map $\mathbb{Z}^{E^0} \to G$ sending $\mathbf{1}$ to g_0. Choose a prime number p with $p \mid m_1$. Tensoring with $\mathbb{Z}/p\mathbb{Z}$, we get a surjective map

$$(\mathbb{Z}/p\mathbb{Z})^{E^0} \cong \mathbb{Z}^{E^0} \otimes (\mathbb{Z}/p\mathbb{Z}) \to G \otimes (\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^{k+n}.$$

This shows that $|E^0| \geq k + n$. Since $\mathbf{1} \otimes 1$ is a nonzero element of $(\mathbb{Z}/p\mathbb{Z})^{E^0}$, if $g_0 \otimes 1 \in G \otimes (\mathbb{Z}/p\mathbb{Z})$ is zero, then the above surjection is not injective. Thus in this case we have $|E^0| > k + n$.

We also note that for arbitrary G and F, there exists $g_0 \in G$ such that we can find a graph E satisfying the condition (3) with $|E^0| = k+n$. However, in the case G is free and $F \neq 0$, or in the case $G = F = 0$, there exists no such E with $C^*(E)$ is simple. Thus for such G and F and for arbitrary $g_0 \in G$, the graph E constructed in the proof of Proposition 2.3.6 has the smallest possible number of vertices, namely $|E^0| = 1 + k + n$, such that $C^*(E)$ is simple and satisfies condition (3'). For a pair

(G, F) other than the ones mentioned above, there exists some $g_0 \in G$ (necessarily nonzero) such that we can find a graph E with $|E^0| = k+n$ satisfying conditions (1) and (2) of Proposition 2.3.5 and condition (3') of Proposition 2.3.6.

We need the next small variation of Proposition 2.3.6 in order to control the unit in the extension. As explained in Remark 2.3.7, if G, F and $g_0 \in G$ satisfies either $g_0 = 0$ or G is free (and F is nonzero) then a graph E as in Proposition 2.3.6 has at least $1 + k + n$ vertices. If G satisfies *both* of the two conditions, then we need one more vertex to get the next result.

Proposition 2.3.8. *Let G, F, and $g_0 \in G$ be as in Proposition 2.3.6. Then there exists a graph E with a finite number of vertices and with E satisfying conditions (1) and (2) of Proposition 2.3.5 and condition (3') of Proposition 2.3.6, as well as*

(4) there exist two vertices $v, w \in E^0$ such that $(R_E - I)(w, v') < (R_E - I)(v, v')$ for all $v' \in E^0_{\text{reg}}$.

When G is written in the form of (2.6) *we may choose the graph E with $|E^0| = 1 + k + n$.*

Proof. First consider the case $g_0 \neq 0$. In this case we show that the graph E constructed in the proof of Proposition 2.3.6 satisfies (4). For $(a_i)_{i=1}^{k+n}$ as in that proof, there exists $i \in \{1, 2, \ldots, k+n\}$ with $a_i < 0$. Then we have $b_i \geq 2$. Hence $A(0, j) = 1 < b_i \leq A(i, j)$ for all $j = 0, 1, \ldots, k + n'$. Since $R_E - I = A$ for the graph E, the vertices $v = i$ and $w = 0$ satisfy (4).

Next consider the case G is not free. Then we can choose $(a_i)_{i=1}^{k+n}$ in the proof of Proposition 2.3.6 such that $a_1 \leq -m_1 < 0$. As in the case $g_0 \neq 0$, the vertices $v = 1$ and $w = 0$ satisfy (4).

Finally suppose $g_0 = 0$ and G is free. Let n and n' be the ranks of G and F respectively. We define a $(n+2) \times (n'+2)$ matrix A by

$$A := \begin{pmatrix} 3 & 2 & \dots & 2 \\ 2 & 1 & \dots & 1 \\ 2 & 1 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 2 & 1 & \dots & 1 \end{pmatrix},$$

whose image is generated by two elements

$$\mathbf{1} := \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = A \begin{pmatrix} -1 \\ 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence $\ker A \cong \mathbb{Z}^{n'}$, $\operatorname{coker} A \cong \mathbb{Z}^n$ and $[\mathbf{1}] = 0$ in $\operatorname{coker} A$. We define a graph E so that

- $E^0 = \{1, 2, \dots, n+2\}$,
- there are infinitely many edges from $i \in E^0$ with $n' + 2 < i \leq n + 2$ to every vertex,

- $1, 2, \dots, n' + 2$ are regular, and the regular vertex matrix R_E of E is $A + I$.

Then E satisfies conditions (1) and (2) of Proposition 2.3.5 and condition (3') of Proposition 2.3.6. Finally, $v = 1$ and $w = 2$ satisfy (4). □

A Cuntz-Krieger algebra $\mathcal{O}_A$ is isomorphic to the C^*-algebra of a finite graph with no sinks or sources. For such a graph E, every vertex is regular. Therefore, Proposition 2.2.3 and (2.4) shows that the K-groups $G = K_0(\mathcal{O}_A)$ and $F = K_1(\mathcal{O}_A)$ of a Cuntz-Krieger algebra $\mathcal{O}_A$ satisfy

- G is a finitely generated abelian group,
- F is a finitely generated free abelian group with $\operatorname{rank} F = \operatorname{rank} G$.

The following proposition shows that these are the only restrictions for the K-groups of simple Cuntz-Krieger algebras, and there is no restriction on the position of $[1_{\mathcal{O}_A}]_0$ in $K_0(\mathcal{O}_A)$.

Proposition 2.3.9. *Let G be a finitely generated abelian group, and let F be a finitely generated free abelian group with* $\operatorname{rank} F = \operatorname{rank} G$. *Let g_0 be an element of G. Then there exists a graph E with finite vertex set E^0 and finite edge set E^1 that satisfies properties (1)–(4) of Proposition 2.3.8. Moreover, if G is of the form of* (2.6), *we may choose E with* $|E^0| = 1 + k + n$.

Proof. The graph E constructed in the proof of Proposition 2.3.6 or Proposition 2.3.8 has finitely many edges because the rank n' of F coincides with the rank n of G. Hence this graph E has the desired properties. □

2.4 A method for realizing six-term exact sequences

In Proposition 2.4.3 of this section we prove a result that will allow us to realize a given six-term exact sequence with four groups already given as kernels and cokernels of certain matrices A and B of a certain form determined by a block matrix $\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right)$, just as in (5) of Proposition 2.2.4. We start by observing that every such sequence has this form, provided only that the relevant index map vanishes:

Proposition 2.4.1. *Suppose $A \in M_{n_1,n_1'}(\mathbb{Z})$ and $B \in M_{n_3,n_3'}(\mathbb{Z})$ for some $n_1, n_1', n_3, n_3' \in \{0,1,2,\ldots,\infty\}$. Then $Y \mapsto \left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right)$ is a bijection from $M_{n_3',n_1}(\mathbb{Z})$ to the set of all matrices $X \in M_{n_1+n_3,n_1'+n_3'}(\mathbb{Z})$ for which the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{n_1'} & \xrightarrow{I'} & \mathbb{Z}^{n_1'} \oplus \mathbb{Z}^{n_3'} & \xrightarrow{P'} & \mathbb{Z}^{n_3'} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle A} & & \vdots{\scriptstyle X} & & \downarrow{\scriptstyle B} & & \\ 0 & \longrightarrow & \mathbb{Z}^{n_1} & \xrightarrow[I]{} & \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_3} & \xrightarrow[P]{} & \mathbb{Z}^{n_3} & \longrightarrow & 0 \end{array}$$

commutes, where I and I' are the obvious inclusions $x \mapsto (x, 0)$, and P and P' are the obvious projections $(x, y) \mapsto y$.

Moreover, for each $Y \in M_{n_3',n_1}(\mathbb{Z})$, the sequence

$$\begin{array}{ccccccc} \operatorname{coker} A & \xrightarrow[I]{} & \operatorname{coker} \left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) & \xrightarrow[P]{} & \operatorname{coker} B & \longrightarrow & 0 \\ \uparrow{\scriptstyle [Y]} & & & & & & \\ \ker B & \xleftarrow{P'} & \ker \left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) & \xleftarrow{I'} & \ker A & \longleftarrow & 0 \end{array} \tag{2.7}$$

is exact, where we use the same notation I, P, I', and P' to denote the induced maps on cokernels or the restricted maps on kernels, and where $[Y]$ is the composition of the restriction of Y to $\ker B$ and the natural surjection $\mathbb{Z}^{n_1} \to \operatorname{coker} A$.

Proof. The former assertion is easy to see, and the latter follows from the Snake Lemma (see [47]). $\square$

Lemma 2.4.2. *Let $n, n' \in \{0, 1, 2, \ldots, \infty\}$ and $A \in M_{n,n'}(\mathbb{Z})$. Let G be an abelian group. Then any homomorphism $\eta\colon \ker A \to G$ extends to $\zeta\colon \mathbb{Z}^{n'} \to G$ such that $\zeta|_{\ker A} = \eta$.*

Proof. Since $\operatorname{im} A$ is a subgroup of the free abelian group $\mathbb{Z}^n$, it follows that $\operatorname{im} A$ is free abelian. Therefore, there exists a homomorphism $S\colon \operatorname{im} A \to \mathbb{Z}^{n'}$ such that

$$A \circ S(x) = x \quad \text{for all } x \in \operatorname{im} A. \tag{2.8}$$

Let $I\colon \mathbb{Z}^{n'} \to \mathbb{Z}^{n'}$ denote the identity map on $\mathbb{Z}^{n'}$. Since $I - SA$ takes values in $\ker A$, we can define $\zeta\colon \mathbb{Z}^{n'} \to G$ by $\zeta = \eta \circ (I - SA)$. It is easy to verify $\zeta|_{\ker A} = \eta$. $\square$

Proposition 2.4.3. *Let $\mathcal{E}$ denote the following exact sequence of abelian groups*

$$\begin{array}{ccccc} G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 \\ {\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle 0} \\ F_3 & \xleftarrow{\gamma'} & F_2 & \xleftarrow{\epsilon'} & F_1 \end{array} \tag{2.9}$$

with F_1, F_2, and F_3 free. Suppose that there exist column-finite matrices $A \in M_{n_1,n_1'}(\mathbb{Z})$ and $B \in M_{n_3,n_3'}(\mathbb{Z})$ for some $n_1, n_1', n_3, n_3' \in \{0, 1, 2, \ldots, \infty\}$ with isomorphisms

$$\alpha_1\colon \operatorname{coker} A \to G_1, \qquad \beta_1\colon \ker A \to F_1,$$

$$\alpha_3\colon \operatorname{coker} B \to G_3, \qquad \beta_3\colon \ker B \to F_3.$$

Then there exist a column-finite matrix $Y \in M_{n_1,n_3'}(\mathbb{Z})$ and isomorphisms

$$\alpha_2\colon \operatorname{coker}\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) \to G_2, \qquad \beta_2\colon \ker\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) \to F_2$$

such that α_i and β_i for $i = 1, 2, 3$ give an isomorphism (see (2.3)*) from the exact sequence*

$$\begin{array}{ccccc} \operatorname{coker} A & \xrightarrow[I]{} & \operatorname{coker}\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) & \xrightarrow[P]{} & \operatorname{coker} B \\ {\scriptstyle [Y]}\big\uparrow & & & & \big\downarrow{\scriptstyle 0} \\ \ker B & \xleftarrow{P'} & \ker\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) & \xleftarrow{I'} & \ker A. \end{array} \tag{2.10}$$

to $\mathcal{E}$, where I, I', P, and P' are induced by the obvious inclusions or projections.

Proof. By a simple calculation, or applying the Snake Lemma, we see that the sequence (2.7) is exact (see Remark 2.4.4).

We define $\pi_1 \colon \mathbb{Z}^{n_1} \to G_1$ (respectively, $\pi_3 \colon \mathbb{Z}^{n_3} \to G_3$) to be the composition of the natural surjection to the cokernel and the isomorphism α_1 (respectively, α_3). We first construct a homomorphism $\pi_2 \colon \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_3} \to G_2$ with

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathbb{Z}^{n_1} & \xrightarrow[I]{} & \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_3} & \xrightarrow[P]{} & \mathbb{Z}^{n_3} & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle \pi_1} & & \vdots{\scriptstyle \pi_2} & & \big\downarrow{\scriptstyle \pi_3} & & \\ \cdots & \xrightarrow{\delta} & & & G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 & \longrightarrow & 0 \end{array} \tag{2.11}$$

commuting, where

$$I \colon \mathbb{Z}^{n_1} \ni x \mapsto (x, 0) \in \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_3}$$

$$P \colon \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_3} \ni (x, y) \mapsto y \in \mathbb{Z}^{n_3}$$

are the obvious inclusions, and projections.

Since $\mathbb{Z}^{n_3}$ is a free abelian group and $\gamma \colon G_2 \to G_3$ is surjective, there exists a homomorphism $\mu \colon \mathbb{Z}^{n_3} \to G_2$ such that

$$\gamma \circ \mu = \pi_3 \qquad \begin{array}{ccc} & & \mathbb{Z}^{n_3} \\ & {\scriptstyle \mu}\swarrow & \big\downarrow{\scriptstyle \pi_3} \\ G_2 & \xrightarrow[\gamma]{} & G_3. \end{array}$$

We define $\pi_2\colon \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_3} \to G_2$ by

$$\pi_2(x,y) := \epsilon(\pi_1(x)) + \mu(y).$$

for $x \in \mathbb{Z}^{n_1}$ and $y \in \mathbb{Z}^{n_3}$. Commutativity in (2.11) can be easily verified as

$$\begin{aligned} \pi_2(I(x)) &= \pi_2(x,0) = \epsilon(\pi_1(x)) \\ \gamma(\pi_2(x,y)) &= \gamma\big(\epsilon(\pi_1(x)) + \mu(y)\big) = 0 + \pi_3(y) = \pi_3(P(x,y)). \end{aligned}$$

Next we construct a homomorphism $Y\colon \mathbb{Z}^{n_3'} \to \mathbb{Z}^{n_1}$ such that

$$\pi_2 \circ \left(\begin{smallmatrix} Y \\ B \end{smallmatrix}\right) = 0 \qquad \mathbb{Z}^{n_3'} \xrightarrow{\left(\begin{smallmatrix} Y \\ B \end{smallmatrix}\right)} \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_3} \xrightarrow{\pi_2} G_2 \tag{2.12}$$

and

$$\pi_1 \circ Y|_{\ker B} = \delta \circ \beta_3 \qquad \begin{array}{ccc} \ker B & \xrightarrow{Y|_{\ker B}} & \mathbb{Z}^{n_1} \\ {\scriptstyle\beta_3}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ F_3 & \xrightarrow{\delta} & G_1. \end{array} \tag{2.13}$$

By Lemma 2.4.2, there exists an extension $\beta_3'\colon \mathbb{Z}^{n_3'} \to F_3$ of the isomorphism $\beta_3\colon \ker B \to F_3$. Since $\pi_1\colon \mathbb{Z}^{n_1} \to G_1$ is surjective and $\mathbb{Z}^{n_3'}$ is a free abelian group, there exists a homomorphism $Y_1\colon \mathbb{Z}^{n_3'} \to \mathbb{Z}^{n_1}$ such that

$$\pi_1 \circ Y_1 = \delta \circ \beta_3' \qquad \begin{array}{ccc} \mathbb{Z}^{n_3'} & \xrightarrow{Y_1} & \mathbb{Z}^{n_1} \\ {\scriptstyle\beta_3'}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ F_3 & \xrightarrow{\delta} & G_1. \end{array} \tag{2.14}$$

Since $\mu|_{\operatorname{im} B}$ takes values in $\ker\gamma = \operatorname{im}\epsilon$, we see that $\mu|_{\operatorname{im} B}\colon \operatorname{im} B \to \operatorname{im}\epsilon$. Also, $\operatorname{im} B$ is a subgroup of a free abelian group and thus $\operatorname{im} B$ is free abelian. Because $\epsilon \circ \pi_1 : \mathbb{Z}^{n_1} \to \operatorname{im}\epsilon$ is surjective, there exists a homomorphism $Y_2\colon \operatorname{im} B \to \mathbb{Z}^{n_1}$ such that

$$\epsilon \circ \pi_1 \circ Y_2 = \mu|_{\operatorname{im} B} \qquad \begin{array}{ccc} \operatorname{im} B & \xrightarrow{\mu|_{\operatorname{im} B}} & G_2 \\ {\scriptstyle Y_2}\downarrow & & \uparrow{\scriptstyle\epsilon} \\ \mathbb{Z}^{n_1} & \xrightarrow{\pi_1} & G_1. \end{array}$$

Define a homomorphism $Y\colon \mathbb{Z}^{n_3'} \to \mathbb{Z}^{n_1}$ by

$$Y := Y_1 - Y_2 \circ B,$$

and, in line with our convention, we will use the same symbol for the matrix $Y \in M_{n_1,n_3'}(\mathbb{Z})$ that implements this homomorphism. For $y' \in \mathbb{Z}^{n_3'}$ we have

$$\begin{aligned}\pi_2(Y(y'), B(y')) &= \epsilon(\pi_1(Y(y'))) + \mu(B(y')) \\ &= \epsilon\big(\pi_1\big(Y_1(y') - Y_2(B(y'))\big)\big) + \mu(B(y')) \\ &= 0 - \mu(B(y')) + \mu(B(y')) \\ &= 0.\end{aligned}$$

This shows (2.12). The equality (2.13) follows from (2.14) because

$$Y|_{\ker B} = Y_1|_{\ker B} - (Y_2 \circ B)|_{\ker B} = Y_1|_{\ker B},$$

and hence we have

$$\pi_2 \circ \left(\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right) = \pi_2 \circ I \circ A = \epsilon \circ \pi_1 \circ A = 0.$$

This and (2.12) show

$$\pi_2 \circ \left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) = 0.$$

Hence the map $\pi_2\colon \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_3} \to G_2$ factors through a map

$$\operatorname{coker}\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) \to G_2$$

which is denoted by α_2.

