SMOOTH INFINITESIMAL RIGIDITY FOR HIGHER RANK PARTIALLY HYPERBOLIC ACTIONS ON STEP-2 NILMANIFOLDS

A Dissertation

Presented to

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Cheng Zhan

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Abstract

If Λ is finitely generated and M is compact, an action $\varphi : \Lambda \times M \to M$ is a C^∞ homomorphism from Λ to $\mathrm{Diff}(M)$. There is a natural formal tangent space at the point $[\varphi]$ determined by φ, which is given by the 1-cocycles over φ with coefficients in the smooth vector fields on M. The 1-coboundaries form a closed subspace of the formal tangent space, and when these two spaces are equal, the action is said to be infinitesimally rigid.

The purpose of this thesis is to use representation theory to prove the infinitesimal rigidity of partially hyperbolic actions on a family of 2-step free nilmanifolds. We start by characterizing the irreducible representations in $L^2(\Gamma\backslash N)$ using the coadjoint orbit method. Then we introduce the obstructions to solving the twisted coboundary equation $\lambda\omega - \omega \circ A = \theta$, and prove how these obstructions vanish for the whole action due to the higher rank condition.

Contents

Chapter 1

Introduction

The rigidity properties of hyperbolic actions of $\mathbb{Z}^k$ or $\mathbb{R}^k$ for $K \geq 2$ have been intensely studied over the last two decades or so. Unlike the classical cases of diffeomorphisms and flows (actions of $\mathbb{Z}$ and $\mathbb{R}$) where only C^0 orbit structure may be stable under small perturbations, algebraic actions of higher rank abelian groups exhibit many rigidity properties.

The approach to prove local rigidity within partially hyperbolic algebraic actions differentiates itself from the *a priori* regularity method that successfully handles the rigidity problem for actions with sufficiently strong hyperbolic preperties. D.Damjanovic and A. Katok used a technique that is analytic in nature and involves an iterative scheme which gives a smooth solution in [DK10]. It also involves decomposing the appropriate function spaces into direct sum of subspaces invariant under the action, solving the cohomological equations separately and then gluing the solutions together. We call this method the harmonic analysis method.

In their paper [DK10], D.Damjanovic and A. Katok showed that there are infinitely many obstruction for solving the linearized equation for a single element of the action, and how these obstructions vanish simultaneously due to the higher rank condition. After that, a generalization of KAM (Kolmogorov-Arnold-Moser) iterative scheme was introduced to prove C^∞ rigidity for $\mathbb{Z}^k$ ($k \geq 2$) higher rank partially hyperbolic actions by toral automorphisms.

In this thesis, we prove the infinitesimal rigidity for higher rank partially hyperbolic action by nilmanifold automorphisms. We use the coadjoint orbit method, which is a geometric way to characterize all the equivalence classes of irreducible representations, and carry this approach on a family of 2-step free nilmanifolds. In particular, the strategy for this harmonic analysis method is to decompose $L^2(\Gamma\backslash N)$ under the right action such that $L^2(\Gamma\backslash N) = \oplus_{\pi\in(\Gamma\backslash N)^\wedge} H_\pi$. Correspondingly, solving the coboundary equation $\lambda\omega - \omega \circ A = \theta$ with tame estimates can be reduced to finding the solution within each subspace H_π and then glue them together. Historically, local C^∞ rigidity of higher rank Anosov actions by automorphisms of nilmanifolds was proved by A.Katok and Spatzier in Theorem 15, Section 3 of [KS97] without tame estimates for the cohomological equation.

We need a precise formula for the projection P_π with good estimate for each $P_\pi(f)$ in order to understand how do nilmanifold automorphisms affect the projections $P_\pi : L^2(\Gamma\backslash N) \to H_\pi$, or in terms of coadjoint orbit, how does the orbit behave under automorphisms, and finally prove that the formal solution $\omega = \sum_{\pi\in(\Gamma\backslash N)^\wedge} \omega_\pi$ converges in some Sobolev space. Therefore, we can actually glue these solutions togother.

1.1 Statement of Results

Let N be a free nilpotent group, Γ be a cocompact lattice in N, and the quotient space $M = \Gamma\backslash N$ is called a nilmanifold.

An action $\alpha : \mathbb{Z}^2 \times M \to M$ defined by automorphisms is given by an embedding $\rho_\alpha : \mathbb{Z}^2 \to \mathrm{Aut}(M)$ so that

$$\alpha(g, x) = \rho_\alpha(g)x$$

for any $g \in \mathbb{Z}^2$ and $x \in M$. We will write simply $\alpha(g)$ for $\rho_\alpha(g)$.

Definition 1.1.1. *The action α is higher rank if $A^\ell B^k$ is ergodic on M and the induced automorphism is ergodic on the commutator $\Gamma\backslash[N, N]$ for every $(\ell, k) \neq (0, 0)$, where A, B are two commutative automorphisms of M defined by $A := \alpha(g_1), B := \alpha(g_2), g_1, g_2 \in \mathbb{Z}^2$.*

In general, let Λ be a finitely generated group and let φ be an action $\varphi : \Lambda \times M \to M$. Any C^∞-diffeomorphism $f : M \to M$ induces a map on a smooth vector field $X \in \mathrm{Vect}^\infty(M)$ by $f_*X = (Df) \circ X \circ f^{-1}$.

A 1-cocycle $Z^1(\Lambda, \mathrm{Vect}^\infty(M))$ assigns to each $\gamma \in \Lambda$ a smooth vector field $\beta(\gamma) \in \mathrm{Vect}^\infty(M)$ satisfying the cocycle condition

$$\beta(\gamma_1\gamma_2) = \beta(\gamma_1) + \varphi(\gamma_1)_*\beta(\gamma_2).$$

A 1-cocycle β is a coboundary $B^1(\Lambda, \mathrm{Vect}^\infty(M))$ if there exists a smooth vector field $\tau \in \mathrm{Vect}(M)$ such that

$$\beta(\gamma) = \varphi(\gamma)_*\tau - \tau \text{ for all } \gamma \in \Lambda.$$

Let $H^1(\Lambda, \mathrm{Vect}^\infty(M))$ denote the quotient group of 1-cocycles $Z^1(\Lambda, \mathrm{Vect}^\infty(M))$ by the 1-coboundaries $B^1(\Lambda, \mathrm{Vect}^\infty(M))$: $H^1(\Lambda, \mathrm{Vect}^\infty(M)) := Z^1(\Lambda, \mathrm{Vect}^\infty(M))/B^1(\Lambda, \mathrm{Vect}^\infty(M))$, and φ is said to be C^∞-infinitesimally rigid if $H^1(\Lambda, \mathrm{Vect}^\infty(M))$ is trivial.

We introduce a cohomology sequence for $\Lambda = \mathbb{Z}^k$

$$C^0(\mathbb{Z}^k, \mathrm{Vect}^\infty(M)) \xrightarrow{\delta_v^1} C^1(\mathbb{Z}^k, \mathrm{Vect}^\infty(M)) \xrightarrow{\delta_v^2} C^2(\mathbb{Z}^k, \mathrm{Vect}^\infty(M)).$$

Let $\phi \in C^0(\mathbb{Z}^k, \mathrm{Vect}^\infty(M)) = \mathrm{Vect}^\infty(M), \beta \in C^1(\mathbb{Z}^k, \mathrm{Vect}^\infty(M))$ (maps from $\mathbb{Z}^k$ to $\mathrm{Vect}^\infty(M)$), and $\gamma \in C^2(\mathbb{Z}^k, \mathrm{Vect}^\infty(M))$ (maps from $\mathbb{Z}^k \times \mathbb{Z}^k$ to $\mathrm{Vect}^\infty(M)$). Coboundary operators are defined as

$$\begin{aligned} \delta_v^1\phi(g) &:= \varphi(g)_*\phi - \phi \\ \delta_v^2\beta(g_1, g_2) &:= (\varphi(g_2)_*\beta(g_1) - \beta(g_1)) - (\varphi(g_1)_*\beta(g_2) - \beta(g_2)). \end{aligned}$$

Remark 1.1.2. *The definition of δ_v^2 is somewhat different from the standard coboundary operator $\mathrm{d}^2 : C^1(\mathbb{Z}^k, \mathrm{Vect}^\infty(M)) \to C^2(\mathbb{Z}^k, \mathrm{Vect}^\infty(M))$ given by $\mathrm{d}^2\,\beta(g_1, g_2) = \varphi(g_1)_*\beta(g_2) - \beta(g_1g_2) + \beta(g_1)$. However, δ_v^2 and d^2 define the same kernel, $\beta(g_1g_2) = \beta(g_2g_1) = \beta(g_2) + \varphi(g_2)_*\beta(g_1)$ for $g_1, g_2 \in \mathbb{Z}^k$. It is obvious that $\mathrm{Ker}\,\mathrm{d}^2 \subset \mathrm{Ker}\,\delta_v^2$, we just need to prove the opposite direction: $\mathrm{Ker}\,\delta_v^2 \subset \mathrm{Ker}\,\mathrm{d}^2$. This is true in general for $\mathbb{Z}^k$ cocycles, but it also follows from our proof of Thm 1.1.3. Indeed, $\mathrm{Range}\,\delta^1 \subset \mathrm{Ker}\,\mathrm{d}^2 \subset \mathrm{Ker}\,\delta^2$, and we prove in the thesis that $\mathrm{Ker}\,\delta^2 = \mathrm{Range}\,\delta^1$, so $\mathrm{Ker}\,\mathrm{d}^2 = \mathrm{Ker}\,\delta^2$.*

Theorem 1.1.3. *If α is a higher rank action by nilmanifolds automorphisms, then it is infinitesimally rigid:*

$$H^1(\mathbb{Z}^k, \mathrm{Vect}^\infty(M)) = 0.$$

Moreover δ_v^1 has a tame inverse on its image such that $\|(\delta_v^1)^{-1}\beta\|_r \leq C_r\|\beta\|_{r+k}$, where $r \geq 0$ is arbitrary, $k > n_2 + 4$, $n_2 = \dim[N, N]$ and $\|\cdot\|_r$ denotes Sobolev norm.

