Mathematics Colloquium

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Zassenhaus Algebras

Abstract

Let A be an abelian group. Then $End(A)$, the set of all endomorphisms of A, is a ring. Over the last fifty years, many results have been published dealing with the converse: Given a ring R, is there some abelian group A such that $R \cong End(A)$? A very remarkable result was obtained by Hans Zassenhaus in 1967: Assume that R is a ring such that R^+, the additive group of R is free of finite rank. Then there exists an abelian group A, sandwiched between R and $\mathbb{Q}R = \mathbb{Q} \otimes_{\mathbb{Z}} R^+$, such that $R \cong End(A)$. In contrast to all other such results, the Abelian group A and its endomorphism ring have the same rank! At the heart of Zassenhaus' proof is his construction of a countable family $\mathcal{F}$ of cyclic left ideals of the ring R such that for any $\varphi \in End(R^+)$ such that $\varphi(X) \subseteq X$ for all $X \in \mathcal{F}$, there is some $r \in R$ with $\varphi(x) = rx$ for all $x \in R$, i.e. φ is the left multiplication $r\cdot$ by some $r \in R$. This result has recently been extended to certain rings R with R^+ a free abelian of countable rank. Moreover, many more rings have been found that allow such a family of left ideals. This makes it natural to consider the following notion. For any ring R with identity, define $\widehat{R}^\ell = \{\varphi \in End(R^+) : \varphi(X) \subseteq X$ for all left ideals X of R $\}$. We call the ring R a Zassenhaus ring, if $\widehat{R}^\ell = R\cdot$. If R happens to be a K-algebra, we define $\widehat{R}^\ell = \{\varphi \in End_K(R^+) : \varphi(X) \subseteq X$ for all left ideals X of R $\}$. We will consider several classes of rings (algebras) and we will determine which rings (algebras) in these classes are Zassenhaus rings (algebras).