Now we shall construct a homomorphism

$$\beta_2\colon\ \ker\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) \to F_2$$

fitting into

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker A & \xrightarrow[I']{} & \ker\left(\begin{smallmatrix} A & Y \\ 0 & B\end{smallmatrix}\right) & \xrightarrow[P']{} & \ker B & & \\
 & & \downarrow{\scriptstyle \beta_1} & & \vdots{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_3} & & \\
0 & \longrightarrow & F_1 & \xrightarrow{\epsilon'} & F_2 & \xrightarrow{\gamma'} & F_3 & \xrightarrow{\delta} & G_1
\end{array}
\tag{2.15}
$$

where I' and P' are the restrictions of obvious inclusion, and projections, as above.

For $(x', y') \in \ker\left(\begin{smallmatrix} A & Y \\ 0 & B\end{smallmatrix}\right)$, we have

$$\delta\big(\beta_3(P'(x', y'))\big) = \pi_1(Y(y')) = \pi_1(-A(x')) = 0.$$

Hence the image of $\beta_3 \circ P'$ is contained in $\ker \delta = \operatorname{im} \gamma'$. The abelian group $\ker\left(\begin{smallmatrix} A & Y \\ 0 & B\end{smallmatrix}\right)$ is free because it is a subgroup of the free group $\mathbb{Z}^{n_1'} \oplus \mathbb{Z}^{n_3'}$. Therefore there exists a homomorphism $\nu \colon \ker\left(\begin{smallmatrix} A & Y \\ 0 & B\end{smallmatrix}\right) \to F_2$ such that

$$
\gamma' \circ \nu = \beta_3 \circ P' \qquad\qquad
\begin{array}{ccc}
\ker\left(\begin{smallmatrix} A & Y \\ 0 & B\end{smallmatrix}\right) & \xrightarrow[P']{} & \ker B \\
{\scriptstyle \nu}\vdots & & \downarrow{\scriptstyle \beta_3} \\
F_2 & \xrightarrow[\gamma']{} & F_3.
\end{array}
$$

Since

$$\gamma' \circ \nu \circ I' = \beta_3 \circ P' \circ I' = 0,$$

the image of $\nu \circ I'$ is contained in $\ker \gamma' = \operatorname{im} \epsilon'$. Since ϵ' is injective, there exists a homomorphism $\eta \colon \ker A \to F_1$ such that $\epsilon' \circ \eta = \nu \circ I'$. By Lemma 2.4.2, there exists an extension $\zeta \colon \mathbb{Z}^{n_1'} \to F_1$ of the homomorphism $\beta_1 - \eta \colon \ker A \to F_1$. We define

$$\beta_2 \colon \ker\left(\begin{smallmatrix} A & Y \\ 0 & B\end{smallmatrix}\right) \ni (x', y') \mapsto \nu(x', y') + \epsilon'(\zeta(x')) \in F_2$$

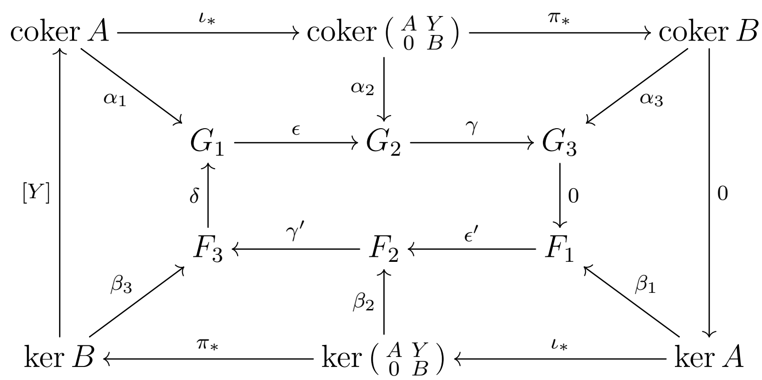
Commutativity of (2.15) can be verified by

$$\beta_2(I'(x)) = \nu(x, 0) + \epsilon'(\zeta(x)) = \nu(I'(x)) + \epsilon'(\beta_1(x) - \eta(x)) = \epsilon'(\beta_1(x))$$

$$\gamma'(\beta_2(x', y')) = \gamma'(\nu(x', y') + \epsilon'(\zeta(x')) = \beta_3(y') + 0 = \beta_3(P'(x', y')).$$

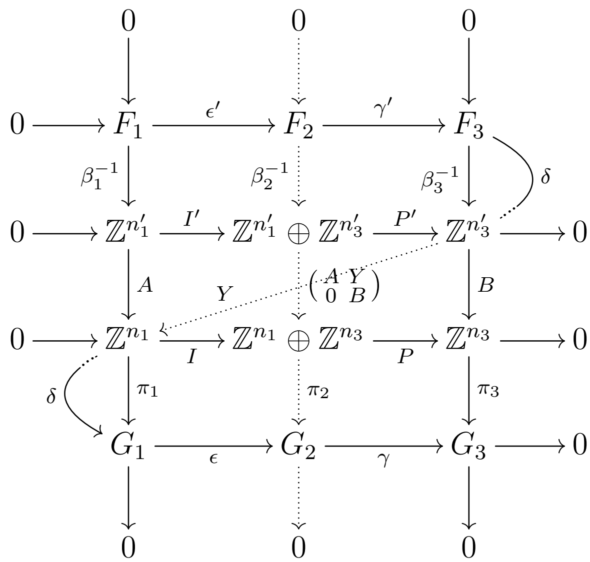
for $x \in \ker A$ and $(x', y') \in \ker \left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right)$.

Thus we constructed $Y \in M_{n_1, n_3'}(\mathbb{Z})$ and two homomorphisms α_2 and β_2. The diagram



commutes by (2.11) for the top two squares, by (2.13) for the left square, by (2.15) for the bottom two squares, and trivially for the right square. Finally, the Five Lemma shows that α_2 and β_2 are isomorphisms. $\square$

Remark 2.4.4. The content of Proposition 2.4.3 can be summarized in the following commutative diagram with exact rows and columns:



(2.16)

The solid lines and $\delta\colon F_3 \to G_1$ are the given data, and the dotted lines indicate the homomorphisms we need to construct. We also need to show that δ factors through $Y\colon \mathbb{Z}^{n'_3} \to \mathbb{Z}^{n_1}$, the middle column is exact, and the six squares commute. Once we establish these facts, the snake lemma implies that the sequence (2.7) is isomorphic to the given one $\mathcal{E}$.

In order to use the matrix Y found above to define a graph, we will need Y to be non-negative. The two ensuing lemmas are the key to arranging this.

Lemma 2.4.5. *Let $n, n' \in \{0, 1, 2, \ldots, \infty\}$. For $A \in M_{n,n'}(\mathbb{Z})$, the following three conditions are equivalent:*

1. *For every $i \in \{1, 2, \ldots, n\}$ there exists $\xi_i \in \mathbb{Z}^{n'}$ such that $A\xi_i - \delta_i \in (\mathbb{Z}^+)^n$.*
2. *There exists $A' \in M_{n',n}(\mathbb{Z})$ such that $AA' - I \in M_{n,n}(\mathbb{Z}^+)$.*
3. *For every $m \in \{0, 1, 2, \ldots, \infty\}$ and every $Y \in M_{n,m}(\mathbb{Z})$, there exists $Q \in M_{n',m}(\mathbb{Z})$ such that $AQ + Y \in M_{n,m}(\mathbb{Z}^+)$.*

Proof. (1)⇒(2): By (1) for each $i \in \{1, 2, \ldots, n\}$ there exists $\xi_i \in \mathbb{Z}^{n'}$ satisfying $A\xi_i - \delta_i \in (\mathbb{Z}^+)^n$. Then $A' := (\xi_1\xi_2 \cdots \xi_n) \in M_{n',n}(\mathbb{Z})$ defined by $A'(\delta_i) = \xi_i$ for all i satisfies $AA' - I \in M_{n,n}(\mathbb{Z}^+)$.

(2)⇒(3): Take m and $Y \in M_{n,m}(\mathbb{Z})$. We set $|Y| \in M_{n,m}(\mathbb{Z})$ to be the entry-wise absolute value of Y; i.e., $|Y|(i,j) := |Y(i,j)|$. Let $A' \in M_{n',n}(\mathbb{Z})$ be as in (2), and set $Q := A'|Y| \in M_{n',m}(\mathbb{Z})$. Then we have

$$AQ + Y = (AA' - I)|Y| + (|Y| + Y) \in M_{n,m}(\mathbb{Z}^+).$$

(3)⇒(1): Take arbitrary $i \in \{1, 2, \ldots, n\}$, and apply (3) to $Y \in M_{n,1}(\mathbb{Z})$ given by $Y(\delta_1) = -\delta_i$ to get $Q \in M_{n',1}(\mathbb{Z})$ with $AQ+Y \in M_{n,1}(\mathbb{Z}^+)$. If we set $\xi := Q(\delta_1) \in \mathbb{Z}^{n'}$ then we have

$$A\xi - \delta_i = (AQ + Y)(\delta_1) \in (\mathbb{Z}^+)^n. \qquad \square$$

The following lemma is analogous to the previous one, and is as easy to prove when n' is finite, but substantially more complicated when n' is infinite. For each $i \in \{1, 2, \ldots, n'\}$, $\delta_i^t \in M_{1,n'}(\mathbb{Z})$ denotes the transpose of δ_i, that is, the ith row of the identity $I \in M_{n',n'}(\mathbb{Z})$.

Lemma 2.4.6. *Let $n, n' \in \{0, 1, 2, \ldots, \infty\}$. For $B \in M_{n,n'}(\mathbb{Z})$, the following three conditions are equivalent:*

1. *For every $i \in \{1, 2, \ldots, n'\}$ there exists $\eta_i \in \mathbb{Z}^n$ such that $\eta_i^t B - \delta_i^t \in M_{1,n'}(\mathbb{Z}^+)$, chosen such that for all finite subsets $F \subset \{1, 2, \ldots, n\}$, there exists a finite subset $G \subset \{1, 2, \ldots, n'\}$ such that for every $i \notin G$ and every $j \in F$, we get $\eta_{i,j} = 0$.*

2. *There exists* $B' \in M_{n',n}(\mathbb{Z})$ *such that* $B'B - I \in M_{n',n'}(\mathbb{Z}^+)$.

3. *For every* $m \in \{0, 1, 2, \ldots, \infty\}$ *and every* $Y \in M_{m,n'}(\mathbb{Z})$, *there exists* $Q \in M_{m,n}(\mathbb{Z})$ *such that* $QB + Y \in M_{m,n'}(\mathbb{Z}^+)$.

Proof. (1)⇒(2): When n' is finite, $B' \in M_{n',n}(\mathbb{Z})$ can be defined so that for each $i \in \{1, 2, \ldots, n'\}$ the ith row of B is $\eta_i^t \in M_{1,n}(\mathbb{Z})$ satisfying $\eta_i^t B - \delta_i^t \in M_{1,n'}(\mathbb{Z}^+)$ which exist by condition (1). When n' is infinite, the condition (1) implies that n is also infinite. For each integer k, let $G_k \subset \{1, 2, \ldots, n'\}$ be a finite set as in (1) for the finite set $F = \{1, 2, \ldots, k\}$. We define a finite set $G'_k \subset \{1, 2, \ldots, n'\}$ for $k = 1, 2, \ldots$ inductively by

$$G'_1 := \{1\} \cup G_1, \quad G'_k := \{k\} \cup G_k \setminus \Big(\bigcup_{j=1}^{k-1} G'_j\Big).$$

From this definition, it is easy to see that the $\{G'_k\}_{k=1}^{\infty}$ are mutually disjoint, that their union is the whole of $\{1, 2, \ldots, n'\} = \mathbb{N}$, and $G'_k \cap G_{k-1} = \emptyset$ for all $k > 1$. For $i \in G'_1$, choose $\eta_i \in \mathbb{Z}^n$ such that $\eta_i^t B - \delta_i^t \in M_{1,n'}(\mathbb{Z}^+)$ by the first condition of (1). For each $i \in G'_k$ for $k > 1$, choose $\eta_i^t \in M_{1,n}(\mathbb{Z})$ such that $\eta_i^t B - \delta_i^t \in M_{1,n'}(\mathbb{Z}^+)$ and $(\eta_i^t)_{1,j} = 0$ for $j = 1, 2, \ldots, k-1$ by the latter condition of (1). We set $B' \in M_{n',n}(\mathbb{Z})$ by $B'_{i,j} = (\eta_i)_{1,j}$. We need to check that B' is column-finite, which follows from the fact that for each $j \in \{1, 2, \ldots, n\}$, $B'_{i,j} \neq 0$ implies i is in the finite set $\bigcup_{k=1}^{j-1} G'_k$. Now it is easy to see $B'B - I \in M_{n',n'}(\mathbb{Z}^+)$.

(2)⇒(1): Take $B' \in M_{n',n}(\mathbb{Z})$ as in (2), For each i, let $\eta_i^t \in \mathbb{Z}^n$ be the ith row of B. Then we have $\eta_i^t B - \delta_i^t \in M_{1,n'}(\mathbb{Z}^+)$. Thus we get the former condition of (1). This choice of η_i^t's also satisfies the latter condition of (1) if for a given finite subset

$F \subset \{1, 2, \dots, n\}$, we choose $G \subset \{1, 2, \dots, n'\}$ by

$$G = \bigcup_{j \in F} \{i : B_{i,j} \neq 0\}$$

which is finite because B is column-finite.

(2)⇒(3): Take m and $Y \in M_{m,n'}(\mathbb{Z})$. We set $|Y| \in M_{m,n'}(\mathbb{Z})$ to be the entry-wise absolute value of Y; i.e., $|Y|(i,j) := |Y(i,j)|$. Let $B' \in M_{n',n}(\mathbb{Z})$ be as in (2), and set $Q := |Y|B' \in M_{m,n}(\mathbb{Z})$. Then we have

$$QB + Y = |Y|(B'B - I) + (|Y| + Y) \in M_{m,n'}(\mathbb{Z}^+).$$

(3)⇒(2): Apply (3) to $m = n$ and $Y = -I$. □

Proposition 2.4.7. *In the situation of Proposition 2.4.3, assume that $Z \in M_{n_1,n_3'}(\mathbb{Z})$ is given. If A satisfies the equivalent conditions of Lemma 2.4.5 or B satisfies the equivalent conditions of Lemma 2.4.6, then the matrix $Y \in M_{n_1,n_3'}(\mathbb{Z})$, along with α_2 and β_2 inducing the isomorphism, may be chosen with the additional property $Y \geq Z$.*

Proof. Let $Y' \in M_{n_1,n_3'}(\mathbb{Z})$ denote a matrix already chosen in Proposition 2.4.3, along with maps α_2' and β_2'. Assume first that A satisfies the conditions of Lemma 2.4.5. Then by (3) of the lemma we may choose $Q \in M_{n_1',n_3'}(\mathbb{Z})$ such that

$$AQ + [Y' - Z] \in M_{n_1,n_3'}(\mathbb{Z}^+)$$

One checks directly that with

$$\begin{aligned} Y &= AQ + Y' \\ \beta_2 &= \beta_2' \circ \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} \\ \alpha_2 &= \alpha_2' \end{aligned}$$

the conditions are all met. (In particular, one can easily verify that in this case $\mathrm{im}\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) = \mathrm{im}\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right)$, so that $\mathrm{coker}\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) = \mathrm{coker}\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right)$, and that multiplication by $\left(\begin{smallmatrix} I & Q \\ 0 & I \end{smallmatrix}\right)$ is an isomorphism from $\ker\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right)$ onto $\ker\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right)$.)

When B satisfies the conditions of Lemma 2.4.6, we choose $Q \in M_{n_1,n_3}(\mathbb{Z})$ such that

$$QB + [Y' - Z] \in M_{n_1,n_3'}(\mathbb{Z}^+)$$

and set

$$\begin{aligned} Y &= Y' + QB \\ \beta_2 &= \beta_2' \\ \alpha_2 &= \alpha_2' \circ \begin{pmatrix} I & -Q \\ 0 & I \end{pmatrix} \end{aligned}$$

and one can check the conditions are met. (In particular, one can easily verify that in this case $\ker\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right) = \ker\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right)$, and that multiplication by $\left(\begin{smallmatrix} I & -Q \\ 0 & I \end{smallmatrix}\right)$ is an isomorphism from $\mathrm{im}\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right)$ onto $\mathrm{im}\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right)$, and hence multiplication by $\left(\begin{smallmatrix} I & -Q \\ 0 & I \end{smallmatrix}\right)$ induces an isomorphism from $\mathrm{coker}\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right)$ onto $\mathrm{coker}\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right)$.) □

Proposition 2.4.8. *In the situation of Proposition 2.4.3, assume that $Z \in M_{n_1,n_3'}(\mathbb{Z})$ is given, that $n_1, n_3 < \infty$, and that $g_2 \in G_2$ is given with $\alpha_3([\mathbf{1}]) = \gamma(g_2)$. If B satisfies the condition that for some $1 \leq i, j < n_3$ we have*

$$B_{ik} < B_{jk} \qquad 1 \leq k < n_3',$$

then the matrix $Y \in M_{n_1,n_3'}(\mathbb{Z})$, along with α_2 and β_2 inducing the isomorphism, may be chosen with the additional properties that $\alpha_2([\mathbf{1}]) = g_2$ and $Y \geq Z$.