Remark 1.1.4. *One can also prove that δ_v^2 has a well defined inverse on its image and $\|(\delta_v^2)^{-1}\gamma\|_r \leq C_r\|\gamma\|_{2r+k}$ for $r \geq 0$, k depends on the dimension of $\Gamma\backslash N$. If one could improve the result to $\|(\delta_v^2)^{-1}\gamma\|_r \leq C_r\|\gamma\|_{r+qr+k}, 0 < q < 1$, then it is possible to run a modified KAM scheme and prove the local rigidity of partially hyperbolic actions by higher rank nilmanifold automorphisms.*

Remark 1.1.5. *The reason that we pick 2-step nilmanifolds as our model is mainly for the computation purpose that they have a rather simple coadjoint orbit structure. It seems possible to extend the result to n-step nilmanifolds.*

Chapter 2

Overview of Previous Research and Motivation

2.1 Cocycles for Higher-rank Group Action and Infinitesimal Rigidity

Cocycles lie at the center of many questions about the rigidity of various smooth actions, existence of invariant structures, and other important properties of the action. For cocycles over actions of higher rank abelian groups the cohomological picture may be very different from that in the rank-one case. For the classes of genuinely higher rank abelian Anosov actions, A.Katok and Spatzier proved that any C^∞-cocycle $\beta : A \times M \to \mathbb{R}^\ell$ is C^∞-cohomologous to a constant cocycle, given a standard Anosov A-action on a manifold M where A is isomorphic to $\mathbb{R}^k$ or $\mathbb{Z}^k$ with $k \geq 2$, in their first paper exploring rigidity properties of hyperbolic actions of $\mathbb{Z}^k$ or $\mathbb{R}^k$ for $k \geq 2$ in [KS94]. Given an action of G on a manifold M and a group H, a map $\beta : G \times M \to H$ is called a cocycle provided $\beta(g_1 g_2, m) = \beta(g_1, g_2 m)\beta(g_2, m)$. If H is a Lie group, two cocycles β and β^* are C^∞-cohomologous if there exists a C^∞ transfer function $P : M \to H$, such that $\beta^*(a, x) = P(ax)^{-1}\beta(a, x)P(x)$ for all $a \in G, x \in M$. Hurder proved infinitesimal rigidity for certain hyperbolic actions in [Hur95]. For

Anosov actions by nilmanifold automorphisms, local rigidity is proved by A.Katok and Spatzier in [KS97] without tame estimates. Their proof does not apply to partially hyperbolic actions. Damjanović and A.Katok proved $H^1 = 0$ on torus and δ^1, δ^2 have tame inverses on their images for partially hyperbolic actions by torus automorphisms in [DK10].

2.2 Motivation

An action α is locally rigid if there exists $\ell > 0$ such that for every small perturbation $\tilde{\alpha}$ of α in C^ℓ topology, there is a C^∞ diffeomorphism h (close to the identity) which conjugates $\tilde{\alpha}$ to α: $h \circ \tilde{\alpha} = \alpha \circ h$. The *a priori* regularity method that successfully proves local rigidity of $\mathbb{Z}^k$ actions with sufficiently strong hyperbolic properties (Anosov actions) on the torus encounters some difficulties in the partially hyperbolic setting. First, the foliation for perturbed action is not necessary smooth. Second, even if one only considers perturbation along the neutral foliation, cocycle rigidity of the unperturbed algebraic action is not sufficient. To overcome these difficulties, Damjanović and A.Katok [DK04] used a different approach, the KAM/Harmonic analysis and proved that partially hyperbolic higher rank abelian actions by ergodic automorphisms on the torus are locally rigid. Instead of starting from conjugacy of low regularity, they constructed one of high regularity by an iterative process as a fixed point of a certain nonlinear operator. It is a new approach to prove local differentiable rigidity for actions of higher rank abelian groups and it relies on the classical approach to perturbation problems. Unlike earlier methods, it does not require previous knowledge of structural stability (existence of topological orbit equivalence) and instead, uses an adapted version of the KAM iterative scheme. Moser first noticed that commutativity along with simultaneous Diophantine condition was sufficient to provide a smooth solution to certain over-determined system of equations; however, a major difference is there are infinitely many obstructions for solving the linearized equation for a single element of the action, while there is only one obstruction to solving linearized conjugacy equation of commuting circle rotation. Because of the "higher rank trick" (no nontrivial rank-one factors), these obstructions vanish for the whole action.

In the present thesis, we apply the higher rank trick used in [DK10] and prove the vanishing of first cohomology for higher rank partially hyperbolic actions on a family of 2-step nilmanifolds.

Chapter 3

Preliminaries

This chapter is devoted to the tools we applied in the thesis. The results are provided for the purpose of making the subject complete and keeping the main result more accessible.

3.1 Representation Theory of Lie Groups

Definition 3.1.1. *1. A representation of G on a complex Hilbert space $\mathcal{H}$ is a homomorphism $\pi : G \to GL(\mathcal{H})$, where $GL(\mathcal{H})$ is the group of bounded linear operators on $\mathcal{H}$ with bounded inverses such that $G \times \mathcal{H} \to \mathcal{H}$ is continuous.*

2. A subspace $\mathcal{K}$ of $\mathcal{H}$ is invariant if $\pi(g)\mathcal{K} \subset \mathcal{K}$ for all $g \in G$.

3. $(\pi, \mathcal{H})$ is irreducible if 0 and $\mathcal{H}$ are the only closed invariant subspaces.

4. π is unitary if $\pi(g)$ is unitary for all g, i.e. $\|\pi(g)v\| = \|v\|$ for all $v \in \mathcal{H}$.

5. Two (unitary) representations $\pi, \mathcal{H}$ and $\pi', \mathcal{H}'$ are (unitary) equivalent iff there exists a bounded linear (unitary) $T : \mathcal{H} \to \mathcal{H}'$, with bounded inverse, such that $T \circ \pi(g) = \pi'(g) \circ T$ for all $g \in G$. Here T is called the intertwining operator.

Lemma 3.1.2. *(Schur's Lemma) A unitary representation $\pi : G \to GL(\mathcal{H})$ is irreducible iff the only bounded linear operators on $\mathcal{H}$ commuting with all $\pi(g)$ are the scalar operators.*

3.2 Nilpotent Lie Groups

A Lie group G is nilpotent iff its Lie algebra (over $\mathbb{R}$) is nilpotent. A Lie algebra $\mathfrak{g}$ is nilpotent iff

$$\ldots \subset [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]] \subset [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$$

eventually vanishes. Alternatively we can introduce the descending series of $\mathfrak{g}$ inductively by

$$\mathfrak{g}^{(1)} = \mathfrak{g}, \mathfrak{g}^{(n+1)} = [\mathfrak{g}, \mathfrak{g}^{(n)}] = span_{\mathbb{C}}\{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}^{(\mathfrak{n})}\}$$

$\mathfrak{g}$ is a nilpotent Lie algebra if there is an integer n such that $\mathfrak{g}^{(n+1)} = (0)$. If $\mathfrak{g}^{(n)} \neq (0)$ as well, so that n is minimal, then $\mathfrak{g}$ is said to be $n-$step nilpotent.

Example 3.2.1. *A typical example of nilpotent group is Heisenberg group. Let $\mathcal{H}_n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with multiplication*

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}\omega((x, y), (x', y')))$$

where ω is the symplectic form on $\mathbb{R}^{2n}$,

$$\omega((x, y), (x', y')) = \langle x, y'\rangle - \langle x', y\rangle.$$

The Heisenberg Lie algebra is $\mathfrak{h}_n = span_{\mathbb{R}}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ with Lie brackets $[X_j, Y_j] = Z, 1 \leq j \leq n$ and all other basis brackets not determined by skew-symmetry are zero. Then $[\mathfrak{h}_n, \mathfrak{h}_n] = span_{\mathbb{R}}\{Z\}$, and $[\mathfrak{h}_n, [\mathfrak{h}_n, \mathfrak{h}_n]] = 0$, so $\mathcal{H}_n$ is a two-step nilpotent group.