Proof. Take Y', β_2' and α_2' as in Proposition 2.4.3 and set

$$g_2' = \alpha_2'([\mathbf{1}]) - g_2.$$

Observe that $\gamma(g_2') = 0$ because

$$\alpha_3([\mathbf{1}]) = \alpha_3\left(\left(\begin{smallmatrix}\mathbf{1}\\ \mathbf{1}\end{smallmatrix}\right)\right) = \gamma(g_2)$$

Hence there exists $\xi \in \mathbb{Z}^{n_1}$ such that $\epsilon(\alpha_1([\xi])) = g_2'$. Choose $Q' \in M_{n_1,n_3}(\mathbb{Z})$ such that $\xi = Q'\mathbf{1}$. Find an integer $c > 0$ so that with $Q'' \in M_{n_1,n_3}(\mathbb{Z})$ defined by

$$(Q'')_{k,\ell} = \begin{cases} 1 & \ell = i \\ -1 & \ell = j \\ 0 & \text{else} \end{cases}$$

we have

$$(Q' + cQ'')B \geq Z - Y'$$

This is possible because each row of $Q''B$ is identically

$$\begin{pmatrix} B_{i,1} - B_{j,1} & B_{i,2} - B_{j,2} & \cdots & B_{i,n_3} - B_{j,n_3} \end{pmatrix},$$

which is strictly positive by assumption on B. Set $Q = Q' + cQ''$ and $Y = Y' + QB$. Then $Y \geq Z$ by how Y was chosen. In addition, if we let

$$\beta_2 = \beta_2' \qquad \text{and} \qquad \alpha_2 = \alpha_2' \circ \begin{pmatrix} I & -Q \\ 0 & I \end{pmatrix},$$

then since $\ker\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right) = \ker\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right)$ and since $\left(\begin{smallmatrix} I & Q \\ 0 & I \end{smallmatrix}\right)$ is an automorphism on $\operatorname{im}\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right)$ (to see this use the fact that $QB = Y - Y'$), we conclude that $\beta_2 : \ker\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right) \to F_2$ and $\alpha_2 : \operatorname{coker}\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right) \to G_2$ are well-defined isomorphisms.

We also have

$$\begin{aligned}
\alpha_2([\mathbf{1}]) &= \alpha_2'\left(\begin{pmatrix} I & -Q \\ 0 & I \end{pmatrix}\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}\right) \\
&= \alpha_2'\left(\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} - \begin{pmatrix} Q\mathbf{1} \\ \mathbf{0} \end{pmatrix}\right) \\
&= \alpha_2'(\mathbf{1}) - \alpha_2'(I(Q\mathbf{1})) \\
&= \alpha_2'(\mathbf{1}) - \epsilon \circ \alpha_1([\xi]) \\
&= \alpha_2'(\mathbf{1}) - g_2' \\
&= g_2
\end{aligned}$$

so the conclusion holds. □

Proposition 2.4.9. *In the situation of Proposition 2.4.3, assume that $n_1, n_3 < \infty$, that $F_3 = 0$, and that there exists a splitting map $\sigma : G_3 \to G_2$ for γ. Let $g_2 \in G_2$ be given, set $g_3 = \gamma(g_2) \in G_3$, and let g_1 be the unique element of G_1 with $\epsilon(g_1) = g_2 - \sigma(g_3)$. If*

$$\alpha_1(\mathbf{1}) = g_1 \qquad \text{and} \qquad \alpha_3(\mathbf{1}) = g_3,$$

then there exist homomorphisms

$$\alpha_2\colon \operatorname{coker}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \to G_2 \qquad \text{and} \qquad \beta_2\colon \ker\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \to F_2$$

such that the collection of homomorphisms α_i and β_i for $i = 1, 2, 3$ provide an isomorphism, and $\alpha_2([\mathbf{1}]) = g_2$.

Proof. Let $\left[\left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right)\right] \in \operatorname{coker}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. Then

$$\alpha_2\left(\left[\left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right)\right]\right) = \epsilon(\alpha_1([\xi])) + \sigma(\alpha_3([\eta]))$$

and

$$\beta_2\left(\left[\begin{pmatrix}\xi\\0\end{pmatrix}\right]\right) = \epsilon'(\beta_1(\xi)).$$

□

2.5 Gluing graphs

Suppose two graphs E_1 and E_3 are given along with groups G_2 and F_2 so that the following diagram $\mathcal{E}$ commutes:

$$\begin{array}{ccccc} K_0(C^*(E_1)) & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & K_0(C^*(E_3)) \\ {\scriptstyle\partial_1}\uparrow & & & & \downarrow{\scriptstyle 0} \\ K_1(C^*(E_3)) & \xleftarrow{\gamma'} & F_2 & \xleftarrow{\epsilon'} & K_1(C^*(E_1)). \end{array} \tag{2.17}$$

In the present section we investigate circumstances under which it is possible to glue together E_1 and E_3 to form a third graph E_2 whose C^*-algebra has $\mathcal{E}$ as its six-term exact sequence in K-theory. We shall see that the results of the previous section allow us to perform such a gluing, and realize the sequence of groups in $\mathcal{E}$, under very modest assumptions on either E_1 or E_3. However, since there are natural obstructions for the pre-ordering on G_2 to originate from a graph C^*-algebra, we will need to impose further restrictions before being able to realize the pre-ordered sequence $\mathcal{E}^+$ consisting of an exact sequence of partially pre-ordered groups. The necessity of these conditions follow from the fullness issues considered in [21] and [23], but for the reader's convenience we shall develop them by much more elementary methods at the end of this section.

2.5.1 Adhesive graphs

Definition 2.5.1. We say that the graph E is *left adhesive* if for any $v_0 \in E^0$ there exist $n \geq 1$ and distinct $e_0, e_1, \ldots, e_n \in E^1$ such that

(i) $r(e_0) = v_0$

(ii) $s(e_k) \in V_{v_0}$ for all $k = 0, 1, \ldots, n$

(iii) $V_{v_0} \subseteq E^0_{\mathrm{reg}}$

where $V_{v_0} := \{r(e_k) \mid k = 1, \ldots, n\}$.

Definition 2.5.2. We say that the graph E is *right adhesive* if for any $v_0 \in E^0_{\mathrm{reg}}$ there exist $n \geq 1$ and distinct $e_0, e_1, \ldots, e_n \in E^1$ such that

(i) $s(e_0) = v_0$

(ii) $r(e_k) \in W_{v_0}$ for all $k = 0, 1, \ldots, n$

(iii) the collection $\{W_{v_0} : v_0 \in E^-_{\mathrm{reg}}\}$ satisfies the property that for each $w \in E^0$, the set $\{v_0 \in E^0_{\mathrm{reg}} \mid w \in W_{v_0}\}$ is finite

where $W_{v_0} := \{r(e_k) \mid k = 1, \ldots, n\}$.

The reader is requested to note the similarities between these concepts, and how there is only partial symmetry. The relevance of adhesiveness in our situation is explained by the following lemma.

Lemma 2.5.3. *When $E = (E^0, E^1, r, s)$ is left adhesive, then $R_E - I$ satisfies the equivalent conditions of Lemma 2.4.5. When E is right adhesive, $R_E - I$ satisfies the equivalent conditions of Lemma 2.4.6.*

Proof. In the first case, define $\xi_{v_0} \in \mathbb{Z}^{E^0_{\text{reg}}}$ by

$$\xi_{v_0} = \sum_{w \in V_{v_0}} \delta_w.$$

The for each $v \in E^0$ we have

$$[(R_E - I)\xi_{v_0}]_v = \begin{cases} |\{e \in E^1 \mid r(e) = v, s(e) \in V_{v_0}\}| - 1 & \text{if } v \in V_{v_0} \\ |\{e \in E^1 \mid r(e) = v, s(e) \in V_{v_0}\}| & \text{if } v \notin V_{v_0}. \end{cases}$$

For every $v \in V_{v_0}$ we have at least one edge starting in V_{v_0} and ending in v, so $(R_E - I)\xi_{v_0} \geq 0$. We also have $(R_E - I)\xi_{v_0} \geq \delta_{v_0}$ in the case when $v \notin V_{v_0}$. When $v_0 \in V_{v_0}$ we have that $v_0 = r(e_i)$ for $e_i \neq e_0$, so that two different edges start in V_{v_0} and end in v_0, as required. Thus $(R_E - I)\xi_{v_0} \geq \delta_{v_0}$ as desired.

In the second case, define $\eta_{v_0} \in \mathbb{Z}^{E^0}$ by

$$\eta_{v_0} = \sum_{w \in W_{v_0}} \delta_w.$$

For any $v \in E^0_{\text{reg}}$, we have

$$[\eta^t_{v_0}(R_E - I)]_v = \begin{cases} |\{e \in E^1 \mid s(e) = v, r(e) \in W_{v_0}\}| - 1 & \text{if } v \in W_{v_0} \cap E^0_{\text{reg}} \\ |\{e \in E^1 \mid s(e) = v, r(e) \in W_{v_0}\}| & \text{if } v \notin W_{v_0} \cap E^0_{\text{reg}} \end{cases}$$

and we obtain the desired conclusion $\eta^t_{v_0}(R_E - I) \geq \delta^t_{v_0}$. We also see by Condition (iii) that for finite $F \subseteq E^0$ we may take

$$G = \bigcup_{w \in F} \{v_0 \in E^0_{\text{reg}} \mid w \in W_{v_0}\}$$

to arrange that $\eta_{v,w} = 0$ when $v \in F$ and $w \notin G$. □

Checking adhesiveness by the definition is rarely necessary, and frequently we will be able to appeal to one of these simpler sets of conditions:

Lemma 2.5.4. *Let $E = (E^0, E^1, r, s)$ be a graph. Then E is left adhesive when any of the conditions hold:*

(ℓ1) *$E^0 = E^0_{\text{reg}}$, and each $v_0 \in E^0$ supports two loops.*

(ℓ2) *For each $v_0 \in E^0$, there exist edges $e_1, \ldots, e_n \in E^1$ forming a cycle so that each $s(e_k) \in E^0_{\text{reg}}$, and $e_0 \in E^1$ so that $e_0 \neq e_1$, $s(e_0) = s(e_1)$, and $r(e_0) = v_0$*

In addition, E is right adhesive when any of the following conditions hold:

(r0) $E^0_{\text{reg}} = \emptyset$

(r1) *Each $v_0 \in E^0_{\text{reg}}$ supports two loops.*

(r2) *E^0_{reg} is finite, and for each $v_0 \in E^0_{\text{reg}}$, there exist edges $e_1, \ldots, e_n \in E^1$ forming a cycle, and $e_0 \in E^1$ so that $e_0 \neq e_1$, $r(e_0) = r(e_1)$, and $s(e_0) = v_0$.*

Proof. In (ℓ1) and (r1), we choose at each $v_0 \in E^0_{\text{reg}}$ the two loops as our e_0, e_1. Condition (r0) implies that the criteria to be right adhesive vacuously hold, and for (ℓ2) and (r2) we choose the indicated sets of edges, noting in the latter case that the finiteness condition is automatically true because E^0_{reg} is finite. □

Proposition 2.5.5. *Let $\mathcal{E}$ denote the exact sequence of abelian groups*

$$\begin{array}{ccccc} G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 \\ \delta\uparrow & & & & \downarrow 0 \\ F_3 & \xleftarrow{\gamma'} & F_2 & \xleftarrow{\epsilon'} & F_1 \end{array} \tag{2.18}$$

with F_1, F_2, and F_3 free abelian. Let $E_1 = (E_1^0, E_1^1, r_{E_1}, s_{E_1})$ and let $E_3 = (E_3^0, E_3^1, r_{E_3}, s_{E_3})$ be graphs such that isomorphisms

$$\alpha_i : K_0(C^*(E_i)) \to G_i \qquad \textit{and} \qquad \beta_i : K_1(C^*(E_i)) \to F_i$$

are given for $i = 1, 3$. If E_1 is left adhesive or E_3 is right adhesive, then there exists a graph $E_2 = (E_2^0, E_2^1, r_{E_2}, s_{E_2})$ with the properties

(1) $E_2^0 = E_1^0 \sqcup E_3^0$;

(2) E_2^1 is equal to the disjoint union of E_1^1 and E_3^1 together with

(a) a finite and nonzero number of edges from each $v \in (E_3^0)_{\mathrm{reg}}$ to vertices in E_1^0

(b) an infinite number of edges from each $v \in (E_3^0)_{\mathrm{sing}}$ to each vertex in E_1^0

so that with $\mathcal{I}$ the ideal of $C^*(E_2)$ given by E_1, the ideal $\mathcal{I}$ is essential, and there exist isomorphisms $\alpha_2 : K_0(C^*(E_2)) \to G_2$ and $\beta_2 : K_1(C^*(E_2)) \to F_2$ so that $K_{\mathrm{six}}(C^*(E_2), \mathcal{I})$ is isomorphic to $\mathcal{E}$ via the maps α_i and β_i, for $i = 1, 2, 3$.

Proof. Let R_{E_1} and R_{E_3} denote the vertex matrices of E_1 and E_3, respectively, and set $A = R_{E_1} - I$ and $B = R_{E_3} - I$. By Lemma 2.5.3, either A satisfies the conditions of Lemma 2.4.5 or B those of Lemma 2.4.6, so we may apply Proposition 2.4.7 to find Y such that $Y \geq Z$ where we set

$$Z = \begin{pmatrix} 1 & 1 & \cdots \\ 0 & 0 & \cdots \\ \vdots & & \end{pmatrix}.$$

Thus we obtain that Y is nonnegative, with a positive entry in each column. Using Y to read off how many edges to add, and adding an infinite number of edges from every $v \in (E_3)^0_{\text{sing}}$ to every $w \in E_1^0$ we create E_3 with regular vertex matrix R_{E_3} so that $R_{E_3} - I$ takes the form $\left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right)$. Note that we have arranged that E_1^0 is saturated and hereditary in E_2, and that

$$(E_2^0)_{\text{reg}} \cap E_3^0 = (E_3^0)_{\text{reg}}.$$

Hence E_1^0 defines a gauge-invariant ideal $\mathcal{I}$. To show that $\mathcal{I}$ is essential, it suffices to prove that $\mathcal{I}$ nontrivially intersects every nonzero gauge-invariant ideal. Let such an ideal be given by a hereditary and saturated set H along with a set of breaking vertices B. It then suffices to prove that $H \cap E_1^0 \neq \emptyset$. This follows by noting that if $H \subseteq E_3^0$, then some $v \in H \cap E_3^0$ may be chosen, and since this vertex has at least one edge to E_1^0, we have found the desired contradiction. □

Proposition 2.5.6. *In the situation of Proposition 2.5.5, if one of the following conditions holds:*

(i) *E_1^0 is the smallest hereditary and saturated subset of itself containing $v_1, \ldots, v_n$ for some finite choice of v_i*

(ii) E_3 is transitive, and either

(1) $(E_3^0)_{\mathrm{sing}} \neq \emptyset$; or

(2) $|E_3^0| = \infty$

then $\mathcal{I}$ may be chosen stenotic.

Proof. For (i), arrange the vertices of E_1 such that $v_1, \dots, v_n$ are listed first. Choosing Y dominating

$$Z = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ \vdots & & & \\ 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & & & \end{pmatrix}$$

in the previous proof we may arrange that there is an edge from each vertex in E_3 to each hereditary and saturated subset in E_1. Let $\mathcal{J}$ be an ideal of $C^*(E_2)$ which, as above, we may assume is gauge invariant and hence given by (H, B). Since no vertex in E_3 is breaking for any subset of E_1, we see that if $\mathcal{J} \not\subseteq \mathcal{I}$, H must intersect E_3^0. But then, by our construction and the condition that H is hereditary, we see that $E_1^0 \subseteq H$, and hence that $\mathcal{I} \subseteq \mathcal{J}$.

In the case (ii)(2) we choose instead Y dominating $Z = I$ and for (ii)(1) recall that every singular vertex of E_3 emits, by our construction, an edge to any vertex of E_1. Now when $\mathcal{J} \not\subseteq \mathcal{I}$ and (H, B) are given as above, we again get that $H \cap E_3^0 \neq \emptyset$, which implies by transitivity that $E_3^0 \subseteq H$. Then if $v \in E_3^0$ is singular, since it emits an edge to each vertex in E_1, we get that $H = E_2^0$ since H is hereditary. Similarly, if $v_1, v_2, \ldots$ are regular in E_3^0, we have that $\{v_1, \ldots, v_n\}$ emits to the first n vertices of E_1 so that again $H = E_2^0$. □

2.5.2 Order obstructions

Proposition 2.5.7. *Let $\mathcal{A}$ be a C^*-algebra with real rank zero, and let $\mathcal{I}$ be an ideal that is simple and purely infinite. If $\pi : \mathcal{A} \to \mathcal{A}/I$ denotes the quotient map, we have*

$$K_0(\mathcal{A})^+ = \{x \in K_0(\mathcal{A}) \mid \pi_*(x) \geq 0\}$$

Proof. The containment $K_0(\mathcal{A})^+ \subset \{x \in K_0(\mathcal{A}) \mid \pi_*(x) \geq 0\}$ is clear since π_* is positive. For the reverse inclusion, suppose $x \in K_0(\mathcal{A})$ with $\pi_*(x) \geq 0$. Then $\pi_*(x) \in K_0(A/I)^+$, and hence $\pi(x)$ is equal to the K_0-class of a projection in $M_n(A/I)$ for some $n \in \mathbb{N}$. Since A has real rank zero, it follows that $M_n(A)$ has real rank zero (see [9, Theorem 2.10]). Furthermore, since $M_n(A/I) \cong M_n(A)/M_n(I)$, and projections in quotients of real rank zero C^*-algebras lift to projections (see [9, Theorem 3.14]), it follows that there exists $x' \in K_0(\mathcal{A})^+$ with $\pi_*(x') = \pi_*(x)$. Thus $\pi(x - x') = 0$ and $x - x' \in K_0(I)$. Because I is simple and purely infinite, we have $K_0(I)^+ = K_0(I)$, so that $x - x' \in K_0(I)^+$, and hence $x - x' \in K_0(A)^+$. We conclude that $x = (x - x') + x' \in K_0(A)^+$. □

Proposition 2.5.8. *Let a C^*-algebra $\mathcal{A}$ be given with an approximate unit of projections, and with $\mathcal{I}$ a largest ideal. When there exists a projection $p \in \mathcal{A}$ such that*

(i) $p \notin \mathcal{I}$

(ii) $[p] = 0$ *in* $K_0(\mathcal{A})$

then $K_0(\mathcal{A})^+ = K_0(\mathcal{A})$.