The last result about nilpotent Lie group is related to particular bases for the nilpotent Lie algebra, and its proof can be found in [CG90a], Thm 1.1.13, by L.Corwin and F.P.Greenleaf.

Theorem 3.2.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \ldots \subseteq \mathfrak{g}_k \subseteq \mathfrak{g}$ be subalgebras, with $\dim \mathfrak{g}_j = m_j$ and $\dim \mathfrak{g} = n$.*
(a) $\mathfrak{g}$ has a basis $\{X_1, ..., X_n\}$ such that

(i) for each $m, \mathfrak{h}_{\mathfrak{m}} = span_{\mathbb{R}}\{X_1, ..., X_m\}$ *is a subalgebra of* $\mathfrak{g}$*,*

(ii) for $1 \leq j \leq k, \mathfrak{h}_{m_j} = \mathfrak{g}_j$*.*

(b) *If the* $\mathfrak{g}_j$ *are ideals of* $\mathfrak{g}$*, then one can pick the* X_j *so that* (i) *is replaced by*

(iii) for each $m, \mathfrak{h}_m = span_{\mathbb{R}}\{X_1, ..., X_m\}$ *is an ideal of* $\mathfrak{g}$*.*

We call a basis satisfying (i) and (ii) a weak Malcev basis for $\mathfrak{g}$ through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$, and one satisfying (ii) and (iii), a strong Malcev basis for $\mathfrak{g}$ through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$.

3.3 Irreducible Components in $L^2(\Gamma \backslash N)$

Some of the references are [CG90b], [CGP77a] and [CG90a].

Let N be a nilpotent group and Γ a lattice, which is a discrete subgroup in N such that the quotient space $\Gamma \backslash N$ has a finite invariant measure (N is a unimodular group and the volume $(\Gamma \backslash N)$ is finite). The lattice is uniform (or cocompact) if the quotient space is compact, and nonuniform otherwise. The homogeneous space of right coset $\Gamma \backslash N$ is called a nilmanifold, and all lattices in nilpotent groups are cocompact.

The nilmanifold $\Gamma \backslash N$ admits a unique probability measure that is invariant under right translations $\Gamma x \to \Gamma xy$ for all $y \in N$, and it is called the Haar measure. In general, Haar measure on a locally compact group G is left invariant, $\mu(gE) = \mu(E)$ for all $g \in G, E$ a Borel set, and unique up to a factor. The measure defined by $\nu_h(E) := \mu(Eh)$ is also left invariant: $\nu_h(gE) = \mu(gEh) = \mu(Eh) = \nu_h(E)$. Define $\delta : G \to (0, \infty)$ by $\nu_h(E) = \delta(h)\mu(E)$. It is not difficult to verify that δ is homomorphism, $\delta(gh) = \delta(g)\delta(h)$, implying $[G, G] \subset \ker(\delta)$. G is called unimodular if $\delta \equiv 1$, and a connected nilpotent Lie group is always a unimodular group, which means that the left invariant measure and the right invariant measure are identical.

The regular representation, or right action U on $L^2(\Gamma \backslash N)$ is defined by

$$U(n)f(x) = f(xn), \quad x \in \Gamma \backslash N, \quad n \in N. \tag{3.1}$$

Under such action $L^2(\Gamma \backslash N)$ decomposes into an orthogonal direct sum $\oplus_{\pi \in (\Gamma \backslash N)^\wedge} H_\pi$ of primary subspaces, where $(\Gamma\backslash N)^\wedge$ represents a discrete subset of $\widehat{N}$ (the set of equivalence classes of irreducible unitary representations of N) for which $H_\pi \neq \{0\}$; each H_π is a direct sum with finite multiplicity of irreducible subspaces associated to a single representation $\sigma_i \in \widehat{N}$.

3.3.1 Orbit Method

Now we introduce a powerful tool: the orbit method. For reference, see [Kir04]. The idea behind this method is to unite harmonic analysis with symplectic geometry, and can be considered as a part of the more general idea of the unification of mathematics and physics. Historically, the orbit method was proposed for the description of unitary dual (i.e. the set of equivalence classes of unitary irreducible representations) of nilpotent Lie group. It turned out that the method not only solves this problem but at the same time, gives simple and visual solutions to all other principal questions in representation theory.

In the dissertation, we will restrict ourselves to nilpotent Lie group, which is also an ideal situation where orbit method works perfectly.

Now we state the Kirillov orbit method that gives a geometric characterization of the set of the equivalence classes of irreducible representations.

Theorem 3.3.1. *(Kirillov Theory) Let N be a connected, simply connected nilpotent Lie group, and $\widehat{N}$ denotes the equivalence classes of unitary irreducible representations of N, then $\widehat{N}$ corresponds to the coadjoint orbits coming from $Ad(N)$ acting on the element in the dual of Lie algebra $\mathfrak{n}^* = Lie(N)^*$, in the following ways:*

(i) For all $\zeta \in \mathfrak{n}^*$, there exists a unitary irreducible representation π_ζ that is unique up to unitary equivalence of irreducible representations.

(ii) For all $\pi \in \widehat{N}$, there exists $\zeta \in \mathfrak{n}^*$, such that $\pi \cong \pi_\zeta$.

(iii) $\pi_\zeta \cong \pi_\eta$ iff ζ, η are in the same coadjoint orbit in $\mathfrak{n}^* = Lie(N)^*$, $Ad^*(N)\zeta = Ad^*(N)\eta$, or $Ad^*(x)\zeta = \eta$ for some $x \in N$.

To illustrate the orbit method, we start with the characterization of unitary irreducible representations on the Heisenberg group.

Let $G = \mathcal{H}_n, \mathfrak{g} = \mathfrak{h}_\mathfrak{n}$. We represent $w \in G, W \in \mathfrak{g}$ by $(n+2) \times (n+2)$ matrices

$$w = \begin{pmatrix} 1 & x_1 & \dots & \dots & x_n & z \\ & \ddots & & & & y_1 \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & \ddots & y_n \\ 0 & & & & & 1 \end{pmatrix}, W = \begin{pmatrix} 0 & a_1 & \dots & \dots & a_n & c \\ & \ddots & & & & b_1 \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & \ddots & b_n \\ 0 & & & & & 1 \end{pmatrix}$$

where $x, y, a, b \in \mathbb{R}^n$ and $z, c \in \mathbb{R}$, by calculation

$$(adw)W = wWw^{-1} = \begin{pmatrix} 0 & a_1 & \dots & \dots & a_n & c+q \\ & \ddots & & & & b_1 \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & \ddots & b_n \\ 0 & & & & & 1 \end{pmatrix}$$

where $q = x\cdot b - y\cdot a$ and $(\cdot)$ is the inner product in $\mathbb{R}^n$.

Now rewrite W as $W = \sum_{i=1}^n (a_i X_i + b_i Y_i) + cZ$, where X_i, Y_i correspond to the entries in the n-tuples a, b. Then $\{Z, Y_1, ..., Y_n, X_1, ..., X_n\}$ is a strong Malcev basis for $\mathfrak{g}$, and for $\ell \in \mathfrak{g}^*$ written in terms of the dual basis $\{Z^*, Y_1^*, ..., Y_n^*, X_1^*, ..., X_n^*\}$, we have

$$\ell = \gamma Z^* + \sum_{j=1}^n (\beta_j Y_j^* + \alpha_j X_j^*) = \ell_{\alpha,\beta,\gamma},$$

$$\ell(W) = c\gamma + \sum_{j=1}^n (\alpha_j a_j + \beta_j b_j).$$

Thus if $w = \exp(zZ + \sum_{i=1}^{n} y_i Y_i + \sum_{i=1}^{n} z_i Z_i)$ and $W \in \mathfrak{g}$ is written as above, we have

$$\begin{aligned}
Ad^*(w)\ell(W) &= \ell(Ad(w^{-1})W) \\
&= \ell[\sum_{j=1}^{n}(a_j X_j + b_j Y_j) + (c + \sum_{j=1}^{n}(y_j a_j - x_j b_j))Z] \\
&= \sum_{j=1}^{n}(a_j \alpha_j + b_j \beta_j) + c\gamma + \sum_{j=1}^{n}(y_j a_j \gamma - x_j b_j \gamma) \\
&= \sum_{j=1}^{n}(a_j(\alpha + y_j\gamma) + b_j(\beta_j - x_j\gamma)) + c\gamma \\
&= \ell_{\alpha+\gamma y, \beta-\gamma x, \gamma}(W).
\end{aligned}$$

For $\gamma \neq 0$

$$(Ad^*G)\ell_{\alpha,\beta,\gamma} = \{\ell_{\alpha',\beta',\gamma} : \alpha', \beta' \in \mathbb{R}^n\},$$

while for $\gamma = 0$, we get

$$(Ad^*G)\ell_{\alpha,\beta} = \{\ell_{\alpha,\beta,0}\}.$$

So, the coadjoint orbits are $2n$ dimensional orbits in $\mathfrak{g}^*$ of the form

$$\gamma Z^* + \zeta^{\perp}$$

for $\gamma \neq 0$, and the zero-dimensional orbits that are points in $\zeta^{\perp} = \mathbb{R}X^* + \mathbb{R}Y^*$ otherwise, where $\zeta^{\perp} = \{\ell \in \mathfrak{g}^* : \ell(Z) = 0\}$.

3.3.2 Maximal Integral Characters

Let Γ be a cocompact lattice in N and $L^2(\Gamma\backslash N) = \oplus_{\pi \in (\Gamma\backslash N)^\wedge} H_\pi$, a natural question is how can one detect which $H_\pi \neq \{0\}$, or which $\pi_\zeta \in (\Gamma\backslash N)^\wedge$? (Here $\zeta \in \text{Lie}(N)^*$, the dual of Lie algebra, and coadjoint orbits are natural parametrization of the equivalence classes of irreducible representations). The answer is: $\pi_\zeta \in (\Gamma\backslash N)^\wedge$ iff the representation π_ζ is induced from a maximal integral character, see Lemma 3.3.3.

Definition 3.3.2. *Maximal Character* $(\mathcal{M}, \chi)$:

(i) m *is a subalgebra and subordinate to* ℓ*, i.e.* $< \ell, [m, m] >= 0$*, for some* $\ell \in Lie(N)^*$*,*

(ii) m *has maximum dimension among the algebras subordinate to* ℓ*,*

(iii) $\mathcal{M} = \exp(m)$*,*

(iv) $\chi = e^{2\pi i\ell}|\mathcal{M}$*.*

In addition, $(\mathcal{M}, \chi)$ *is called a Maximal Integral Character if*

(i) $\Gamma \cap \mathcal{M}\backslash\mathcal{M}$ *is compact,*

(ii) $\chi|\Gamma \cap \mathcal{M} = 1$*.*

Furthermore, there is an explicit way to decompose a function $f \in L^2(\Gamma\backslash N)$ into its primary components. Let P_π be the projection of $L^2(\Gamma\backslash N)$ onto H_π (in [CGP77b])

$$P_\pi f(\Gamma n) = \sum_{\gamma \in \Gamma\cap\mathcal{M}\backslash\Gamma} \int_{m \in \Gamma\cap\mathcal{M}\backslash\mathcal{M}} f(\Gamma m\gamma n)\overline{\chi(m)}d\dot{m} \tag{3.2}$$

for all $f \in C^\infty(\Gamma\backslash N)$, where $d\dot{m}$ = normalized invariant measure on $\Gamma \cap \mathcal{M}\backslash\mathcal{M}$. The sum is absolutely convergent.

In addition to the explicit projection formula, the size of each f_π can be estimated in Sobolev norm as follows

$$\|P_\pi f\|_r \leq C\|\pi\|^{-k}\|f\|_{r+k} \tag{3.3}$$

in [CG90b]. This estimate plays a crucial role in the subsequence chapter when we try to glue the solutions from every subspace H_π into a global one.

3.3.3 Induced Action on the Irreducible Representations

Based on formula (3.2), there is a straightforward computation for $P_\pi(f \circ A)$, where A is an automorphism of the nilmanifold.

$$
\begin{aligned}
P_\pi(f \circ A)(\Gamma n) &= \sum_{\gamma \in \Gamma \cap \mathcal{M} \backslash \Gamma} \int_{\Gamma \cap \mathcal{M} \backslash \mathcal{M}} f(A\Gamma A m A \gamma A n) \overline{\chi(m)} d_{\mathcal{M}} \dot{m} \\
&= \sum_{\gamma \in \Gamma \cap A\mathcal{M} \backslash \Gamma} \int_{\Gamma \cap A\mathcal{M} \backslash A\mathcal{M}} f(\Gamma m \gamma A n) \overline{\chi(A^{-1}m)} d_{\mathcal{M}} A^{-1} \dot{m}
\end{aligned}
$$

for all $f \in \mathbb{C}^\infty(\Gamma \backslash N)$, where $d_{\mathcal{M}}\dot{m}$ is the normalized invariant measure on $\Gamma \cap \mathcal{M} \backslash \mathcal{M}$. Let $\nu = d_{\mathcal{M}} A^{-1}\dot{m}$, which is a measure on $\Gamma \backslash A\mathcal{M}$, we would like to verify that this is the normalized Haar measure. If this is true, then $d_{\mathcal{M}} A^{-1}\dot{m} = d_{A\mathcal{M}}\dot{m}$ by the uniqueness of such measure. It is not difficult to see that ν is the right Haar measure, i.e $\nu(Eg) = d_{\mathcal{M}} A^{-1}\dot{m}(Eg) = d_{\mathcal{M}}\dot{m}(A^{-1}(Eg)) = d_{\mathcal{M}}\dot{m}(A^{-1}EA^{-1}g) = \nu(E)$, for $g \in A\mathcal{M}$, and a connected nilpotent Lie group is a unimodular group. Therefore $d_{\mathcal{M}} A^{-1}\dot{m} = d_{A\mathcal{M}}\dot{m}$, and

$$
(f \circ A)_\pi = f_{A^*\pi} \circ A, A^* = (A^t)^{-1}.
$$

3.3.4 Counting Multiplicity

One central problem in harmonic analysis on nilmanifold is: What is the multiplicity of a given irreducible representation $\sigma \in \widehat{N}$ (the set of equivalent classes of irreducible representation of N) in the regular representation on $L^2(\Gamma \backslash N)$?

This question was answered by Richardson [Ric70] and Howe [How71] independently, and can be summarized as follow.

Lemma 3.3.3 (Multiplicity Formula)**.** *A representation $\pi_\zeta \in \widehat{N}$ occurs in $L^2(\Gamma \backslash N)$ iff $\pi_\zeta = U^\chi = Ind(\mathcal{M} \uparrow N, \chi)$ for some Maximal Integral Character $(\mathcal{M}, \chi)$. The multiplicity of π equals the number of closed double cosets $\mathcal{M}x\Gamma$ for which $\chi|\mathcal{M} \cap x\Gamma x^{-1} = 1|\mathcal{M} \cap x\Gamma x^{-1}$ (call these the integral double cosets). See [CGP77b]*

Let $(\mathcal{M}, \chi)$ be a Maximal Integral Character, and realize $\pi_\zeta = U^\chi = Ind(\mathcal{M} \uparrow N, \chi)$ in the usual way, modeled in the space $H(U^\chi)$ of functions $f : N \to \mathbb{C}$ such that

(i) $f(mn) = \chi(m)f(n)$ (for $m \in \mathcal{M}, n \in N$),

(ii) $\int_{\mathcal{M}\backslash N} |f(n)|^2 dn < \infty$.

Let $H(U^\chi)_{00}$ be the dense subspace of such functions which are continuous and compactly supported in $\mathcal{M}\backslash N$. In [CG76] Corwin-Greenleaf studied the intertwining operators and showed the following result.

Theorem 3.3.4. *Let $(\mathcal{M}, \chi)$ be any Maximal Integral Character for $\Gamma\backslash N$. The following has only finitely many nonzero terms*

$$BF(\Gamma n) = \sum_{\gamma \in \Gamma \cap \mathcal{M}\backslash \Gamma} F(\gamma n)$$

if $F \in H(U^\chi)_{00}$, and the map $B : H(U^\chi)_{00} \to L^2(\Gamma\backslash N)$ extends uniquely to an interwining isometry from $H(U^\chi)$ to an irreducible invariant subspace $H_{(\mathcal{M},\chi)} \subseteq L^2(\Gamma\backslash N)$.

Each $x \in N$ acts on $(\mathcal{M}, \chi)$ to give a new maximal character $(\mathcal{M}, \chi) \cdot x = (\mathcal{M}^x, \chi^x)$ defined by

$$\chi^x(s) = \chi(xsx^{-1}) \forall s \in \mathcal{M}^x = x^{-1}\mathcal{M}x.$$

Let $((\mathcal{M}, \chi) \cdot N)_\sharp$ be the set of Maximal Integral Characters in the orbit $(\mathcal{M}, \chi) \cdot N$.Then

(i) For two integral points in $((\mathcal{M}, \chi) \cdot N)_\sharp$,the range spaces $H_{(\mathcal{M},\chi)\cdot x}$ and $H_{(\mathcal{M},\chi)\cdot y}$ are equal if $\mathcal{M}x\Gamma = \mathcal{M}y\Gamma$ and are orthogonal otherwise.