Proof. We have noted that $K_0(\mathcal{A}) = K_0(\mathcal{A})^+ - K_0(\mathcal{A})^+$ whenever $\mathcal{A}$ has an approximate unit of projections, so it suffices to prove that $-K_0^+(\mathcal{A}) \subseteq K_0(\mathcal{A})$. Note that $p \in \mathcal{A} \subseteq \mathcal{A} \otimes \mathbb{K}$ is full since it is not an element of the largest ideal $\mathcal{I}$. Hence for any projection $q \in A \otimes \mathbb{K}$ there exists n with the property that q is Murray-von Neumann equivalent to a subprojection r of $p^{\oplus n}$, and we get that

$$-[q]_0 = n[p]_0 - [q]_0 = [p^{\oplus n} - r]_0 \geq 0$$

in $K_0(\mathcal{A})$. □

Proposition 2.5.9. *Let* $C^*(E)$ *be a graph* C^**-algebra with a gauge-invariant ideal* $\mathcal{I}$ *such that* $C^*(E)/\mathcal{I}$ *is purely infinite and simple. Then there exists a projection* $p \in C^*(E)$ *such that* $p \notin \mathcal{I}$ *and such that* $[p]_0 = 0$ *in* $K_0(\mathcal{A})$.

Proof. We may choose a subgraph $E_3 \subseteq E$ so that $C^*(E)/I \cong C^*(E_3)$, and in the subgraph E_3 there exists a vertex $v \in E_3^0$ supporting two different cycles $\xi, \eta \in E_3^*$. Let $p = p_v - s_\xi s_\xi^* \in C^*(E)$. Then $[p]_0 = 0$. But since $p \geq s_\eta s_\eta^*$ we have that $p \notin \mathcal{I}$. □

Corollary 2.5.10. *Let* E *be a graph, let* $\mathcal{I}$ *be an ideal of* $C^*(E)$*, and let* $\pi : C^*(E) \to C^*(E)/\mathcal{I}$ *be the quotient map. Then the following two statements hold.*

(1) If $C^*(E)$ *has real rank zero and* $\mathcal{I}$ *is purely infinite and simple, then*

$$K_0(C^*(E))^+ = \{x \in K_0(C^*(E)) \mid \pi_*(x) \geq 0\}.$$

(2) If $\mathcal{I}$ is the largest ideal of $C^*(E)$ and $C^*(E)/\mathcal{I}$ is purely infinite and simple, then

$$K_0(C^*(E))^+ = K_0(C^*(E)).$$

2.6 Six-term exact sequences realized by graph C^*-algebras

In this section we consider the range of the ordered six-term exact sequence from K-theory for various classes of graph C^*-algebras that are classified by this invariant.

2.6.1 Range of the Non-Unital Invariant

It was proven in [23, Theorem 4.7] that the six-term exact sequence $K_{\text{six}}^+(C^*(E), \mathcal{I}_{\max})$ is a complete stable isomorphism invariant when $C^*(E)$ is a graph C^*-algebra, $\mathcal{I}_{\max}$ is a largest ideal in $C^*(E)$, and $\mathcal{I}_{\max}$ is an AF-algebra. It was also proven in [20, Corollary 6.4] that the six-term exact sequence $K_{\text{six}}^+(C^*(E), \mathcal{I}_{\min})$ is a complete stable isomorphism invariant when $C^*(E)$ is a graph C^*-algebra, $\mathcal{I}_{\min}$ is a smallest nontrivial ideal in $C^*(E)$, and $C^*(E)/\mathcal{I}_{\min}$ is an AF-algebra. In the first case, $C^*(E)/\mathcal{I}_{\max}$ is simple and hence either purely infinite or AF. If $C^*(E)/\mathcal{I}_{\max}$ is AF, then $C^*(E)$ is an AF-algebra and the ordered group $K_0(C^*(E))$ is a complete stable isomorphism invariant. Thus the case that we are concerned with the six-term exact sequence is when $C^*(E)/\mathcal{I}_{\max}$ is purely infinite. Likewise, $\mathcal{I}_{\min}$ is simple and hence either purely infinite or AF. When $\mathcal{I}_{\min}$ is AF, then $C^*(E)$ is AF and the ordered group

$K_0(C^*(E))$ is again a complete stable isomorphism invariant. Thus the case that we are concerned with is when $\mathcal{I}_{\min}$ is purely infinite.

Theorem 2.6.1. *Let $C^*(E)$ be a graph C^*-algebra with a largest nontrivial ideal $\mathcal{I}$ such that $\mathcal{I}$ is an AF-algebra and $C^*(E)/\mathcal{I}$ is purely infinite. Then $K_{\mathrm{six}}^+(C^*(E), \mathcal{I})$ is a complete stable isomorphism invariant within this class, and the range of this invariant is all six-term exact sequences of countable abelian groups*

$$\begin{array}{ccccc} R_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 \\ {\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle 0} \\ F_3 & \xleftarrow{\gamma'} & F_2 & \xleftarrow{0} & 0 \end{array}$$

where F_2 and F_3 are free abelian groups, R_1 is a Riesz group, and G_2 and G_3 have the trivial pre-ordering (i.e., $G_i^+ = G_i$ for $i = 2, 3$).

Proof. It follows from [23, Theorem 4.7] that $K_{\mathrm{six}}^+(C^*(E), \mathcal{I})$ is a complete stable isomorphism invariant within this class. The necessity of the form of the exact sequence stated above follows from the fact that the descending map $\partial_0 : K_0(C^*(E)/\mathcal{I}) \to K_1(\mathcal{I})$ is always zero [11, Theorem 4.1], by well-known facts about the ordered K-theory of AF or purely infinite C^*-algebras, and from Corollary 2.5.10(2).

To see that all such exact sequences are attained, we know by Proposition 2.3.1 that there exists a row-finite graph with no sinks E_1 such that $C^*(E_1)$ is an AF-algebra and $K_0(C^*(E_1))$ is order isomorphic to R_1. By Proposition 2.3.3 there exists a row-finite graph with no sinks E_3 satisfying Conditions (1)–(3) of Proposition 2.3.3, with $K_0(C^*(E_3)) \cong G_3$ and $K_1(C^*(E_3)) \cong F_3$. Then $C^*(E_3)$ is purely infinite and simple. By Condition (2) of Proposition 2.3.3 each vertex of the graph E_3 contains two loops, and hence E_3 is right adhesive appealing to condition $(r1)$ of Lemma 2.5.4. We have arranged that $|E_3^0| = \infty$ and that E_3 is transitive, so by Proposition 2.5.6(ii)(2) we may glue together E_1 and E_3 in such a way that the ideal $\mathcal{I}$ of $C^*(E_2)$ corresponding to E_1 is stenotic. Since $C^*(E_2)/\mathcal{I} \cong C^*(E_3)$ is simple, it follows that $\mathcal{I}$ is the largest ideal of $C^*(E_2)$. We have arranged that $\mathcal{I}$ is AF. Moreover, Corollary 2.5.10(2) implies $K_0(C^*(E_2))^+ = K_0(C^*(E_2))$, so since we have assumed that $G_2^+ = G_2$, our constructed map α_2 will automatically be an order isomorphism. Hence $K_{\text{six}}^+(C^*(E_2), \mathcal{I})$ is order isomorphic to the required six-term exact sequence. □

Theorem 2.6.2. *Let $C^*(E)$ be a graph C^*-algebra with a smallest nontrivial ideal $\mathcal{I}$ such that $C^*(E)/\mathcal{I}$ is an AF-algebra and $\mathcal{I}$ is purely infinite. Then $K_{\text{six}}^+(C^*(E), \mathcal{I})$ is a complete stable isomorphism invariant within this class, and the range of this invariant is all six-term exact sequences of countable abelian groups*

$$\begin{array}{ccccc} G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & R_3 \\ {\scriptstyle 0}\uparrow & & & & \downarrow{\scriptstyle 0} \\ 0 & \xleftarrow{0} & F_2 & \xleftarrow{\epsilon'} & F_1 \end{array}$$

where F_1 and F_2 are free abelian groups, R_3 is a Riesz group, the group G_1 has the trivial pre-ordering $G_1^+ = G_1$, and G_2 has the pre-ordering $G_2^+ = \{x \in G_2 : \gamma(x) \geq 0\}$.

Proof. It follows from [20, Corollary 6.4] that $K_{\text{six}}^+(C^*(E), \mathcal{I})$ is a complete stable isomorphism invariant within this class. The necessity of the form of the exact sequence stated above follows from the fact that the descending map $\partial_0 : K_0(C^*(E)/\mathcal{I}) \to K_1(\mathcal{I})$ is always zero [11, Theorem 4.1], the fact that the K_0-group of an AF-algebra is always a Riesz group, the fact that the K_1-group of an AF-algebra is always zero, the fact that the homomorphisms that appear are always order homomorphisms, and from Corollary 2.5.10(2).

To see that all such exact sequences are attained, we know by Proposition 2.3.1 that there exists a row-finite graph with no sinks E_3 such that $C^*(E_3)$ is an AF-algebra and $K_0(C^*(E_3))$ is order isomorphic to R_3. By Proposition 2.3.3 there exists a row-finite graph with no sinks E_1 satisfying Conditions (1)–(3) of Proposition 2.3.3, with $K_0(C^*(E_1)) \cong G_1$ and $K_1(C^*(E_1)) \cong F_1$. Then $C^*(E_1)$ is purely infinite and simple. By Condition (2) of Proposition 2.3.3 each vertex of the graph E_1 contains two loops. Thus the regular vertex matrix $R_{E_1} - I$ has non-negative entries and positive entries down its diagonal. It follows from Lemma 2.5.4 that E_1 is left-adhesive. Thus Proposition 2.5.5 shows that there exists a graph E_2 satisfying Conditions (1)–(4) of Proposition 2.5.5. Furthermore, $C^*(E_2)/\mathcal{I}$ is an AF-algebra, and $\mathcal{I}$ is Morita equivalent to $C^*(E_1)$ and thus purely infinite and simple. Because every vertex of E_3^0 has a finite and nonzero number of edges from this vertex to E_1^0, and because E_1 is strongly connected, it follows that any nonempty hereditary subset of E_2 must contain E_1^0. Thus $\mathcal{I}$ is the smallest ideal of $C^*(E_2)$. Moreover, Corollary 2.5.10(1) implies $K_0(C^*(E_2))^+ = \{x \in K_0(C^*(E_2)) : \pi_*(x) \geq 0\}$ and thus in the commutative diagram (2.3) we have that ϕ is an order isomorphism. Hence $K_{\text{six}}^+(C^*(E_2), \mathcal{I})$ is order isomorphic to the required six-term exact sequence. □

Next we consider graph C^*-algebras with a unique nontrivial ideal. Recall that if E is a graph and $\mathcal{I}$ is a unique nontrivial ideal in $C^*(E)$, then $\mathcal{I}$ and $C^*(E)/\mathcal{I}$ are simple. Since $\mathcal{I}$ and $C^*(E)/\mathcal{I}$ are both graph C^*-algebras, by [13, Lemma 1.3] and [6, Corollary 3.5], and since any simple graph C^*-algebra is either an AF-algebra or a Kirchberg algebra, there are four cases to consider:

Type $[\infty\infty]$: $\mathcal{I}$ and $C^*(E)/\mathcal{I}$ are both Kirchberg algebras.

Type $[\mathbf{1}\infty]$: $\mathcal{I}$ is an AF-algebra and $C^*(E)/\mathcal{I}$ is a Kirchberg algebra.

Type $[\infty\mathbf{1}]$: $\mathcal{I}$ is a Kirchberg algebra and $C^*(E)/\mathcal{I}$ is an AF-algebra.

Type $[\mathbf{11}]$: $\mathcal{I}$ and $C^*(E)/\mathcal{I}$ are both AF-algebras.

Theorem 2.6.3. *If $C^*(E)$ is a graph C^*-algebra with a unique nontrivial ideal $\mathcal{I}$, then $K_{\text{six}}^+(C^*(E), \mathcal{I})$ is a complete stable isomorphism invariant within this class. The range of this invariant for the four possible types are precisely the collections of six-term exact sequences*

$$\begin{array}{ccccc} G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 \\ {\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle 0} \\ F_3 & \xleftarrow{\gamma'} & F_2 & \xleftarrow{\epsilon'} & F_1 \end{array}$$

where each of F_1, F_2, and F_3 are free abelian groups, and the following additional properties are satisfied for each type:

Type $[\infty\infty]$: All the groups G_1, G_2, and G_3 have the trivial pre-ordering (i.e., $G_i^+ = G_i$ for $i = 1, 2, 3$).

Type $[\mathbf{1}\infty]$: $F_1 = 0$, G_1 is a simple Riesz group, and both G_2 and G_3 have the trivial pre-ordering (i.e., $G_i^+ = G_i$ for $i = 2, 3$).

Type $[\infty\mathbf{1}]$ *:* $F_3 = 0$*,* G_3 *is a simple Riesz group, the group* G_1 *has the trivial pre-ordering (i.e.,* $G_1^+ = G_1$*), and* G_2 *has the pre-ordering* $G_2^+ = \epsilon(G_1) \sqcup \{x \in G_2 : \gamma(x) > 0\}$*.*

Type $[\mathbf{11}]$ *:* $F_1 = 0$*,* $F_2 = 0$*,* $F_3 = 0$*, all three of the groups* G_1*,* G_2*, and* G_3 *are Riesz groups,* G_1 *and* G_3 *are simple ordered groups, and the sequence is lexicographically ordered (see [29]); i.e.,* $\epsilon(G_1^+) = \epsilon(G_1) \cap G_2^+$ *and* $\gamma(G_2^+) = G_3^+$*.*

Proof. It follows from [23, Theorem 4.5] that $K_{\text{six}}^+(C^*(E), \mathcal{I})$ is a complete stable isomorphism invariant for the class of graph C^*-algebras with a unique nontrivial ideal. Also, the necessity of the forms of the exact sequences stated for the four types follows from the fact that the descending map $\partial_0 : K_0(C^*(E)/\mathcal{I}) \to K_1(\mathcal{I})$ is always zero [11, Theorem 4.1], the fact that the K_0-group of an AF-algebra is always a Riesz group, the fact that the K_1-group of an AF-algebra is always zero, the fact that the homomorphisms that appear are always order homomorphisms, and from Corollary 2.5.10. To see that all sequences in the four types are attained, we consider the four cases separately.

Case $[\infty\infty]$**:** By Proposition 2.3.3 there exist row-finite graphs with no sinks E_1 and E_3, each satisfying Conditions (1)–(3) of Proposition 2.3.3, with $K_0(C^*(E_1)) \cong G_1$ and $K_1(C^*(E_1)) \cong F_1$ and with $K_0(C^*(E_3)) \cong G_3$ and $K_1(C^*(E_3)) \cong F_3$. Then $C^*(E_1)$ and $C^*(E_3)$ are purely infinite and simple, and in fact both are left and right adhesive. Thus it follows from Proposition 2.5.5 that E_1 and E_3 may be glued to obtain a graph E_2 with an essential ideal $\mathcal{I}$ such that $C^*(E_2)/\mathcal{I} \cong C^*(E_3)$ is purely infinite and simple, and $\mathcal{I}$ is Morita equivalent to $C^*(E_1)$ and thus also purely infinite and simple. It follows that $\mathcal{I}$ is the nontrivial ideal of $C^*(E_2)$ and that $C^*(E_2)$ is of type $[\infty\infty]$. Moreover, the commutative diagram that appears in (2.3) has that ϕ is an order isomorphism due to the fact that $K_0(C^*(E_2))^+ = K_0(C^*(E_2))$ by Corollary 2.5.10 and $G_2^+ = G_2$. Thus $K_{\text{six}}^+(C^*(E_2), \mathcal{I})$ is order isomorphic to the required six-term exact sequence.

Case $[\mathbf{1}\infty]$**:** This is a special case of Theorem 2.6.1.

Case $[\infty\mathbf{1}]$**:** This is a special case of Theorem 2.6.2.

Case $[\mathbf{11}]$**:** It follows from Proposition 2.3.1 that there exists a row-finite graph with no sinks E such that $C^*(E)$ is an AF-algebra and $K_0(C^*(E))$ is order isomorphic to R_2. Since the extension $0 \to R_1 \to R_2 \to R_3 \to 0$ is order exact and R_1 and R_3 are simple, it follows that R_2 has exactly one nontrivial order ideal, namely $\epsilon(R_1)$. Thus there exists a unique nontrivial ideal $\mathcal{I} \triangleleft C^*(E)$ with $K_{\text{six}}(C^*(E), \mathcal{I})$ isomorphic to the given sequence. Furthermore, since ideals and quotients of AF-algebras are AF-algebras, we see that $C^*(E)$ is of type $[\mathbf{11}]$. □

2.6.2 Range of the Unital Invariant

Next we consider the range of the ordered six-term exact sequence for graph C^*-algebras that are unital extensions of Kirchberg algebras. The ordered six-term exact sequence together with the position of the unit in the K_0-group of the extension has been shown to be a complete isomorphism invariant for unital graph C^*-algebras with a unique nontrivial ideal (see [19] and [22, Theorem 5.3 and Corollary 5.4]). We describe the range of this invariant here.

Theorem 2.6.4. *If $C^*(E)$ is the C^*-algebra of a graph with a finite number of vertices that contains a unique nontrivial ideal $\mathcal{I}$, then $K_{\text{six}}^+(C^*(E), \mathcal{I})$ together with the element $[1_{C^*(E)}]_0 \in K_0(C^*(E)$ is a complete isomorphism invariant within this class. The range of this invariant for this class is all six-term exact sequences*

$$\begin{array}{ccccc} G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 \\ \uparrow{\scriptstyle \delta} & & & & \downarrow{\scriptstyle 0} \\ F_3 & \xleftarrow{\gamma'} & F_2 & \xleftarrow{\epsilon'} & F_1 \end{array}$$

as in Theorem 2.6.3, satisfying the further conditions:

(1) F_1, F_3, G_1, *and* G_3 *are finitely generated abelian groups,*

(2) $\operatorname{rank} F_1 \leq \operatorname{rank} G_1$ *and* $\operatorname{rank} F_3 \leq \operatorname{rank} G_3$,

(3) if either (G_1, G_1^+) *or* (G_3, G_3^+) *is a Riesz group, that group is* $(\mathbb{Z}, \mathbb{Z}^+)$.

In addition, the the order unit g_2 of $K_0(C^*(E))$ can be chosen to be any element of G_2 satisfying

(4) if $(G_3, G_3^+) = (\mathbb{Z}, \mathbb{Z}^+)$, *then* $\gamma(g_2) > 0$.

Moreover, if $G_1 \cong \mathbb{Z}_{m_1} \oplus \ldots \oplus \mathbb{Z}_{m_k} \oplus \mathbb{Z}^m$ and $G_3 \cong \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_l} \oplus \mathbb{Z}^n$, then E may be chosen with $m + k + n + l + 2$ or fewer vertices.

Proof. The fact that $K_{\text{six}}^+(C^*(E), \mathcal{I})$ with the element $[1_{C^*(E)}]_0 \in K_0(C^*(E)$ is a complete isomorphism invariant follows from [19] and [22, Theorem 5.3 and Corollary 5.4].

The necessary conditions on the six-term exact sequence from Theorem 2.6.3 are of course also relevant here, and we saw in the discussion preceding Proposition 2.3.6 that (1) and (2) must hold in this case. In addition, we note that the only unital simple graph C^*-algebras that are AF are of the form $M_n(\mathbb{C})$, proving that when G_3 is a Riesz group, it must be $\mathbb{Z}$. The same reasoning holds for G_1 since the ideal must be Morita equivalent to $M_n(\mathbb{C})$. And finally, when $g_2 = [1_{C^*(E)}]_0$ is given, we get that $\gamma(g_2)$ is given by the unit of the quotient, which is a strictly positive element of $K_0(M_n(\mathbb{C}))$ in case that is AF, proving necessity of (4).

To realize the invariant we argue separately in each of the following cases.

Case $[\infty\infty]$**:** By appealing to Proposition 2.3.6 rather than Proposition 2.3.3, we obtain a graph E_1 realizing G_1 and F_1 as above, and a graph E_3 realizing G_3 and F_3 as above but with the added assumptions that $|E_1^0| = m + k + 1$, $|E_3^0| = n + l + 1$, and that $[\mathbf{1}] = \gamma(g_2)$ in $K_0(C^*(E_3))$. Moreover, Proposition 2.3.8 shows that we may choose E_3 so that condition (4) of Proposition 2.3.8 holds. For $1 = 1, 3$, we get that $C^*(E_i)$ is purely infinite and simple, as desired. Condition (4) of Proposition 2.3.8 shows that the hypotheses of Proposition 2.4.8 are satisfied by $B = R_{E_3} - I$. Thus Proposition 2.4.8 implies we may obtain E_2 by gluing E_1 and E_3 in a way that the obtained isomorphism α_2 sends $\mathbf{1}$ to g_2. The total number of vertices in E_2 is $|E_2^0| = |E_1^0| + |E_3^0| = m + k + n + l + 2$.

Case $[\mathbf{1}\infty]$**:** In this case $G_1 = \mathbb{Z}$ and $F_1 = 0$, which we may realize by a graph E_1 with one vertex and no edges. We apply Proposition 2.3.3 to obtain a graph E_3 realizing G_3 and F_3 and with the added assumptions $|E_3^0| = n + l + 1$, and that $[\mathbf{1}] = \gamma(g_2)$ in $K_0(C^*(E_3))$. Moreover, Proposition 2.3.8 shows that we may choose E_3 so that condition (4) of Proposition 2.3.8 holds. We get that $C^*(E_3)$ is purely infinite and simple, as desired. Condition (4) of Proposition 2.3.8 shows that the hypotheses of Proposition 2.4.8 are satisfied by $B = R_{E_3} - I$. Thus Proposition 2.4.8 implies we may obtain E_2 by gluing E_1 and E_3 in a way that the obtained isomorphism α_2 sends $\mathbf{1}$ to g_2. The total number of vertices in E_2 is $|E_2^0| = |E_1^0| + |E_3^0| = 1+n+l+1 = n+l+2$. Since $k = 0$ and $m = 1$ in this case, we have $|E_2^0| = n + l + 2 < m + k + n + l + 2$.

Case $[\infty\mathbf{1}]$**:** In this case $G_3 = \mathbb{Z}$ and $F_3 = 0$, so there is a splitting map σ for γ. We realize G_3 by a graph E_1 with two vertices $\{v, w\}$ and $n - 1$ edges from v to w when $n > 1$, or by a solitary vertex if $n = 1$. We apply Proposition 2.3.3 to obtain a graph E_1 realizing G_1 and F_1 and with the added assumptions $|E_1^0| = m + k + 1$, and that $[\mathbf{1}] = g_2 - \sigma \circ \gamma(g_2)$ in $K_0(C^*(E_1))$. Letting E_2 be the union of E_1 and E_3 with infinitely many edges from the sink in E_3 to each vertex in E_1, we obtain a graph C^*-algebra $C^*(E_2)$ with a unique ideal. We then appeal to Proposition 2.4.9 to see that the invariant has the desired form. We also see that the graph E_2 has $|E_2^0| = |E_1^0| + |E_3^0| < m + k + 1 + 2 = m + k + 3$. Since $l = 0$ and $n = 1$ in this case, we have $|E_2^0| < m + k + n + l + 2$.

Case $[\mathbf{11}]$**:** One checks by elementary methods that a graph E_2 of the form

$$\bullet \xrightarrow{\ x\ } \bullet \overset{\infty}{\Longrightarrow} \bullet \xrightarrow{\ y\ } \bullet$$

where x and y are finite nonzero numbers indicating the number of edges drawn, can realize all possible choices of order and unit in the given extension. In this case $k = l = 0$ and $m = k = 1$, and we see that $|E_2^0| = 4 = m + k + n + l + 2$. $\square$

Using basic group theory, and the fact that if $\phi : G \to H$ is a group homomorphism, then $\operatorname{rank} G = \operatorname{rank} \operatorname{im} \phi + \operatorname{rank} \ker \phi$, we see that the following relations are also satisfied:

- F_2 and G_2 are finitely generated abelian groups
- $\operatorname{rank} F_1 \leq \operatorname{rank} F_2 \leq \operatorname{rank} F_1 + \operatorname{rank} F_3$
- $\operatorname{rank} F_2 - \operatorname{rank} G_2 = \operatorname{rank} F_3 - \operatorname{rank} G_3 + \operatorname{rank} F_1 - \operatorname{rank} G_1$ (so that, in particular, $\operatorname{rank} F_2 \leq \operatorname{rank} G_2$).

Recall that the class of Cuntz-Krieger algebras of matrices satisfying Condition (II) coincides with the class of C^*-algebras of finite graphs with no sinks that satisfy Condition (K), which also coincides with the C^*-algebras of finite graphs with no sinks that have a finite number of ideals (see [4, Theorem 3.13]). The following result can therefore be interpreted as determining the range of the six-term exact sequence for Cuntz-Krieger algebras with a unique nontrivial ideal.

Theorem 2.6.5. *If $C^*(E)$ is the C^*-algebra of a finite graph with no sinks and with a unique nontrivial ideal $\mathcal{I}$, then $K_{\text{six}}(C^*(E), \mathcal{I})$ and $[1_{C^*(E)}]$ is a complete isomorphism invariant within this class. The range of this invariant is all six-term exact sequences*

$$\begin{array}{ccccc} G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 \\ \delta \uparrow & & & & \downarrow 0 \\ F_3 & \xleftarrow{\gamma'} & F_2 & \xleftarrow{\epsilon'} & F_1 \end{array}$$

satisfying the conditions (1),(3),(4) of Theorem 2.6.4 as well as satisfying:

(2') $\operatorname{rank} G_1 = \operatorname{rank} F_1$ *and* $\operatorname{rank} G_3 = \operatorname{rank} F_3$.

The order unit of $C^*(E)$ can be any element of G_2. Moreover, if $G_1 \cong \mathbb{Z}_{m_1} \oplus \ldots \oplus \mathbb{Z}_{m_k} \oplus \mathbb{Z}^m$ and $G_3 \cong \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_l} \oplus \mathbb{Z}^n$, then E may be chosen with no more than $m+k+n+l+2$ vertices.

Proof. Proceed as in the proof of Case $[\infty\infty]$ of Theorem 2.6.3, but use Proposition 2.3.9 in place of Proposition 2.3.3. Also observe that $C^*(E)$, as well as the ideal and quotient by the ideal, are purely infinite in this case, and hence the ordering on the K_0-groups is unnecessary in the invariant. □

From basic group theory, and using the fact that if $\phi : G \to H$ is a group homomorphism, then $\operatorname{rank} G = \operatorname{rank} \operatorname{im} \phi + \operatorname{rank} \ker \phi$, we see that the following relations are also satisfied:

- F_2 and G_2 are finitely generated abelian groups
- $\operatorname{rank} F_1 \leq \operatorname{rank} F_2 \leq \operatorname{rank} F_1 + \operatorname{rank} F_3$
- $\operatorname{rank} F_2 = \operatorname{rank} G_2$.

2.7 Permanence

Consider a class of C^*-algebras $\mathfrak{C}$ with the property that whenever $\mathcal{A} \in \mathfrak{C}$, any ideal $\mathcal{I}$ and any quotient $\mathcal{A}/\mathcal{I}$ also lies in $\mathfrak{C}$. A **permanence result** for $\mathfrak{C}$ is a result that gives conditions for any extension

$$0 \longrightarrow \mathcal{I} \xrightarrow{\iota} \mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{I} \longrightarrow 0$$

to have the property that $\mathcal{I}, \mathcal{A}/\mathcal{I} \in \mathfrak{C}$ implies $\mathcal{A} \in \mathfrak{C}$, in terms of the six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{I}) & \xrightarrow{\iota_*} & K_0(\mathcal{A}) & \xrightarrow{\pi_*} & K_0(\mathcal{A}/\mathcal{I}) \\ {\scriptstyle\partial_1}\uparrow & & & & \downarrow{\scriptstyle\partial_0} \\ K_1(\mathcal{A}/\mathcal{I}) & \xleftarrow[\pi_*]{} & K_1(\mathcal{A}) & \xleftarrow[\iota_*]{} & K_1(\mathcal{I}) \end{array}$$

from K-theory.

Two well-known permanence results are of direct relevance for the following. First, Brown in [10] proved that if $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are AF algebras, then so is $\mathcal{A}$. And second, it follows from [9, Theorem 3.14 and Corollary 3.16] that if $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are of real rank zero, then $\mathcal{A}$ is of real rank zero precisely when $\partial_0 = 0$.

We now set out to prove a permanence result for the class $\mathfrak{C}$ of stable graph C^*-algebras of real rank zero. It is known that when $\mathcal{A} \in \mathfrak{C}$, then so is any ideal $\mathcal{I}$ and any quotient $\mathcal{A}/\mathcal{I}$. We offer the following permanence result under the added assumptions that $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are either AF or simple, and $\mathcal{I}$ is a stenotic ideal of $\mathcal{A}$.

Our strategy for doing so is as follows: Given an extension of graph C^*-algebras satisfying the necessary conditions, we build, using the results in the previous section, a graph realizing its K-theoretic data. And then we appeal to work of the first named author, Restorff and Ruiz to be able to prove by classification that the given extension C^*-algebra is in fact isomorphic to the one given by the constructed graph.

Theorem 2.7.1. *Let*

$$0 \longrightarrow C^*(E_1) \longrightarrow \mathcal{A} \longrightarrow C^*(E_3) \longrightarrow 0$$

be a stenotic extension with $C^*(E_1)$ *and* $C^*(E_2)$ *both stable and either simple or AF.*

The following three statements are equivalent:

(i) $\mathcal{A}$ *is a graph* C^**-algebra.*

(ii) $\mathcal{A}$ *is a graph* C^**-algebra of real rank zero.*

(iii) (1) $\partial_0 = 0$*; and*

(2) $K_0(C^*(E_3))^+ = K_0(C^*(E_3)) \Longrightarrow K_0(\mathcal{A})^+ = K_0(\mathcal{A})$.

Proof. We first note that by Brown's extension result combined with Proposition 2.3.1, all of the claims hold true in the [**11**] case, with (iii)(2) being vacuously true.

Turning to the remaining three cases, let $\mathcal{I} = C^*(E_1)$ considered as an ideal of $\mathcal{A}$. In these cases, either $\mathcal{I}$ or $\mathcal{A}/\mathcal{I}$ is simple, so that $\mathcal{I}$ is necessarily gauge-invariant. As we have seen, this forces $\partial_0 = 0$, and since both $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ have real rank zero, we conclude the same about $\mathcal{A}$, proving $(i) \Longrightarrow (ii)$. That $(ii) \Longrightarrow (iii)(1)$ is also clear from [9], and that $(ii) \Longrightarrow (iii)(2)$ follows from Corollary 2.5.10(2) since when $K_0(C^*(F))$ is trivially ordered, $C^*(F)$ must in our case be simple and purely infinite.

The remaining implication $(iii) \Longrightarrow (i)$ is the most difficult. To begin, we assume that (iii) holds. Since $C^*(E_1)$ is a graph C^*-algebra that is either simple or AF, it follows that $C^*(E_1)$ has the corona factorization property. Because $C^*(E_3)$ is a separable stable C^*-algebra, and $C^*(E_1)$ has the corona factorization property, it follows from [40] that $\mathcal{A}$ is stable.

In case $[\infty\infty]$, we know that $K_0(\mathcal{A})$ is trivially ordered, and may hence use Theorem 2.6.3 to realize $K^+_{\mathrm{six}}(\mathcal{A},\mathcal{I})$ by some graph E_2. Since $M(\mathcal{I})/\mathcal{I}$ is simple, the extension is automatically full, and it follows from [20, Theorem 4.6] that $\mathcal{A} \cong$

$C^*(E_2)$, and $\mathcal{A}$ is a graph C^*-algebra. The case $[\infty\mathbf{1}]$ is solved precisely the same way by appealing instead to Theorem 2.6.2 to realize $K^+_{\mathrm{six}}(\mathcal{A},\mathcal{I})$. In case $[\mathbf{1}\infty]$ the result follows similarly, appealing to Theorem 2.6.1. This time the extension is not full a priori, but turns out to be so because of condition (iii)(2) combined with [21, Corollary 3.17], so again we obtain the desired result. □

The condition (iii)(2) is vacuously true in the $[\mathbf{11}]$ and $[\infty\mathbf{1}]$ cases, and automatically true in the $[\infty\infty]$ case as seen in Proposition 2.5.7. It is necessary in the $[\mathbf{1}\infty]$ case as noted in [21, Example 4.4].

The class of unital graph C^*-algebras is not closed under taking ideals. Nevertheless, we have the following.

Theorem 2.7.2. *Let*

$$0 \longrightarrow C^*(E_1) \otimes \mathbb{K} \longrightarrow \mathcal{A} \longrightarrow C^*(E_3) \longrightarrow 0$$

be a unital essential extension with $C^*(E_1)$ and $C^*(E_3)$ both unital, simple, and purely infinite C^*-algebras. The following are equivalent:

(i) $\mathcal{A}$ is a graph C^*-algebra.

(ii) $\mathcal{A}$ has real rank zero.

Proof. That (i) implies (ii) follows as above. To prove the other implication, set $\mathcal{I} = C^*(E_1) \otimes \mathbb{K}$ and first realize $K_{\mathrm{six}}(\mathcal{A},\mathcal{I})$ along with the given element $[1_{\mathcal{A}}]_0$ by some finite graph E_2, using Theorem 2.6.4. By [19], we get that $\mathcal{A} \cong C^*(E_2)$. □

Similarly, by Theorem 2.6.5 we obtain the following result.

Theorem 2.7.3. *Let*

$$0 \longrightarrow \mathcal{O}_{B_1} \otimes \mathbb{K} \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}_{B_3} \longrightarrow 0$$

be a unital essential extension with $\mathcal{O}_{B_1}$ and $\mathcal{O}_{B_3}$ both simple Cuntz-Krieger algebras. The following are equivalent:

(1) $\mathcal{A}$ is a Cuntz-Krieger algebra.

(2) $\mathcal{A}$ is a graph C^*-algebra.

(3) $\mathcal{A}$ has real rank zero.

We see no reason why this theorem should not hold when $\mathcal{O}_{B_1}$ and $\mathcal{O}_{B_3}$ are given of real rank zero and with an arbitrary ideal lattice, but at the moment proving this seems outside reach. Substantial progress has been reported in [2] and [3].

CHAPTER 3

Observing Lyapunov Exponents of Infinite-dimensional Systems

3.1 Introduction

The paper below was published in The Journal of Statistical Physics in December 2015 [53]. The three authors of the paper are William Ott, Mauricio A. Rivas, and myself.

This paper is about observing Lyapunov exponents of infinite-dimensional dynamical systems by projecting the dynamics into $\mathbb{R}^N$. We focus on discrete-time infinite-dimensional dynamics produced by maps on real Hilbert spaces. Important types of such maps include time-T maps and Poincaré return maps generated by the solution semigroups of evolution partial differential equations.

Let H be a real Hilbert space and let $f : H \to H$ be a C^1 (continuously Fréchet-differentiable) map. A ***Lyapunov exponent*** $\omega(x, v)$ is a limit of the form

$$\omega(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n v\| , \tag{3.1}$$

where $x \in H$ and $v \in T_x H$ is a tangent vector.