(ii) An element $\chi \in N$ gives an integral point $(\mathcal{M}, \chi) \cdot x \Leftrightarrow \mathcal{M}x\Gamma$ is an integral double coset. Furthermore, distinct integral double cosets correspond to distinct Maximal Integral Characters.

(iii) The orthogonal sum

$$\oplus\{H_{(\mathcal{M},\chi'):(\mathcal{M},\chi')\in((\mathcal{M},\chi)\cdot N)_\sharp}\} = \oplus_{x\in(\mathcal{M}\backslash N/\Gamma)^*} H_{(\mathcal{M},\chi)\cdot x}$$

where $(\mathcal{M}\backslash N/\Gamma)^*$ = the integral double cosets, is precisely the primaty subspace of $L^2(\Gamma\backslash N)$ corresponding to $\sigma = U^\chi$.

3.4 Computations Using the Orbit Method

3.4.1 The 5-dimensional Example

Because the Heisenberg group has 1-dimensional center and the automorphism group on $S^1 = \mathbb{Z}\backslash\mathbb{R}$ does not include any ergodic transformation, we need nilpotent Lie groups with higher dimensional center. We found an example that fits for our purpose well, constructed by Homolya-Kowalski in [HK06].

Consider a 5-dimensional nilpotent Lie group with center of dimension 2, consider a base of the Lie algebra

$$e_1, e_2, e_3, e_4, e_5 \in \mathfrak{n}_5 \text{ with } [e_1, e_2] = e_4,\ [e_1, e_3] = e_5.$$

Correspondingly, there is a 5×5 matrix to realize this Lie algebra

$$\begin{pmatrix} 0 & 0 & x & z_1 & z_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $[X, Y_1] = Z_1$, $[X, Y_2] = Z_2$.

Compute the coadjoint orbits, which naturally parametrize the equivalence classes of unitary irreducible representations (a good reference for the parametrization would be [CG90a]). If we represent $w \in G, W \in \mathfrak{g}$ by 5×5 matrices

$$w = \begin{pmatrix} 1 & 0 & x & z_1 & z_2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & y_1 & y_2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, W = \begin{pmatrix} 0 & 0 & a & c_1 & c_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $y = (y_1, y_2), z = (z_1, z_2), b = (b_1, b_2), c = (c_1, c_2) \in \mathbb{R}^2$ and $x, a \in \mathbb{R}$, a straightforward calculation gives us

$$(Adw)W = wWw^{-1} = \begin{pmatrix} 0 & 0 & a & c_1 + b_1 x - a y_1 & c_2 + b_2 x - a y_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Rewrite W as $W = \sum_{i=1}^{2}(b_i Y_i + c_i Z_i) + aX$, where the Y_i, Z_i correspond to the entries in the 5-tuples b,c. Then X, Y_1, Y_2, Z_1, Z_2 is a basis for g, and if $l \in g^*$ is expressed in terms of the dual basis $X^*, Y_1^*, Y_2^*, Z_1^*, Z_2^*$, we have

$$\ell = \alpha X^* + \sum_{j=1}^{2}(\beta_j Y_j^* + \gamma_j Z_j^*) = \ell_{\alpha,\beta,\gamma}, \tag{3.4}$$

$$\ell(W) = a\alpha + \sum_{j=1}^{2}(\beta_j b_j + \gamma_j c_j). \tag{3.5}$$

Thus if $w = \exp(\sum_{i=1}^{2}(b_i Y_i + c_i Z_i) + aX)$ and $W \in G$ is written as above, we have

$$\begin{aligned} Ad^*(w)\ell(W) &= \ell(Ad(w^{-1})W) \\ &= a(\alpha + \sum_{j=1}^{2}\beta_j y_j) + b_1(\beta_1 - x\gamma_1) + b_2(\beta_2 - x\gamma_2) + c_1\gamma_1 + c_2\gamma_2 \\ &= \ell_{\alpha + y\cdot\gamma, \beta - x\gamma, \gamma}(W). \end{aligned} \tag{3.6}$$

So there are four types of coadjoint orbits

(a) $\gamma = (\gamma_1, \gamma_2) = (0, 0)$, we get

$$(Ad^*G)\ell_{\alpha,\beta,0} = \ell_{\alpha,\beta,0}.$$

(b) $\gamma = (\gamma_1, 0), \gamma_1 \neq 0$

$$(Ad^*G)\ell_{(\alpha,\beta_1,\beta_2,\gamma_1,0)} = \ell_{(\alpha + y_1\gamma_1, \beta_1 - x\gamma_1, \beta_2, \gamma_1, 0)}.$$

(c) $\gamma = (0, \gamma_2), \gamma_2 \neq 0$

$$(Ad^*G)\ell_{(\alpha,\beta_1,\beta_2,0,\gamma_2)} = \ell_{(\alpha+y_2\gamma_2,\beta_1,\beta_2-x\gamma_2,0,\gamma_2)}.$$

(d) $\gamma = (\gamma_1, \gamma_2), \gamma_1, \gamma_2 \neq 0$

$$(Ad^*G)\ell_{(\alpha,\beta_1,\beta_2,\gamma_1,\gamma_2)} = \ell_{(\alpha+y_1\gamma_1+y_2\gamma_2,\beta_1-x\gamma_1,\beta_2-x\gamma_2,\gamma_1,\gamma_2)}.$$

A natural question arises: How are these coadjoint orbits related, and is it possible to move one type of the orbit to another by an automorphism?

Let $A \in \mathrm{Aut}(\mathfrak{g})$, since A preserves the Lie bracket, and maps the center to center, some restrictions on A follow quite naturally.

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$$

with $a_{44} = a_{11}a_{22}, a_{45} = a_{11}a_{23}, a_{54} = a_{11}a_{32}, a_{55} = a_{11}a_{33}$.

It is not difficult to tell that A^* maps the elements in (a), one-point orbits, to itself. (b),(c) and (d) are 2 dimensional planes in $\mathbb{R}^5$, and we index each of them by its normal direction and the distance from origin to the plane. After we fix the center, the plane can be characterized in $\mathbb{R}^3$. There are three types of coadjoint orbits:

(i) $\gamma = (\gamma_1, 0), \gamma_1 \neq 0$, the normal direction of the plane in $\mathbb{R}^3$ is $(0, 0, 1)$,

and the distance is $|\beta_2|$,

(ii) $\gamma = (0, \gamma_2), \gamma_2 \neq 0$, $n = (0, 1, 0), d = |\beta_1|$,

(iii) $\gamma = (\gamma_1, \gamma_2), \gamma_1, \gamma_2 \neq 0$, $n = (0, \gamma_2, -\gamma_1), d = \frac{|\beta_1\gamma_2 - \beta_2\gamma_1|}{(\gamma_1^2+\gamma_2^2)^{1/2}}$.

If $\gamma_2 = 0$ or $\gamma_1 = 0$, (iii) reduces to (i) or (ii). So essentially, there are two types of coadjoint orbits, one-point orbits and hyperplane orbits.

3.4.2 Free 2-step Nilpotent Group with 3 Generators (Dimension 6)

The example above will have some conflict with ergodicity, because the condition $[Y_1, Y_2] = 0$ and $\det(A) = 1$ will force the matrix representation of the Lie algebra automorphism to have root of unity in the spectrum, as $a_{11}(a_{22}a_{33} - a_{23}a_{32})(a_{44}a_{55} - a_{45}a_{54}) = 1$ implies $a_{11} = \pm 1$. Therefore, it is impossible to put the ergodic assumption on the induced automorphism. So we replace $[Y_1, Y_2] = 0$ by $[Y_1, Y_2] = Z_3$, and there is no reason to distinguish X and Y_1, Y_2 as they play the same role now. Therefore we change the notation of the basis $\{X, Y_1, Y_2, Z_1, Z_2, Z_3\}$ to $\{X_1, X_2, X_3, Z_1, Z_2, Z_3\}$ with $[X_1, X_2] = Z_1, [X_1, X_3] = Z_2, [X_2, X_3] = Z_3$, and realize the new Lie algebra by a 6×6 matrix:

$$\begin{pmatrix} 0 & x_2 & x_1 & z_1 & z_2 & z_3 \\ 0 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & x_2 & x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Another way to look at the matrix is to rewrite them as: $X_1 = e_{13}, X_2 = e_{12} + e_{34}, X_3 = e_{26} + e_{35}$. We compute the coadjoint orbit for a given element $\ell = \sum_{j=1}^{3} \alpha_j X_j^* + \sum_{i=1}^{3} \gamma_i Z_i^* = \ell_{\alpha,\gamma}$,

$$\begin{aligned}
&(Ad^*(w)\ell)(W) \\
&= \ell(Ad(w^{-1})W) \\
&= \sum_{n=1}^{3} a_n \alpha_n - a_1 \sum_{j=1}^{2} \gamma_j x_{j+1} + a_2 \sum_{i=0}^{1} (-1)^i x_{2i+1} \gamma_{2i-1} + a_3 \sum_{k=1}^{2} x_k \gamma_{k+1} + \sum_{m=1}^{3} c_m \gamma_m \\
&= \ell_{(\alpha_1 - x_2\gamma_1 - x_3\gamma_2, \alpha_2 + x_1\gamma_1 - x_3\gamma_3, \alpha_3 + x_1\gamma_2 + x_2\gamma_3, \gamma_1, \gamma_2, \gamma_3)}(W).
\end{aligned}$$