3.1.1 Lyapunov exponents in finite dimensions

Lyapunov exponents play a central role in the theory of nonuniformly hyperbolic dynamical systems in finite dimensions (here the domain of f is a compact Riemannian manifold M). They are deeply related to a number of dynamical quantities of interest, including entropy, dimension, and rates of escape in open systems. Although Lyapunov exponents encode information about the infinitesimal behavior of f, a vast array of results demonstrates that local and even global information about the nonlinear dynamics of f can be deduced from them (see *e.g.* [5, 8, 71]).

The limit in (3.1) does not necessarily exist for every (x, v) in the tangent bundle TM; nevertheless, Lyapunov exponents exist for almost every $x \in M$ assuming stationarity. Oseledec [51] proves that if μ is an f-invariant Borel probability measure, then for μ almost every $x \in M$, there exist numbers

$$\omega_1(x) > \omega_2(x) > \cdots > \omega_{q(x)}(x)$$

with corresponding multiplicities $m_1(x), \ldots , m_{q(x)}(x)$ such that

(a) for every tangent vector $v \in T_xM$, $\omega(x, v)$ exists and equals $\omega_j(x)$ for some j;

(b) $\sum_{i=1}^{q(x)} m_i(x) = \dim(M)$;

(c) $\sum_{i=1}^{q(x)} \omega_i(x) m_i(x) = \lim_{n\to\infty} \left(\frac{1}{n}\right) \log |\det(Df_x^n)|$.

Further, if f is a diffeomorphism of M, then the tangent space T_xM admits a decomposition

$$T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{q(x)}(x)$$

with $\dim(E_i(x)) = m_i(x)$ and $\omega(x, v) = \omega_i(x)$ for every $v \in E_i(x)$. If μ is ergodic, then the $\omega_i(\cdot)$ are constant μ almost everywhere; in this case we call the ω_i the Lyapunov exponents of the system (f, μ).

While the Oseledec multiplicative ergodic theorem makes conclusions about Lyapunov exponents given an invariant measure, the *existence* of important invariant measures for dynamical systems that exhibit some degree of hyperbolicity is another matter entirely. Researchers actively work to identify mechanisms that may produce nonuniform hyperbolicity and then prove that these mechanisms do produce nonuniform hyperbolicity for concrete systems of interest in the physical and biological sciences. This program has been carried out for limit cycles and homoclinic orbits in [70] and [69], respectively.

3.1.2 Lyapunov exponents in infinite dimensions

Here one starts with a dynamical system $\sigma : \Omega \to \Omega$, selects a Banach space B, and then assigns to each $\omega \in \Omega$ a bounded linear operator $\mathcal{L}_\omega$ on B. The assignment $\omega \mapsto \mathcal{L}_\omega$ is known as a cocycle over the dynamical system. Having defined the cocycle, one then hopes to prove a multiplicative ergodic theorem in the spirit of Oseledec that applies to the compositions $\mathcal{L}_\omega^{(n)} = \mathcal{L}_{\sigma^{n-1}(\omega)} \circ \cdots \circ \mathcal{L}_{\sigma(\omega)} \circ \mathcal{L}_\omega$. For a smooth map f on a real Hilbert space H, Ruelle proves a multiplicative ergodic theorem for the derivative cocycle assuming Df_x is compact [63]. Cocycles into operators on Banach spaces (possibly with nontrivial essential spectrum) are treated in [26, 41, 48, 67].

Transfer operator techniques have led to substantial understanding of the statistical properties of deterministic autonomous dynamical systems. With an eye on applications, multiplicative ergodic theorists in recent years have sought to extend transfer operator techniques to nonautonomous and random dynamical systems. This effort has led to beautiful multiplicative ergodic theorems for transfer operator cocycles [26, 27].

The program aimed at deducing global dynamical information about infinite-dimensional systems from Lyapunov exponent data is in its early stages of development. Results in this direction include the existence of Sinai-Ruelle-Bowen (SRB) measures for periodically-kicked supercritical Hopf bifurcations in a concrete PDE context [46] and the existence of horseshoes in a general context [42, 43].

3.1.3 Observation of Lyapunov exponents

Suppose $A \subset H$ satisfies $f(A) = A$ (we call A an invariant set). For example, A may be the global attractor of a dissipative PDE such as the two-dimensional incompressible Navier-Stokes system. We are interested in observing Lyapunov exponents of the restriction $f|A$ by projecting the dynamics into $\mathbb{R}^N$. For a map $\varphi : H \to \mathbb{R}^N$ (we call φ an observable or measurement map), we say that φ induces dynamics on $\varphi(A)$ if there exists a map $f_* : \varphi(A) \to \varphi(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\varphi} \\ \varphi(A) & \xrightarrow{f_*} & \varphi(A) \end{array}$$

Question 1. For a 'typical' observable φ, if φ induces dynamics on $\varphi(A)$ and if $\omega(z, w)$ is a Lyapunov exponent for f_*, do there exist $x \in A$ and a vector v such that $\omega(x, v)$ is a Lyapunov exponent for $f|A$ and $\omega(x, v) = \omega(z, w)$?

Ott and Yorke [54] develop an affirmative answer to Question 1 for the case $H = \mathbb{R}^D$. In this work we treat the infinite-dimensional case by developing an embedding result of the following type: For a 'typical' observable φ, if φ induces dynamics on $\varphi(A)$, then φ embeds A into $\mathbb{R}^N$. We then use this embedding result (Theorem 3.3.10) to answer Question 1 in the affirmative (Corollary 3.3.11). Keep the following in mind as we develop the theory.

(a) (Placement of hypotheses) Since we develop a theory of observation, we strive to place the hypotheses on the observed set $\varphi(A)$ and the induced dynamics thereon rather than on f and A. Indeed, we view f and A as objects that are not known *a priori*.

(b) (Notion of 'typical') We use the measure-theoretic notion of prevalence [12, 33, 34, 35, 55]. Prevalence is suitable for spaces of observables such as $C^1(H, \mathbb{R}^N)$. See Section 3.2 for a brief overview.

(c) (Generalized tangent spaces) We expect the set A to have fractal properties. We therefore use a generalized notion of tangent space suitable for such sets (Definition 3.3.2).

We finish the introduction by briefly examining an alternate approach to the embedding problem: Use assumptions about dimension to embed A into $\mathbb{R}^N$ (in the spirit of the Whitney embedding theorem) rather than formulating results in terms of induced dynamics. As we will see, one encounters an unresolved challenge when using dimension characteristics.

3.1.4 Embedding results via dimension characteristics

Here we discuss a prototype result that makes use of dimension.

Prototype Theorem 3.1.1. *Let H be a real Hilbert space and let $A \subset H$ be compact. Fix $N \in \mathbb{N}$. For almost every (in the sense of prevalence) $\varphi \in C^1(H, \mathbb{R}^N)$, if* $\dim(\varphi(A)) < N/2$*, then φ is one-to-one on A.*

Observe that the hypothesis involving dimension is placed on $\varphi(A)$ rather than A. We do not know if there exists a dimension characteristic for which the prototype theorem holds. Natural candidates include box-counting dimension $\dim_{\mathfrak{B}}$ and

Hausdorff dimension $\dim_{\mathfrak{H}}$. Sets with finite box-counting dimension project well:

Theorem 3.1.2 ([32]). *Let H be a real Hilbert space and let $A \subset H$ be a compact set with $\dim_{\mathfrak{B}}(A) = d < \infty$ and with thickness exponent $\tau(A)$ (see [32, Definition 3.4] for the definition of thickness exponent). Let $N > 2d$ be an integer and let $\alpha \in \mathbb{R}$ satisfy*

$$0 < \alpha < \frac{N - 2d}{N(1 + \tau(A)/2)}.$$

For almost every (in the sense of prevalence) C^1 map $\varphi : H \to \mathbb{R}^N$, there exists $K > 0$ such that for all $x, y \in A$, we have

$$K \left\|\varphi(x) - \varphi(y)\right\|^{\alpha} \geqslant \left\|x - y\right\|.$$

That is, φ is one-to-one on A with Hölder-continuous inverse.

Remark 3.1.3. Theorems 3.1.2 and 3.1.4 remain true when one replaces the thickness exponent of A with the Lipschitz deviation $\operatorname{dev}(A)$ [57]. Roughly speaking, $\tau(A)$ measures how well A can be approximated by finite-dimensional subspaces of H, while $\operatorname{dev}(A)$ measures how well A can be approximated by the graphs of Lipschitz functions defined on finite-dimensional subspaces of H (with lower values of $\tau(A)$ and $\operatorname{dev}(A)$ indicating better approximability). One always has $\operatorname{dev}(A) \leqslant \tau(A)$.

However, it is difficult to infer the box-counting dimension of a set from that of its images. Sauer and Yorke [65] construct a compact set $Q \subset \mathbb{R}^{10}$ with $\dim_{\mathfrak{B}}(Q) = 5$ such that $\dim_{\mathfrak{B}}(\varphi(Q)) < 4$ for every C^1 map $\varphi : \mathbb{R}^{10} \to \mathbb{R}^6$. See [25, 36] for additional examples in the same spirit. By contrast, Hausdorff dimension is preserved by typical smooth maps (for sets with thickness exponent zero, a condition automatically satisfied when H is finite-dimensional).

Theorem 3.1.4 ([52]). *Let H be a real Hilbert space and let $A \subset H$ be a compact set with thickness exponent $\tau(A)$. Let $N \in \mathbb{N}$. For almost every (in the sense of prevalence) C^1 map $\varphi : H \to \mathbb{R}^N$, we have*

$$\dim_{\mathfrak{H}}(\varphi(A)) \geqslant \min\left\{N, \frac{\dim_{\mathfrak{H}}(A)}{1+\tau(A)/2}\right\}.$$

In particular, $\dim_{\mathfrak{H}}(\varphi(A)) = \dim_{\mathfrak{H}}(A)$ if $\tau(A) = 0$ and $N \geqslant \dim_{\mathfrak{H}}(A)$.

However, sets with low Hausdorff dimension may be difficult to project in a one-to-one way. Kan [64, Appendix] constructs a set $X \subset \mathbb{R}^D$ with Hausdorff dimension zero such that every linear map $\varphi : \mathbb{R}^D \to \mathbb{R}^N$ fails to be one-to-one on X if $N < D$.

For $H = \mathbb{R}^D$, the difficulties associated with Hausdorff dimension and box-counting dimension can be overcome by using the notion of tangent dimension $\dim_{\mathfrak{T}}(Y)$. Introduced in [54], $\dim_{\mathfrak{T}}(Y)$ is given for $Y \subset \mathbb{R}^D$ by

$$\dim_{\mathfrak{T}}(Y) = \sup_{x \in Y} \dim(T_x Y),$$

where $T_x Y$ denotes the tangent space at x relative to Y (Definition 3.3.2). Ott and Yorke formulate a 'Platonic' version of the Whitney embedding theorem using tangent dimension.

Theorem 3.1.5 ([54]). *Let A be a compact subset of $\mathbb{R}^D$ and let $N \in \mathbb{N}$. For almost every (in the sense of prevalence) $\varphi \in C^1(\mathbb{R}^D, \mathbb{R}^N)$, if $\dim_{\mathfrak{T}}(\varphi(A)) < N/2$, then φ is one-to-one on A.*

The proof of Theorem 3.1.5 uses the fact that if $A \subset \mathbb{R}^D$ is compact, then $\dim_{\mathfrak{B}}(A) \leqslant \dim_{\mathfrak{T}}(A)$. This inequality is a consequence of a manifold extension theorem [54, Theorem 3.5]: For every $x \in A$, there exists a neighborhood $N(x)$ of x in $\mathbb{R}^D$ and a C^1 manifold M such that $M \supset A \cap N(x)$ and $T_x A = T_x M$. The manifold extension theorem, however, does not hold in general for compact subsets of infinite-dimensional real Hilbert spaces.

3.2 Linear prevalence

Prevalence is a measure-theoretic notion of genericity for infinite-dimensional spaces. We summarize the theory here in the context of complete metric linear spaces. For more information, see [12, 33, 34, 35, 55].

Definition 3.2.1. Let V be a complete metric linear space. A Borel set $S \subset V$ is said to be ***shy*** if there exists a Borel measure μ on V satisfying

(a) $0 < \mu(K) < \infty$ for some compact set $K \subset V$;

(b) $\mu(S + x) = 0$ for all $x \in V$.

We say that such a measure is transverse to S. More generally, a set S is said to be shy if it is contained in a shy Borel set. The complement of a shy set is said to be a ***prevalent*** set.

Prevalence has the following properties [34].

(LP1) All prevalent sets are dense.

(LP2) Every subset of a shy set is shy.

(LP3) Every translate of a shy set is shy.

(LP4) The union of a countable collection of shy sets is shy.

(LP5) A set $S \subset \mathbb{R}^m$ is shy if and only if it has Lebesgue measure zero.

Property (LP5) shows that prevalence generalizes the translation-invariant notion of *Lebesgue almost every* to infinite-dimensional complete metric linear spaces.

It is useful to view a Borel measure μ on V as an object that defines a family of perturbations. From this point of view, a Borel set $E \subset V$ is prevalent if there exists a Borel measure μ such that for every $x \in V$, $x + y \in E$ for μ almost every y in the support of μ. An often useful choice for μ is Lebesgue measure on a finite-dimensional subspace of V.

Definition 3.2.2. Let V be a complete metric linear space. A finite-dimensional subspace $P \subset V$ is said to be a ***probe*** for a Borel set $E \subset V$ provided

$$\lambda_P(\{p \in P : x + p \notin E\}) = 0$$

for every $x \in V$, where λ_P denotes Lebesgue measure on P.

Notice that if a Borel set $E \subset V$ has a probe, then E is prevalent.

3.3 Projection of dynamics: the Hilbert space case

Throughout this section, let H be a real Hilbert space with norm $\|\cdot\|$ induced by the inner product $\langle \cdot, \cdot \rangle$ and let H^* denote the dual of H. Let $f : H \to H$ be a map and let $A \subset H$ satisfy $f(A) = A$ (we call A an invariant set). For a map $\varphi : H \to \mathbb{R}^N$, we say that φ induces dynamics on $\varphi(A)$ if there exists a map $f_* : \varphi(A) \to \varphi(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ \big\downarrow \varphi & & \big\downarrow \varphi \\ \varphi(A) & \xrightarrow{f_*} & \varphi(A) \end{array}$$

We focus on the following question. For a typical observable $\varphi : H \to \mathbb{R}^N$, does the existence of an induced map f_* with specified properties imply that A is 'equivalent' to $\varphi(A)$ and that the dynamical systems (f, A) and $(f_*, \varphi(A))$ are 'equivalent'? This question has been answered affirmatively in the continuous category.

3.3.1 Continuous observables

We establish some notation before stating the result. For a map $g : X \to X$ and $k \in \mathbb{N}$, let $\mathrm{Per}_k(g)$ denote the set of periodic points of g of period k. More precisely,

$$\mathrm{Per}_k(g) = \{x \in X : g^k(x) = x \text{ and } g^i(x) \neq x \text{ for } 1 \leqslant i \leqslant k-1\}.$$

Theorem 3.3.1 ([45])**.** *Let H be a separable real Hilbert space and let $f : H \to H$ be a map. Suppose that $A \subset H$ is a compact set satisfying $f(A) = A$. Let $N \in \mathbb{N}$ and let V be any closed subspace of $C^0(H, \mathbb{R}^N)$ that contains the bounded linear functions. For prevalent $\varphi \in V$, if f induces a map f_* on $\varphi(A)$ satisfying $f_* \circ \varphi = \varphi \circ f$ on A and if*

(a) $f_* : \varphi(A) \to \varphi(A)$ *is invertible;*

(b) $\mathrm{Per}_1(f_*) \cup \mathrm{Per}_2(f_*)$ *is countable;*

then $\varphi|A$ *is a homeomorphism and the dynamical systems* (f, A) *and* $(f_*, \varphi(A))$ *are topologically conjugate.*

3.3.2 Observing differentiable dynamics

In order to formulate versions of Theorem 3.3.1 for differentiable dynamics, we must first define a notion of differentiability suitable for maps defined on arbitrary subsets of real Hilbert spaces. We call this notion quasi-differentiability; it is defined in terms of generalized tangent spaces.

Definition 3.3.2. Let X be a real Hilbert space and let $E \subset X$. For $x \in E$, let $\Delta_x E$ be the set of all directions $v \in X$ for which there exist sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ in E such that $x_i \to x$, $y_i \to x$, $x_i \neq y_i$ for all i, and

$$\lim_{i\to\infty} \frac{y_i - x_i}{\|y_i - x_i\|} = v.$$

The ***tangent space*** at x relative to E, denoted $T_x E$, is the smallest closed subspace of X that contains $\Delta_x E$. The tangent bundle over E is defined by $TE = \{(x, v) : x \in E,\ \ v \in T_x E\}$.

Definition 3.3.3. Let X be a real Hilbert space. A map $f : X \to X$ is said to be ***quasi-differentiable*** on a set $E \subset X$ if for each $x \in E$ there exists a bounded linear operator Df_x on X such that

$$\lim_{i\to\infty} \frac{f(y_i) - f(x_i) - Df_x(y_i - x_i)}{\|y_i - x_i\|} = 0$$

for all sequences $(x_i)_{i=1}^{\infty}$ in E and $(y_i)_{i=1}^{\infty}$ in E satisfying $x_i \to x$, $y_i \to x$, and $x_i \neq y_i$ for all i. We call the operator Df_x a quasi-derivative of f at x.

Now assume for the remainder of Subsection 3.3.2 that $f : H \to H$ is C^1 and recall that $A \subset H$ satisfies $f(A) = A$. We will address the following question.

(Q1) For a prevalent C^1 observable $\varphi : H \to \mathbb{R}^N$, if f induces a quasi-differentiable map f_* on $\varphi(A)$, does φ embed A into $\mathbb{R}^N$?

As we now explain, care must be taken when choosing a notion of embedding.

3.3.2.1 Notions of embedding

Our first notion of embedding is motivated by classical differential topology.

Definition 3.3.4. A C^1 map $\varphi : H \to \mathbb{R}^N$ is said to be an ***immersion*** on a set $E \subset H$ if $D\varphi_x : T_xE \to T_{\varphi(x)}\varphi(E)$ is injective for every $x \in E$. An injective immersion φ is said to be an ***embedding*** of E if $\varphi|E$ maps E homeomorphically onto $\varphi(E)$.

Suppose that C^1 $\varphi : H \to \mathbb{R}^N$ embeds a set $E \subset H$ into $\mathbb{R}^N$ and let $p \in E$ be an accumulation point of E. If H is finite-dimensional, then the fact that $D\varphi_p : T_pE \to T_{\varphi(p)}\varphi(E)$ is injective implies that it is surjective as well. This follows from the fact that the unit sphere in any finite-dimensional Hilbert space is compact. However, injectivity of $D\varphi_p$ on T_pE does not imply surjectivity of $D\varphi_p$ on T_pE if H is infinite-dimensional because the unit sphere in such an H is no longer compact. The following example illustrates the phenomenon.

Let X be an infinite-dimensional separable real Hilbert space with orthonormal basis $\{\boldsymbol{e}_i : i \in \mathbb{N}\}$ and let $p \in X$. Define $Q = \{p + \boldsymbol{e}_i/i : i \in \mathbb{N}\} \cup \{p\}$. The direction set Δ_pQ is empty and therefore $T_pQ = \{0\}$ despite the fact that p is an accumulation point of Q. Now suppose that C^1 $\varphi : X \to \mathbb{R}^N$ embeds Q into $\mathbb{R}^N$. Since $\varphi(p)$ is an accumulation point of $\varphi(Q)$, the compactness of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ implies that $\Delta_{\varphi(p)}\varphi(Q)$ is nonempty and therefore $\dim(T_{\varphi(p)}\varphi(Q)) > 0$.

Motivated by this example, we formulate a second, stronger notion of embedding.

Definition 3.3.5. A C^1 map $\varphi : H \to \mathbb{R}^N$ is said to be a ***strong embedding*** of a set $E \subset H$ if φ is an embedding of E and if $D\varphi_x : T_xE \to T_{\varphi(x)}\varphi(E)$ is bijective for every $x \in E$.

Note that if H is finite-dimensional, then an embedding of E is a strong embedding of E.

3.3.2.2 Embedding theorems: general case

We formulate conditions under which (Q1) has an affirmative answer in the sense of Definition 3.3.4.

Lemma 3.3.6. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose $A \subset H$ is such that $f(A) = A$ and Df_x is injective on T_xA for every $x \in A \setminus \mathrm{Per}_1(f|A)$. Let $N \in \mathbb{N}$. For every ball $\mathcal{B} = B(y, r)$ in H, the set $W_{\mathcal{B}}$ of observables $\varphi \in C^1(H, \mathbb{R}^N)$ satisfying*

(a) *there exists $(x, v) \in TA$ such that $v \neq 0$, $x \notin B(y, 2r)$, $f(x) \in B(y, r)$, and $D\varphi_x v = 0$;*

(b) *for every such element of TA we have $(D\varphi_{f(x)} \circ Df_x)v = 0$;*

is a shy subset of $C^1(H, \mathbb{R}^N)$.

Proof of Lemma 3.3.6. Assume H is infinite-dimensional. It suffices to consider the case $N = 1$. We will construct a measure μ that is transverse to $W_{\mathcal{B}}$. Choose a C^∞ bump function $\beta : \mathbb{R} \to \mathbb{R}$ such that

$$0 \leqslant \beta \leqslant 1, \qquad \beta \equiv 1 \text{ on } \{|s| < 25/16\}, \qquad \operatorname{supp}(\beta) = \{|s| \leqslant 9/4\}.$$

Define $\beta_{\mathcal{B}} : H \to \mathbb{R}$ by

$$\beta_{\mathcal{B}}(x) = \beta\left(\frac{\|x-y\|^2}{r^2}\right).$$

The function $\beta_{\mathcal{B}}$ has the following properties:

$$0 \leqslant \beta_{\mathcal{B}} \leqslant 1, \qquad \beta_{\mathcal{B}}|B(y, 5r/4) \equiv 1, \qquad \operatorname{supp}(\beta_{\mathcal{B}}) = \overline{B(y, 3r/2)}.$$

Now let $\{\boldsymbol{e}_m^* : m \in \mathbb{N}\}$ be an orthonormal basis for H^*. Define

$$Q = \left\{\beta_{\mathcal{B}} \sum_{m=1}^{\infty} m^{-1}\gamma_m \boldsymbol{e}_m^* : |\gamma_m| \leqslant 1 \text{ for all } m\right\}.$$

Notice that Q is compact. Let μ be the probability measure on Q that results from choosing the γ_m independently and uniformly on $[-1, 1]$. We claim that μ is transverse to $W_{\mathcal{B}}$.

Let $\psi \in C^1(H, \mathbb{R})$. Suppose that there exists $(x, v) \in TA$ such that $v \neq 0$, $x \notin B(y, 2r)$, $f(x) \in B(y, r)$, and $D\psi_x v = 0$. (If no such element of TA exists, then $\{\eta \in Q : \psi + \eta \in W_{\mathcal{B}}\} = \emptyset$.) Let $z = Df_x v$. We represent z as a sequence $(z_i)_{i=1}^{\infty}$ where $z_i = \langle z, \boldsymbol{e}_i\rangle$. Let $\ell \in \mathbb{N}$ be such that $z_\ell \neq 0$. For $(\gamma_m) \in [-1, 1]^{\mathbb{N}}$, we have

$$\begin{aligned} D\left(\psi + \beta_{\mathcal{B}} \sum_{m=1}^{\infty} m^{-1}\gamma_m \boldsymbol{e}_m^*\right)_{f(x)} z &= D\psi_{f(x)} z + \sum_{m \neq \ell} m^{-1}\gamma_m \langle \boldsymbol{e}_m, z\rangle + \ell^{-1}\gamma_\ell \langle \boldsymbol{e}_\ell, z\rangle \\ &= D\psi_{f(x)} z + \sum_{m \neq \ell} m^{-1}\gamma_m \langle \boldsymbol{e}_m, z\rangle + \ell^{-1}\gamma_\ell z_\ell. \qquad (3.2) \end{aligned}$$

Consequently, if we fix γ_m for all $m \neq \ell$, then the right side of (3.2) is equal to 0 for at most one value of γ_ℓ. The Fubini/Tonelli theorem therefore implies that $\mu(\{\eta \in Q : \psi + \eta \in W_{\mathcal{B}}\}) = 0$. We conclude that μ is transverse to $W_{\mathcal{B}}$. □

Lemma 3.3.7. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose $A \subset H$ is such that $f(A) = A$ and suppose $x \in \mathrm{Per}_1(f|A)$. Let $N \in \mathbb{N}$. If*

(Op1) *the operator $Df_x|T_xA$ is not a scalar multiple of the identity;*

(Op2) *the real point spectrum σ_p of $(Df_x|T_xA)^*$ is countable;*

then the set Z_x of observables $\varphi \in C^1(H, \mathbb{R}^N)$ satisfying

(Ker1) $\ker(D\varphi_x) \cap T_xA \neq \{0\}$*;*

(Ker2) $Df_x\big(\ker(D\varphi_x) \cap T_xA\big) \subset \ker(D\varphi_x)$*;*

is a shy subset of $C^1(H, \mathbb{R}^N)$.

Proof of Lemma 3.3.7. Assume H is infinite-dimensional. It suffices to consider the case $N = 1$. If $\dim(T_xA) = 1$, then (Ker1) is satisfied by only a shy subset of $C^1(H, \mathbb{R})$. Condition (Ker1) is always satisfied if $\dim(T_xA) > 1$; in this case we show that (Ker2) is a shy condition.

Let $L = Df_x|T_xA$ and assume that L is not a scalar multiple of the identity. Suppose that $0 \neq w^* \in (T_xA)^*$ satisfies $L(\ker(w^*)) \subset \ker(w^*)$. For all $v \in \ker(w^*)$, we have $\langle w, v\rangle = 0$ and $\langle w, Lv\rangle = \langle L^*w, v\rangle = 0$. The vector w is therefore an eigenvector of L^*.

We show that Z_x is shy by using Lebesgue measure on a 1-dimensional subspace of $C^1(H, \mathbb{R})$. For $\gamma \in \sigma_p$, let E_γ be the eigenspace associated with γ. Since L is not a scalar multiple of the identity, neither is L^*. Let

$$y \in T_x A \setminus \bigcup_{\gamma \in \sigma_p} E_\gamma.$$

We view $y^* \in (T_x A)^*$ as an element of H^* by composing y^* with the orthogonal projection π from H onto $T_x A$: define $y^*(v) = \langle y, \pi(v) \rangle$ for all $v \in H$. Let Y be the 1-dimensional subspace of $C^1(H, \mathbb{R})$ spanned by y^*. Let $\varphi \in C^1(H, \mathbb{R})$. We claim that

$$\lambda_Y(\{c \in \mathbb{R} : \varphi + cy^* \in Z_x\}) = 0. \tag{3.3}$$

Let $\gamma \in \sigma_p$. Suppose that $c_1, c_2 \in \mathbb{R}$ are such that $\varphi + c_1 y^* \in Z_x$ and $\varphi + c_2 y^* \in Z_x$. Suppose further that the vectors in $T_x A$ associated with $D\varphi_x \circ \pi + c_1 y^*$ and $D\varphi_x \circ \pi + c_2 y^*$ via the Riesz representation theorem are both elements of E_γ. This implies that $(c_1 - c_2) y \in E_\gamma$. Since $y \in T_x A \setminus E_\gamma$, we conclude that $c_1 = c_2$. The set σ_p is countable and therefore $\{c \in \mathbb{R} : \varphi + cy^* \in Z_x\}$ is countable. This establishes (3.3).

□

The following proposition provides a preliminary answer to (Q1).

Proposition 3.3.8. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose that $A \subset H$ is a compact set such that $f(A) = A$. Assume*

(H1) $\mathrm{Per}_1(f|A) \cup \mathrm{Per}_2(f|A)$ *is countable;*

(H2) $f|A$ *is injective;*

(H3) *Df_x is injective on T_xA for every $x \in A \setminus \mathrm{Per}_1(f|A)$;*

(H4) *for every $x \in \mathrm{Per}_1(f|A)$, the operator $Df_x|T_xA$ is not a scalar multiple of the identity;*

(H5) *for every $x \in \mathrm{Per}_1(f|A)$, the real point spectrum of $(Df_x|T_xA)^*$ is countable.*

Let $N \in \mathbb{N}$. For prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, if f induces a quasi-differentiable map f_* on $\varphi(A)$, then φ embeds A into $\mathbb{R}^N$ in the sense of Definition 3.3.4.

Proof of Proposition 3.3.8. Let $N \in \mathbb{N}$. Applying Proposition 3.5 of [45], there exists a prevalent set $\Gamma_1 \subset C^1(H, \mathbb{R}^N)$ such that for $\varphi \in \Gamma_1$, if f induces a map f_* on $\varphi(A)$ satisfying $f_* \circ \varphi = \varphi \circ f$ on A, then φ maps A homeomorphically onto its image.

Let $\{B_i : i \in \mathbb{N}\}$ be a collection of open balls in H that forms a basis for the topology on H. Define the following sets:

$$\Gamma_2 = \bigcap_{i=1}^{\infty} C^1(H, \mathbb{R}^N) \setminus W_{B_i}, \qquad \Gamma_3 = \bigcap_{x \in \mathrm{Per}_1(f|A)} C^1(H, \mathbb{R}^N) \setminus Z_x.$$

The set Γ_2 is prevalent by Lemma 3.3.6 and property (LP4). The set Γ_3 is prevalent by Lemma 3.3.7 and (LP4). Property (LP4) applies here because (H1) gives that $\mathrm{Per}_1(f|A)$ is countable.

Let $\Gamma = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$. For $\varphi \in \Gamma$, if f induces a quasi-differentiable map f_* on $\varphi(A)$ satisfying $f_* \circ \varphi = \varphi \circ f$ on A, then φ embeds A into $\mathbb{R}^N$. □

We obtain an improved version of Proposition 3.3.8 by transferring (H1)–(H3) onto the induced dynamics.

Theorem 3.3.9. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose that $A \subset H$ is a compact set such that $f(A) = A$. Assume (H4) and (H5). Let $N \in \mathbb{N}$. For prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, if f induces a quasi-differentiable map f_* on $\varphi(A)$ satisfying*

(H1)* $\mathrm{Per}_1(f_*) \cup \mathrm{Per}_2(f_*)$ *is countable;*

(H2)* f_* *is injective;*

(H3)* $(Df_*)_z$ *is injective on $T_z\varphi(A)$ for every $z \in \varphi(A) \setminus \mathrm{Per}_1(f_*)$;*

then φ embeds A into $\mathbb{R}^N$ in the sense of Definition 3.3.4.

Proof of Theorem 3.3.9. If (H1)–(H3) hold, then Theorem 3.3.9 follows from Proposition 3.3.8. If (H1) does not hold, then $\mathrm{Per}_1(f|A) \cup \mathrm{Per}_2(f|A)$ is uncountable. For prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, $\varphi(\mathrm{Per}_1(f|A) \cup \mathrm{Per}_2(f|A))$ is uncountable (see Proposition 2.6 of [45]); for any such φ, f cannot induce a map on $\varphi(A)$ satisfying (H1)*. If (H2) (respectively (H3)) fails to hold, then a quasi-differentiable induced map satisfying (H2)* (respectively (H3)*) cannot exist for prevalent $\varphi \in C^1(H, \mathbb{R}^N)$. □

3.3.2.3 Embedding theorems: strong case

We now formulate conditions under which (Q1) has an affirmative answer in the sense of Definition 3.3.5. The key idea here is to place a mild hypothesis on the tangent dimension of the image $\varphi(A)$.

Theorem 3.3.10. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose that $A \subset H$ is a compact set such that $f(A) = A$. Assume (H5).*

Let $N \in \mathbb{N}$. For prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, if

$$\dim_{\mathfrak{T}}(\varphi(A)) < N, \tag{DimT}$$

and if f induces a quasi-differentiable map f_* on $\varphi(A)$ satisfying (H1)–(H3)* as well as*

(H4)* *for every $z \in \mathrm{Per}_1(f_*)$, the operator $Df_*|T_z\varphi(A)$ is not a scalar multiple of the identity,*

then φ strongly embeds A into $\mathbb{R}^N$.

Proof of Theorem 3.3.10. First assume that for every $q \in A$ and for every pair of sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ in A with $x_i \to q$, $y_i \to q$, and $x_i \neq y_i$ for all i, the sequence of normalized differences $((y_i - x_i)/\|y_i - x_i\|)$ in the unit sphere S of H has a converging subsequence. Under this assumption, if $\varphi \in C^1(H, \mathbb{R}^N)$ is an embedding of A, then φ is a strong embedding of A.

If (H4) holds as well, then the proof of Theorem 3.3.9 works for Theorem 3.3.10 as well. If (H4) does not hold, there exists $p \in \mathrm{Per}_1(f|A)$ such that $Df_p|T_pA$ is a scalar multiple of the identity. We consider two cases. First, if $\dim(T_pA) \geqslant N$, then for prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, we have that $\dim(T_{\varphi(p)}\varphi(A)) = N$ and $D\varphi_p$ maps T_pA surjectively onto $T_{\varphi(p)}\varphi(A)$. Any such φ cannot induce a quasi-differentiable map on $\varphi(A)$ satisfying (H4)*. Second, if $\dim(T_pA) < N$, then for prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, $D\varphi_p$ maps T_pA injectively (and therefore bijectively by our sequential precompactness assumption) onto $T_{\varphi(p)}\varphi(A)$. If any such φ induces a quasi-differentiable map f_* on $\varphi(A)$, we would have $(Df_*)_{\varphi(p)} = (D\varphi \circ Df \circ D\varphi^{-1})_{\varphi(p)}$ on $T_{\varphi(p)}\varphi(A)$. This precludes the possibility of (H4)*.

For the second part of the proof, assume that there exist $\hat{q} \in A$ and sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ in A such that $x_i \to \hat{q}$, $y_i \to \hat{q}$, $x_i \neq y_i$ for all i, and the sequence of normalized differences $(v_i = (y_i - x_i)/\|y_i - x_i\|)$ has no converging subsequences. This implies that

$$\rho := \lim_{M\to\infty} \inf_{\substack{i,j\geqslant M \\ i\neq j}} \angle(v_i, v_j) > 0.$$

By passing to a subsequence, we may assume that $\angle(v_i, v_{i'}) \geqslant \rho/2$ for all $i \neq i'$.

We use the sequence (v_i) to construct a probe. Let $V = \overline{\operatorname{span}\{v_i : i \in \mathbb{N}\}}$ and let $\pi_V : H \to V$ denote orthogonal projection onto V. Let $L_0 : V \to V$ be a bounded linear map such that $\langle L_0 v_i, L_0 v_{i'} \rangle = 0$ for all $i \neq i'$. Define $L = L_0 \circ \pi_V$. Let $\{\boldsymbol{e}_n : 1 \leqslant n \leqslant N\}$ be an orthonormal basis for $\mathbb{R}^N$. Define the bounded linear map $\psi : H \to \mathbb{R}^N$ by

$$\psi = \sum_{n=1}^{N} \left(\sum_{m=0}^{\infty} (L(y_{mN+n} - x_{mN+n}))^* \circ L \right) \boldsymbol{e}_n, \tag{3.4}$$

where we may assume (by passing to a subsequence if necessary) that $\|L(y_i - x_i)\|$ decreases monotonically to zero as $i \to \infty$ and that this happens quickly enough to guarantee that the sums in (3.4) converge.