On one hand, for $\gamma_i \neq 0$, $i = 1, 2$, or 3, the coadjoint orbits are two-dimensional planes, which can be characterized by the normal directions and the distance from the origin to the plane. Because the last 3 coordinates in the coadjoint orbit do not change, we can restrict the normal direction to the first 3 coordinates: the normal direction $\vec{n} = (\gamma_3, -\gamma_2, \gamma_1)$ and the distance $d = \|\gamma\| +$

$\frac{|\gamma_1\alpha_3-\gamma_2\alpha_2+\gamma_3\alpha_1|}{\|\gamma\|} = \|\gamma\| + \frac{|\vec{n}\cdot\vec{\alpha}|}{\|\gamma\|}$, where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \|\gamma\| = \sqrt{\gamma_1^2+\gamma_2^2+\gamma_3^2}$ or $\max\{\gamma_i | i = 1, 2, 3\}$, depending on which one fits our purpose better.

On the other hand, if $\gamma_1 = \gamma_2 = \gamma_3 = 0$, the representations are trivial on the commutator.

Since the automorphism A preserves the Lie bracket, and maps the center of the nilpotent group to the center, the matrix representation has to be as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix}$$

with

$$\begin{aligned} a_{44} &= a_{11}a_{22} - a_{12}a_{21}, & a_{45} &= a_{11}a_{23} - a_{13}a_{21}, & a_{46} &= a_{12}a_{23} - a_{13}a_{22}, \\ a_{54} &= a_{11}a_{32} - a_{12}a_{31}, & a_{55} &= a_{11}a_{33} - a_{13}a_{31}, & a_{56} &= a_{12}a_{33} - a_{13}a_{32}, \\ a_{64} &= a_{21}a_{32} - a_{22}a_{31}, & a_{65} &= a_{21}a_{33} - a_{23}a_{31}, & a_{66} &= a_{22}a_{33} - a_{23}a_{32}. \end{aligned}$$

For simplicity, assume $a_{ij} = 0, 4 \leq i \leq 6, 1 \leq j \leq 3$, so the matrix consists of two blocks A_1, A_2, both of which are 3×3 matrices.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} \end{pmatrix}.$$

Now we compute how coadjoint orbits change under automorphisms. Let $\|\pi\|$ be the distance from the origin to O_π in the dual space $\mathfrak{g}^*$ (which is d, the distance from the origin to the 2-dimensional plane as we computed before and O_π is the associated $Ad^*(N)$-orbit for each irreducible representation π). After some calculation, we have $\|A\pi\| = \|A_2\vec{\gamma}\| + \frac{|\det(A_1)||\vec{n}\cdot\vec{\alpha}|}{\|A_2\vec{\gamma}\|}$. If $\det(A_1) = 1$, $\|A^n\pi\| = \|A_2^n\vec{\gamma}\| + \frac{|\vec{n}\cdot\vec{\alpha}|}{\|A_2^n\vec{\gamma}\|}$. When it comes to choosing a proper norm, Sobolev norm will allow us to estimate the size of $P_\pi(f)$ by $\|\pi\|$ and f, and would be an ideal setting.

For a general nilpotent Lie goup G, it admits a uniform subgroup Γ, which is a discrete and $\Gamma\backslash G$ is compact, iff the Lie algebra $\mathfrak{g}$ has a rational structure, meaning that $\mathfrak{g} \cong \mathfrak{g}_\mathbb{Q} \otimes \mathbb{R}$, where $\mathfrak{g}_\mathbb{Q} = span_\mathbb{Q}\{X_1, ..., X_n\}$ and the structure constants are rational: $[X_i, X_j] = \sum_{i=1}^n c_{ijk}X_k, c_{ijk} \in \mathbb{Q}$. See [CG90a].

Pick the standard lattice $\Gamma = \mathbb{Z}^3 \times (\frac{1}{2}\mathbb{Z})^3$, and we can detect which irreducible representations show up in $L^2(\Gamma\backslash N)$ by the Multiplicity Formula 3.3.3. A detailed explanation for this specific case is provided in the later section 3.4.4.

3.4.3 Free 2-Step Nilpotent Group with n Generators (Dimension $\frac{n(n+1)}{2}$)

We generalize to higher dimensional nilmanifolds for the purpose of existence of genuinely partially hyperbolic actions. Let $\mathfrak{n}$ be the Lie algebra of a nilpotent Lie group N, with a basis $\{X_1, X_2 \dots X_n, Z_{12}, Z_{13} \dots Z_{1n}, Z_{23} \dots Z_{2n}, Z_{34} \dots Z_{n-1n}\}$, whose brackets are zero except for

$$[X_i, X_j] = Z_{ij}, 1 \le i < j \le n.$$

There is a $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ matrix representation for it.

$$\left(\begin{array}{ccccccccccc}
0 & x_{n-1} & \dots & x_2 & x_1 & z_{12} & z_{13} & \dots & z_{n-2,n-1} & z_{n-2,n} & z_{n-1,n} \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & x_n \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & x_{n-1} & x_n & 0 \\
 & & & & & \dots & \dots & & & & \\
 & & & & & \dots & \dots & & & & \\
0 & 0 & \dots & 0 & 0 & x_2 & x_3 & 0 & \dots & 0 & 0 \\
 & & & & & \dots & \dots & & & & \\
 & & & & & \dots & \dots & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0
\end{array}\right).$$

The purpose of writing this big matrix is to compute the coadjoint orbits.

Given $\ell = \sum_{k=1}^{n} \alpha_k X_k^* + \sum_{1\le i<j\le n} \gamma_{ij} Z_{ij}^* = \ell_{\vec{\alpha},\vec{\gamma}}$,

$$\begin{aligned}
(Ad^*(w)\ell)(W) &= \ell(Ad(w^{-1})W) \\
&= a_1(\alpha_1 + \gamma_{12}x_2 + \gamma_{13}x_3 + \ldots + \gamma_{1n}x_n) + a_2(\alpha_2 - \gamma_{12}x_1 + \gamma_{23}x_3 + \ldots + \gamma_{2n}x_n) \\
&+ a_3(\alpha_3 - \gamma_{13}x_1 - \gamma_{23}x_2 + \gamma_{34}x_4 + \ldots + \gamma_{3n}x_n) + \ldots\ldots \\
&+ a_{n-1}(\alpha_{n-1} - \gamma_{1,n-1}x_1 - \gamma_{2,n-1}x_2 - \ldots - \gamma_{n-2,n-1}x_{n-1} + \gamma_{n-1,n}x_n) \\
&+ a_n(\alpha_n - \gamma_{1n}x_1 - \gamma_{2n}x_2 - \ldots - \gamma_{n-1,n}x_{n-1}) \\
&+ \sum_{1\le i<j\le n} c_{ij}\gamma_{ij} \\
&= \ell_{(\alpha_1+\gamma_{12}x_2+\gamma_{13}x_3+\ldots+\gamma_{1n}x_n\ldots\ldots\alpha_n-\gamma_{1n}x_1-\gamma_{2n}x_2-\ldots-\gamma_{n-1,n}x_{n-1},\gamma_{12},\gamma_{13},\ldots,\gamma_{n-1,n})}(W).
\end{aligned}$$

For $\vec{\gamma} = 0$, the coadjoint orbits are one-point orbits, and the representations of N are one-dimensional, since the quotient group $[N,N]\backslash N$ is abelian.

If $\vec{\gamma} \neq 0$, the coadjoint orbits are hyperplanes in $\mathbb{R}^{\frac{n(n+1)}{2}}$ (or essentially in $\mathbb{R}^n$, since the last

$\frac{n(n-1)}{2}$ coordinates are fixed) with normal direction

$$\vec{n} = \det \begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} & \gamma_{14} & \cdots & \gamma_{1n} \\ -\gamma_{12} & 0 & \gamma_{23} & \gamma_{24} & \cdots & \gamma_{2n} \\ -\gamma_{13} & -\gamma_{23} & 0 & \gamma_{34} & \cdots & \gamma_{3n} \\ & & \cdots & \cdots & & \\ -\gamma_{1,n-1} & -\gamma_{2,n-1} & -\gamma_{3,n-1} & \cdots & 0 & \gamma_{n-1,n} \\ e_1 & e_2 & e_3 & \cdots & e_{n-1} & e_n \end{pmatrix}$$

where $\{e_i, 1 \leq i \leq n\}$ is the standard basis for $\mathbb{R}^n$. The distance from the origin to the plane is $d = \|\vec{\gamma}\| + \frac{|\vec{\alpha} \cdot \vec{n}|}{\|\vec{\gamma}\|}$, where $\cdot$ is the standard inner product and $\|.\|$ is the Euclidean distance.

Now we want to see how are these hyperplanes moved by the automorphisms. Let $A \in \mathrm{Aut}(\mathfrak{n})$, since $[AX_i, AX_j] = A[X_i, X_j] = AZ_{ij}, 1 \leq i < j \leq n$, the restriction of the automorphism on $\{X_1, X_2 \ldots X_n\}$ is A_1, which actually determines the automorphism A_2 acting on the center $\{Z_{ij}, 1 \leq i < j \leq n\}$. So we can focus on A_1 corresponding to a lower dimension matrix.

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n-1} & a_{2n} \\ & & \cdots & \cdots & & \\ & & \cdots & \cdots & & \\ a_{n-11} & a_{n-12} & a_{n-13} & \cdots & a_{n-1n-1} & a_{n-1n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn-1} & a_{nn} \end{pmatrix}$$

Let A_2 denote the $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ matrix corresponding to the automorphism restricted to the center $\{Z_{ij}, 1 \leq i < j \leq n\}$. We already know that $\|\pi\| = \|\vec{\gamma}\| + \frac{|\vec{\alpha} \cdot \vec{n}|}{\|\vec{n}\|}$, what about $\|A\pi\|$? Similar to the 6 dimensional nilmanifold, for $|\det(A_1)| = 1, \|A^n \pi\| = \|A_2^n \vec{\gamma}\| + \frac{|\vec{\alpha} \cdot \vec{n}|}{\|A_2^n \vec{\gamma}\|}$.

3.4.4 The Elements π Occuring in $(\Gamma \backslash N)^\wedge$

The Howe-Richardson occurence condition states that $\pi_\ell \in (\Gamma \backslash N)^\wedge$ iff it is induced from a Maximal Integral Character associated to some element in the coadjoint orbit of ℓ. Here we write π_ℓ for the representation associated to the orbit in $\mathrm{Lie}(N)^*$ that contains the element ℓ. For Maximal Integral Character, see Definition 3.3.2

3.4.4.1 The Free 2-step Nilpotent Group of Dimension 6

For the standard lattice $\Gamma = \mathbb{Z}^3 \times (\frac{1}{2}\mathbb{Z})^3$, and the condition $\chi|\Gamma \cap \mathcal{M} = 1$, we conclude that $\gamma_i \in 2\mathbb{Z}, i = 1, 2, 3$. As we computed before, the coadjoint orbits are two-dimensional planes, and the irreducible representations occurring in $(\Gamma\backslash N)^\wedge$, or $\pi \in (\Gamma\backslash N)^\wedge$ are equivalent to the corresponding coadjoint orbits containing integer points. The plane equation is

$$x = \alpha - t_3\gamma_1 - t_3\gamma_2, y = \beta_1 + t_1\gamma_1 - t_3\gamma_3, z = \beta_2 + t_1\gamma_2 + t_2\gamma_3, t_i \in \mathbb{R}, \gamma_i \in 2\mathbb{Z}.$$

Another way to look at it is starting with the plane passing the origin

$$\gamma_3 x + (-\gamma_2)y + \gamma_1 z = 0$$

and it contains integer points due to the even integer coefficient. We can move this plane parallelly along its normal direction by some distance such that it intersects integer points again. Suppose that the plane contains an integer point (a, b, c), we have

$$\gamma_3(x - a) + (-\gamma_2)(y - b) + \gamma_1(z - c) = 0,$$

$$\gamma_3 x + (-\gamma_2)y + \gamma_1 z = a\gamma_3 - b\gamma_2 + c\gamma_1.$$

Since $\mathbb{Z}$ is a principal ideal domain, and $I = \gamma_1\mathbb{Z} + \gamma_2\mathbb{Z} + \gamma_3\mathbb{Z}$ is an ideal in $\mathbb{Z}$, so $I = r\mathbb{Z}$, $r = \gcd(\gamma_1, \gamma_2, \gamma_3)$. The minimum distance is $d = \gcd(\gamma_1, \gamma_2, \gamma_3)/(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)^{1/2}$, in order for it to intersect integer points again. All these planes that contain integer points are characterized by the normal direction $\vec{n} = (\gamma_3, -\gamma_2, \gamma_1)$, and the distance from the origin to the plane: $d_k = kd$, $k \in \mathbb{Z}$. So for each fixed $(\gamma_1, \gamma_2, \gamma_3)$, we can identify all the coadjoint orbits corresponding to the Maximal Integral Characters, leading to the geometric characterization of all the irreducible representations occurring in $(\Gamma\backslash N)^\wedge$.

Pick an element $\ell = \alpha X^* + \sum_{j=1}^{2}(\beta_j Y_j^* + \gamma_j Z_j^*) = \ell_{\alpha,\beta,\gamma}$ in the coadjoint orbit, which is in the form of $\ell_{(\alpha - y_1\gamma_1 - y_2\gamma_2, \beta_1 - x\gamma_1 - y_2\gamma_3, \beta_2 + x\gamma_2 + y_1\gamma_3, \gamma_1, \gamma_2, \gamma_3)}$. One of the Maximal Integral Characters $(\chi, \mathcal{M})$ is given by

$$\chi = e^{2\pi i \ell_{\alpha,\beta,\gamma}} | \mathcal{M}, m = \operatorname{span}_{\mathbb{Z}}\{\gamma_3 X + \gamma_2 Y_1, 2\gamma_3 X + \gamma_1 Y_2, Z_1, Z_2, Z_3\}, (\gamma_1, \gamma_2 \neq 0, \gamma_i \in 2\mathbb{Z}).$$

A more general Maximal Integral Character $(\mathcal{M}, \chi)$ can be described in the following way $\dim \mathcal{M} = 5$, $m = \operatorname{span}_{\mathbb{Z}}\{aX + bY_1 + cY_2, dX + eY_1 + fY_2, Z_1, Z_2, Z_3\}$ with $a, b, c, d, e, f \in \mathbb{R}$, and the condition that m is subordinate to ℓ, i.e. $< \ell, [m, m] >= 0$ implies that

$$\det \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ f & e & d \\ c & b & a \end{pmatrix} = 0.$$

In other words, we need to find two vectors in $\mathbb{R}^3$ such that these three vectors $(\gamma_1, \gamma_2, \gamma_3), (f, e, d), (c, b, a)$ lie in the same plane. One simple solution is $(a, b, c) = (\gamma_3, \gamma_2, 0), (d, e, f) = (0, 0, \gamma_1)$, or their linear combination (the plane generated by these two vectors).

3.4.4.2 The General Free 2-step Nilpotent Group

Pick the standard lattice $\Gamma = \mathbb{Z}^n \times (\frac{1}{2}\mathbb{Z})^{\frac{n(n-1)}{2}}$, and all the representations in $(\Gamma\backslash N)^\wedge$ come from the induced representations of Maximal Integral Characters. Similar to the 6 dimensional case, we interpret the conditions of being a Maximal Integral Character that $(\Gamma \cap \mathcal{M})\backslash\mathcal{M}$ is compact and $\chi|\Gamma \cap \mathcal{M} = 1$ as $\gamma_{ij} \in 2\mathbb{Z}, d_{min} = \frac{gcd(\gamma_{ij})}{\sqrt{\sum_{1 \leq i < j \leq n} \gamma_{ij}^2}}, 1 \leq i < j \leq n$. So we have a clear picture of all the representations $\pi \in (\Gamma\backslash N)^\wedge$ for these nilmanifolds.

Chapter 4

Main Results

4.1 Solving Coboundary Equations with Tame Estimates

4.1.1 Necessary Conditions for Solving the Coboundary Equation

Lemma 4.1.1. *Given a smooth function $\theta \in C^\infty(\Gamma \backslash N)$, the equation*

$$\lambda\omega - \omega \circ A = \theta$$

for $\lambda \neq 1$ admits a smooth solution ω only if the obstruction

$$\vartheta_\pi^A(\theta) := \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)} \theta_{A^{*i}\pi} \circ A^i$$

vanishes for all $\pi \in (\Gamma \backslash N)_0^\wedge$.

Proof. Without loss of generality, we can always assume $|\lambda| \geq 1$. This is because for $0 < |\lambda| < 1$, the equation $\lambda\omega - \omega \circ A = \theta$ can be transformed to $\omega \circ A^{-1} - \lambda^{-1}\omega = \lambda^{-1}\theta \circ A^{-1}$ with $|\lambda|^{-1} \geq 1$.