Let $\varphi \in C^1(H, \mathbb{R}^N)$. We claim that the set

$$Z_\varphi = \left\{ c \in \mathbb{R} : \dim(T_{(\varphi+c\psi)(\hat{q})}(\varphi + c\psi)(A)) < N \right\}$$

is countable. To see this, let $c_0 \in \mathbb{R}$ be such that there exist N distinct vectors

$w_1, \ldots, w_N$ in the direction set $\Delta_{(\varphi+c_0\psi)(\hat{q})}(\varphi + c_0\psi)(A) \subset \mathbb{S}^{N-1}$ and N strictly increasing sequences $(m_j^{(n)})_{j=1}^{\infty}$ in $\mathbb{Z}^+$ satisfying

$$\lim_{j\to\infty} \frac{(\varphi + c_0\psi)(y_{m_j^{(n)}N+n}) - (\varphi + c_0\psi)(x_{m_j^{(n)}N+n})}{\left\|(\varphi + c_0\psi)(y_{m_j^{(n)}N+n}) - (\varphi + c_0\psi)(x_{m_j^{(n)}N+n})\right\|} = w_n$$

for every $1 \leqslant n \leqslant N$. Note that any sufficiently large c_0 will have this property. Using c_0 as a starting point, define maps $s \mapsto w_n(s)$ by

$$w_n(s) = \lim_{j\to\infty} \frac{(\varphi + s\psi)(y_{m_j^{(n)}N+n}) - (\varphi + s\psi)(x_{m_j^{(n)}N+n})}{\left\|(\varphi + s\psi)(y_{m_j^{(n)}N+n}) - (\varphi + s\psi)(x_{m_j^{(n)}N+n})\right\|}.$$

Each map $s \mapsto w_n(s)$ is defined on all of $\mathbb{R}$ except for perhaps one exceptional value of s.

Let $1 \leqslant n_1 < n_2 \leqslant N$. Our choice of ψ implies that $s \mapsto \angle(\boldsymbol{e}_{n_1}, w_{n_1}(s))$ is decreasing (and strictly decreasing on the preimage of $(0, \pi)$), while $s \mapsto \angle(\boldsymbol{e}_{n_1}, w_{n_2}(s))$ is increasing. Similarly, $s \mapsto \angle(\boldsymbol{e}_{n_2}, w_{n_2}(s))$ is decreasing (and strictly decreasing on the preimage of $(0, \pi)$), while $s \mapsto \angle(\boldsymbol{e}_{n_2}, w_{n_1}(s))$ is increasing. It follows that $w_{n_1}(s) = w_{n_2}(s)$ for at most one value of s. The vectors $w_1(s), \ldots, w_N(s)$ are therefore all distinct except for at most $N(N-1)/2$ values of s. We have shown that Z_φ is finite.

The set $\{\varphi \in C^1(H, \mathbb{R}^N) : \dim_{\mathfrak{T}}(\varphi(A)) = N\}$ is prevalent. Every map in this set fails to satisfy (DimT).

□

3.3.2.4 Implications for Lyapunov exponents

We answer the question that motivates this work - Question 1 - using Theorem 3.3.10.

Corollary 3.3.11. *Let H be a separable real Hilbert space and let $f : H \to H$ be a C^1 map. Suppose that $A \subset H$ is a compact set such that $f(A) = A$. Assume (H5). Let $N \in \mathbb{N}$. For prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, if $\dim_{\mathfrak{T}}(\varphi(A)) < N$ and if f induces a quasi-differentiable map f_* on $\varphi(A)$ satisfying (H1)*–(H4)*, then Lyapunov exponents of f_* correspond to Lyapunov exponents of f as follows. If $z \in \varphi(A)$ and $\omega(z, w)$ is a Lyapunov exponent of f_* with $w \in T_z\varphi(A)$, then $\omega(\varphi^{-1}z, (D\varphi^{-1})_z w)$ is a Lyapunov exponent of $f|A$ and $\omega(\varphi^{-1}z, (D\varphi^{-1})_z w) = \omega(z, w)$.*

Remark 3.3.12. A Lyapunov exponent $\omega(z, w)$ of f_* with $w \in T_z\mathbb{R}^N \setminus T_z\varphi(A)$ may be spurious - it may be an artifact of φ that does not correspond to a Lyapunov exponent of f.

3.4 Discussion

Invariant sets associated with evolution PDEs often live in finite-dimensional submanifolds of the ambient function space, such as inertial manifolds or center manifolds. For example, the genuinely nonuniformly hyperbolic attracting sets produced when certain parabolic PDEs are forced periodically live in two-dimensional center manifolds [46]. It is interesting to consider if observational data can be used to determine whether or not a given invariant set of interest is contained in a finite-dimensional submanifold of the ambient Hilbert space. More precisely:

Definition 3.4.1. Let H be a real Hilbert space. A subset $E \subset H$ is said to be ***locally embeddable*** if for every $x \in E$, there exists a neighborhood U of x in H and a finite-dimensional C^1 submanifold M of H (without boundary) such that

$U \cap E \subset M$. If a finite-dimensional C^1 submanifold M contains every element of E that lies within some neighborhood of x and if the dimension of M is minimal with respect to this property, then we call M a ***local enveloping manifold*** for E at x (see [54, Section 3] for more about local enveloping manifolds when $H = \mathbb{R}^D$).

Question 2. Let H be a real Hilbert space. Let $f : H \to H$ be a C^1 map and suppose that $A \subset H$ is a compact set satisfying $f(A) = A$. Let $N \in \mathbb{N}$. For prevalent $\varphi \in C^1(H, \mathbb{R}^N)$, if f induces a quasi-differentiable map f_* on $\varphi(A)$, does it follow that A is locally embeddable?

This question may well have an affirmative answer given the nature of existing theorems on the regularity of embeddings of subsets of infinite-dimensional spaces into Euclidean spaces. Theorem 3.1.2, for example, guarantees only Hölder continuity of φ^{-1} on $\varphi(A)$, and therefore guarantees only Hölder continuity for an induced map $f_* = \varphi \circ f \circ \varphi^{-1}$ induced by a C^1 map $f : H \to H$.

Bibliography

[1] P. Ara, M.A. Moreno, and E. Pardo, *Nonstable K-theory for graph algebras*, Algebr. Represent. Theory **10** (2007), 157–178.

[2] S. Arklint, R. Bentmann, and T. Katsura, *The K-theoretical range of Cuntz-Krieger algebras*, J. Funct. Anal., to appear.

[3] S. Arklint, R. Bentmann, and T. Katsura, *Reduction of filtered K-theory and a characterization of Cuntz-Krieger algebras*, preprint.

[4] S. Arklint and E. Ruiz, *Corners of Cuntz-Krieger algebras*, Trans. Amer. Math. Soc., to appear.

[5] L. Barreira and Y. Pesin, *Nonuniform hyperbolicity*, vol. 115 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2007. Dynamics of systems with nonzero Lyapunov exponents.

[6] T. Bates, J. H. Hong, I. Raeburn, and W. Szymański, *The ideal structure of the C^*-algebras of infinite graphs*, Illinois J. Math **46** (2002), 1159–1176.

[7] B. Blackadar, K-theory for Operator Algebras, 2^{nd} ed., Cambridge University Press, Cambridge, 1998.

[8] C. Bonatti, L. J. Díaz, and M. Viana, *Dynamics beyond uniform hyperbolicity*, vol. 102 of Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin, 2005. A global geometric and probabilistic perspective, Mathematical Physics, III.

[9] L.G. Brown and G.K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149.

[10] L.G. Brown, *Extensions of AF-algebras: the projection lifting problem*, Operator Algebras and Applications: Symp. Pure Math. **38** (1982), 175–176.

[11] T. M. Carlsen, S. Eilers, and M. Tomforde, *Index maps in the K-theory of graph algebras*, J. K-theory **9** (2012), 385–406.

[12] J. P. R. Christensen, *On sets of Haar measure zero in abelian Polish groups*, in Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972), vol. 13, 1972, 255–260 (1973).

[13] K. Diecke, J. H. Hong, W. Szymański, *Stable rank of graph algebras. Type I graph algebras and their limits.*, Indiana Univ. Math. J. **52** (2003), 963–979.

[14] D. Drinen, *Viewing AF-algebras as graph algebras*, Proc. Amer. Math. Soc. **128** (2000), 1991–2000.

[15] D. Drinen and M. Tomforde, *The C^*-algebras of arbitrary graphs*, Rocky Mountain J. Math. **35** (2005), 105–135.

[16] D. Drinen and M. Tomforde, *Computing K-theory and Ext for graph C^*-algebras*, Illinois J. Math. **46** (2002), 81–91.

[17] E. Effros, D. Handelman, C. L. Shen, *Dimension groups and their affine representations*, Amer. J. Math. **102** (1980), 385–407.

[18] S. Eilers, T. Katsura, M. Tomforde, and J. West, *The Ranges of* **K***-theoretic Invariants for Nonsimple Graph Algebras*, Trans. Amer. Math. Soc. **368** (2016), 3811–3847

[19] S. Eilers and G. Restorff, *On Rørdam's classification of certain C^*-algebras with one nontrivial ideal*, S. Eilers and G. Restorff. Operator algebras: The Abel symposium 2004, 87-96. Abel Symposia **1**, Springer-Verlag, 2006.

[20] S. Eilers, G. Restorff, and E. Ruiz, *Classifying C^*-algebras with both finite and infinite subquotients*, J. Funct. Anal. **265** (2013), 449–468.

[21] S. Eilers, G. Restorff, and E. Ruiz, *The ordered K-theory of a full extension*, submitted for publication.

[22] S. Eilers, G. Restorff, and E. Ruiz, *Strong classification of extensions of classifiable C^*-algebras*, submitted for publication.

[23] S. Eilers and M. Tomforde, *On the classification of nonsimple graph C^*-algebras*, Math. Ann. **346** (2010), 393–418.

[24] G. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29–44.

[25] K. J. Falconer and J. D. Howroyd, *Projection theorems for box and packing dimensions*, Math. Proc. Cambridge Philos. Soc., **119** (1996), 287–295.

[26] G. Froyland, S. Lloyd, and A. Quas, *A semi-invertible Oseledets theorem with applications to transfer operator cocycles*, Discrete Contin. Dyn. Syst., **33** (2013), 3835–3860.

[27] C. González-Tokman and A. Quas, *A semi-invertible operator Oseledets theorem*, Ergodic Theory Dynam. Systems, **34** (2014), 1230–1272.

[28] K. R. Goodearl and D. E. Handelman, *Stenosis in dimension groups and AF C^*-algebras*, J. Reine Angew. Math. **332** (1982), 1–98.

[29] D. Handelman, *Extensions for AF C^* algebras and dimension groups*, Trans. Amer. Math. Soc. **271** (1982), 537–573.

[30] D. Huang, *Flow equivalence of reducible shifts of finite type and Cuntz-Krieger algebras*, J. reine angew. Math. **462** (1995), 185–217.

[31] D. Huang, *The classification of two-component Cuntz-Krieger algebras*, Proc. Amer. Math. Soc. **124** (1996), 505–512.

[32] B. R. Hunt and V. Y. Kaloshin, *Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces*, Nonlinearity, **12** (1999), 1263–1275.

[33] B. R. Hunt and V. Y. Kaloshin, *Chapter 2 - prevalence*, in Handbook of Dynamical Systems, F. T. Henk Broer and B. Hasselblatt, eds., vol. 3 of Handbook of Dynamical Systems, Elsevier Science, 2010, 43 – 87.

[34] B. R. Hunt, T. Sauer, and J. A. Yorke, *Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces*, Bull. Amer. Math. Soc. (N.S.), **27** (1992), 217–238.

[35] ——, *Prevalence. An addendum to: "Prevalence: a translation-invariant 'almost every' on infinite-dimensional spaces" [Bull. Amer. Math. Soc. (N.S.)* **27** *(1992), no. 2, 217–238; MR1161274 (93k:28018)]*, Bull. Amer. Math. Soc. (N.S.), **28** (1993), 306–307.

[36] M. Järvenpää, *On the upper Minkowski dimension, the packing dimension, and orthogonal projections*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, (1994), 34.

[37] T. Katsura, A. Sims, and M. Tomforde, *Realizations of AF-algebras as graph algebras, Exel-Laca algebras, and ultragraph algebras*, J. Funct. Anal. **257** (2009), 1589–1620.

[38] E. Kirchberg, The Classification of Purely Infinite C^*-algebras using Kasparov's Theory, to appear in the Fields Institute Communications series.

[39] E. Kirchberg and M. Rørdam, *Nonsimple purely infinite C^*-algebras*, Amer. J. Math. **122** (2000), 637–666.

[40] D. Kucerovskyand P.W. Ng, *S-regularity and the corona factorization property*, Math. Scand. **99** (2006), 204–216.

[41] Z. Lian and K. Lu, *Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space*, Mem. Amer. Math. Soc., **206** (2010), vi+106.

[42] Z. Lian and L.-S. Young, *Lyapunov exponents, periodic orbits and horseshoes for mappings of Hilbert spaces*, Ann. Henri Poincaré, **12** (2011), 1081–1108.

[43] ——, *Lyapunov exponents, periodic orbits, and horseshoes for semiflows on Hilbert spaces*, J. Amer. Math. Soc., **25** (2012), 637–665.

[44] H. Lin, *The simplicity of the quotient algebra $M(A)/A$ for a simple C^*-algebra*, Math. Scand. **65** (1989), 119–128.

[45] J. Lin and W. Ott, *Observing infinite-dimensional dynamical systems*, SIAM J. Appl. Dyn. Syst., **9** (2010), 1229–1243.

[46] K. Lu, Q. Wang, and L.-S. Young, *Strange attractors for periodically forced parabolic equations*, Mem. Amer. Math. Soc., **224** (2013), vi+85.

[47] S. Mac Lane, *Homology*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1963.

[48] R. Mañé, *Lyapounov exponents and stable manifolds for compact transformations*, in Geometric dynamics (Rio de Janeiro, 1981), vol. 1007 of Lecture Notes in Math., Springer, Berlin, 1983, 522–577.

[49] R. Meyer and R. Nest, *C^*-algebras over topological spaces: filtrated K-theory*, to appear in Canad. J. Math.

[50] P. W. Ng, *The corona factorization property.* Operator theory, operator algebras, and applications, 97–110, Contemp. Math., **414**, Amer. Math. Soc., Providence, RI, 2006.

[51] V. I. Oseledec, *A multiplicative ergodic theorem. lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc, **19** (1968), 197–231.

[52] W. Ott, B. Hunt, and V. Kaloshin, *The effect of projections on fractal sets and measures in Banach spaces*, Ergodic Theory Dynam. Systems, **26** (2006), 869–891.

[53] W. Ott, M.A. Rivas, and J. West, *Observing Lyapunov exponents of infinite-dimensional dynamical systems*, J Stat Phys **161** (2015), 1098–1111.

[54] W. Ott and J. A. Yorke, *Learning about reality from observation*, SIAM J. Appl. Dyn. Syst., **2** (2003), 297–322 (electronic).

[55] ——, *Prevalence*, Bull. Amer. Math. Soc. (N.S.), **42** (2005), 263–290 (electronic).

[56] N. C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. **5** (2000), 49–114.

[57] E. Pinto de Moura and J. C. Robinson, *Lipschitz deviation and embeddings of global attractors*, Nonlinearity, **23** (2010), 1695–1708.

[58] I. Raeburn and W. Szymański, *Cuntz-Krieger algebras of infinite graphs and matrices*, Trans. Amer. Math. Soc. **356** (2004), 39–59.

[59] G. Restorff, *Classification of Cuntz-Krieger algebras up to stable isomorphism*, J. Reine Angew. Math. **598** (2006), 185–210.

[60] M. Rørdam, *Classification of extensions of certain C^*-algebras by their six term exact sequences in K-theory*, Math. Ann. **308** (1997), 93–117.

[61] M. Rørdam, *Classification of nuclear, simple C^*-algebras*, Classification of nuclear C^*-algebras. Entropy in operator algebras, Encyclopaedia Math. Sci., vol. 126, Springer, Berlin, 2002, 1–145.

[62] M. Rørdam, F. Larsen, and N. J. Laustsen, An Introduction to K-theory for C^*-algebras, London Mathematical Society Student Texts **49**, Cambridge University Press, Cambridge, 2000.

[63] D. Ruelle, *Characteristic exponents and invariant manifolds in hilbert space*, Annals of Mathematics, (1982), 243–290.

[64] T. Sauer, J. A. Yorke, and M. Casdagli, *Embedology*, J. Statist. Phys., **65** (1991), 579–616.

[65] T. D. Sauer and J. A. Yorke, *Are the dimensions of a set and its image equal under typical smooth functions?*, Ergodic Theory Dynam. Systems, **17** (1997), 941–956.

[66] W. Szymański, *The range of K-invariants for C^*-algebras of infinite graphs*, Indiana Univ. Math. J. **51** (2002), 239–249.

[67] P. Thieullen, *Fibrés dynamiques asymptotiquement compacts. Exposants de Lyapounov. Entropie. Dimension*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **4** (1987), 49–97.

[68] M. Tomforde, *The ordered K_0-group of a graph C^*-algebra*, C. R. Math. Acad. Sci. Soc. R. Can. **25** (2003), 19–25.

[69] Q. Wang and W. Ott, *Dissipative homoclinic loops of two-dimensional maps and strange attractors with one direction of instability*, Comm. Pure Appl. Math., **64** (2011), 1439–1496.

[70] Q. Wang and L.-S. Young, *Strange attractors in periodically-kicked limit cycles and Hopf bifurcations*, Comm. Math. Phys., **240** (2003), 509–529.

[71] L.-S. Young, *Mathematical theory of lyapunov exponents*, Journal of Physics A: Mathematical and Theoretical, **46** (2013), 254001.

[72] S. Zhang, *A property of purely infinite simple C^*-algebras*, Proc. Amer. Math. Soc. **109** (1990), 717–720.