Suppose there exists a C^∞ solution ω to the coboundary equation $\lambda\omega - \omega \circ A = \theta$. Then we project the equation to the subspace H_π associated to a given irreducible representation π

$$P_\pi(\lambda\omega - \omega \circ A) = P_\pi(\theta),$$

which is equivalent to

$$\lambda\omega_\pi - \omega_{A^*\pi} \circ A = \theta_\pi.$$

Iterating forward

$$\omega_\pi = \sum_{i=0}^{n-1} \lambda^{-(i+1)} \theta_{(A^*)^i\pi} \circ A^i + \lambda^{-n}\omega_{(A^*)^n\pi} \circ A^n$$

and let $n \to \infty$,

$$\omega_{\pi+} = \sum_{i=0}^{\infty} \lambda^{-(i+1)} \theta_{(A^*)^i\pi} \circ A^i + \lim_{n\to\infty} \lambda^{-n}\omega_{(A^*)^n\pi} \circ A^n.$$

If the second term goes to zero, $\omega_{\pi+} = \sum_{i=0}^{\infty} \lambda^{-(i+1)} \theta_{(A^*)^i\pi} \circ A^i$.

Similarly, $\omega_{\pi-} = -\sum_{i=-1}^{-\infty} \lambda^{-(i+1)} \theta_{(A^*)^i\pi} \circ A^i$ if we iterate backward. Now the obstruction for solving the coboundary equation arises, and the existence of a solution implies that $\omega_{\pi+} = \omega_{\pi-}$ for all $\pi \in (\Gamma\backslash N)_0^\wedge$ (obstructions vanish).

The following argument is to verify all obstructions $\vartheta_\pi^A(\theta) = \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)} \theta_{A^{*i}\pi} \circ A^i$ converge in C^0 norm, so the vanishing makes sense.

$$\| \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)} \theta_{A^{*i}\pi} \circ A^i \|_0 = \| \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)} \theta_{A^{*i}\pi} \|_0 \le \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)} \|A^{*i}\pi\|^{-k} \|\theta\|_k.$$

We can always find $k > 0$ such that $\sum_{i\in\mathbb{Z}} \lambda^{-(i+1)} \|A^{*i}\pi\|^{-k} < \infty$ for all $\pi \neq 0$. For $|\lambda| \ge 1$, the $\sum_{i\ge0}$ part converges because we can choose an exponent k to ensure $\sum_{\pi\in(\Gamma\backslash G)_0^\wedge} \|\pi\|^{-k} < \infty$. As for the other part $\sum_{i<0}$, we use the fact that every non-zero integer vector γ representing the center component of the element $\ell \in \mathfrak{g}^*$ always has a nontrivial projection to the expanding directions with respect to A due to the ergodicity assumption:

$$\begin{aligned}
&\sum_{i<0} |\lambda|^{-(i+1)}(\|A_2^{*i}\gamma\| + \frac{|m \cdot \gcd(\gamma_1, \gamma_2, \gamma_3)|}{\|A_2^{*i}\gamma\|})^{-k} \\
&\leq \sum_{i<0} |\lambda|^{-(i+1)} \|A_2^{*i}\gamma_s\|^{-k} \leq C \sum_{i<0} |\lambda|^{-(i+1)}(\rho)^{ik}\|\gamma_s\|^{-k} \\
&< \infty
\end{aligned}$$

where k is large enough such that $\lambda^{-1}\rho^k > 1$ and for some $m \in \mathbb{Z}$, determined by the corresponding coadjoint orbit of the irreducible representation π.

When it comes to smoothness of $\vartheta_\pi^A(\theta) = \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)}\theta_{A^{*i}\pi} \circ A^i$, we need to increase the absolute value of the exponent k of $\|\pi\|$ to get convergence in C^r norm for $r > 0$.

□

4.1.2 Partial Norms

Below all the derivatives are understood in the distributional sense.

Definition 4.1.2. *Let r be a non-negative integer. The Sobolev space $\mathbb{H}^r(\mathbb{R}^n)$ is defined by*

$$H^r(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \partial^\alpha f \in L^2(\mathbb{R}^n) \textit{ for all } |\alpha| \leq r\}$$

Proposition 4.1.3. *$f \in H^r(\mathbb{R}^n)$ iff $(1+|\xi|^2)^{r/2}\hat{f} \in L^2(\mathbb{R}^n)$, and the following norms are equivalent:*

$f \mapsto [\sum_{|\alpha|\leq r} \|\partial^\alpha f\|^2_{L^2(\mathbb{R}^n)}]^{\frac{1}{2}}$ and $f \mapsto [\int_{\mathbb{R}^n}(1+|\xi|^2)^r|\hat{f}(\xi)|^2 d\xi]^{\frac{1}{2}} = \|(1+|\xi|^2)^{\frac{r}{2}}\hat{f}\|_{L^2(\mathbb{R}^n)}$. In short, $\|\partial^\alpha f\|_{L^2} = \|\widehat{\partial^\alpha f}\|_{L^2} = \|\xi^\alpha \hat{f}\|_{L^2}$, and $\|(1+|\xi|^2)^{\frac{r}{2}}\hat{f}\|_{L^2}$ is equivalent to $\|(1+|\xi|^r)\hat{f}\|_{L^2}$

The proof that $[\sum_{|\alpha|\leq r} \|\partial^\alpha f\|^2_{L^2(\mathbb{R}^n)}]^{\frac{1}{2}}$ and $[\int_{\mathbb{R}^n}(1+|\xi|^2)^r|\hat{f}(\xi)|^2 d\xi]^{\frac{1}{2}}$ are equivalent norms is mainly because of the following inequalities

$$C^{-1}(1+|x|)^{2r} \leq \sum_{|\alpha|\leq r} (x^\alpha)^2 \leq C(1+|x|)^{2r}, x \in \mathbb{R}^n, \tag{4.1}$$

where C depends only on r and n.

Since $(f \circ A)_\pi = f_{A*\pi} \circ A$ for $f \in C^\infty(\Gamma\backslash N)$, in order to get a better estimate of its growth under Sobolev norm, we introduce the concept of partial norm here. For a linear transformation A, there is a decomposition of $\mathbb{R}^n$

$$\mathbb{R}^n = V_s \oplus V_c \oplus V_u, \xi \mapsto \xi_s + \xi_c + \xi_u \text{ with } |\xi|^2 \approx |\xi_s|^2 + |\xi_c|^2 + |\xi_u|^2.$$

Define partial norms $\|f\|_r^{(s)} := \||\xi_s|^r \hat{f}\|_{L^2}, \|f\|_r^{(c)} := \||\xi_c|^r \hat{f}\|_{L^2}$ and $\|f\|_r^{(u)} := \||\xi_u|^r \hat{f}\|_{L^2}$, which play a crucial role in our computation. We will see how the partial norms are used to control the growth of $f \circ A^i$

$$\|f \circ A^i\|_r^{(s)} = \||\xi_s|^r \widehat{f \circ A^i}\|_{L^2} = [\int_{\widehat{\mathbb{R}^n}} |\xi_s|^{2r} |\hat{f} \circ A^{-it}(\xi)|^2 d\xi]^{\frac{1}{2}} \tag{4.2}$$

$$= [\int_{\widehat{\mathbb{R}^n}} |A^{it}(\eta)_s|^{2r} |\hat{f}(\eta)|^2 d\eta]^{\frac{1}{2}} \leq [\int_{\widehat{\mathbb{R}^n}} \|A^{it}|_{V^s}\|^{2r} |\eta_s|^{2r} |\hat{f}(\eta)|^2 d\eta]^{\frac{1}{2}} \leq \|A^t|_{V^s}\|^r \|f\|_r^{(s)}.$$

Geometrically, we can decompose the differential operator, the Laplacian, instead of the space $\mathbb{R}^n$. Similar to (4.1), for $f \in C^\infty(\mathbb{R}^n)$

$$C^{-1}\|(1+|\triangle|)^r f\|_{L^2} \leq \sum_{|\alpha|\leq r} \|(D^\alpha)^2 f\|_{L^2} \leq C\|(1+\triangle)^r f\|_{L^2} \tag{4.3}$$

where $\triangle = \sum_{1\leq i\leq n} \frac{\partial^2}{\partial x_i^2}$, and C depends only on r and n. Similar to

$$\mathbb{R}^n = V_s \oplus V_c \oplus V_u,$$

we have

$$\triangle = \triangle_s + \triangle_c + \triangle_u.$$

based on the correspondence between $\frac{\partial}{\partial x_i}$ and ξ_i, where $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$. How can we formulate them in terms of vector fields? To apply the decomposition, we choose an orthonormal basis $\{Y_1, ..., Y_n\}$ which can be separated to $\{Y_{1,u}, ..., Y_{k_u,u}\}, \{Y_{1,c}, ..., Y_{k_c,c}\}$ and $\{Y_{1,s}, ..., Y_{k_s,s}\}$ such that

$AY_{i,u} = \lambda_i Y_{i,u}, AY_{j,c} = \mu_j Y_{j,c}, AY_{\ell,s} = \nu_\ell Y_{\ell,s}$ with $|\lambda_i| > 1, |\mu_j| = 1, |\nu_\ell| < 1, 1 \le i \le k_u, 1 \le j \le k_c, 1 \le \ell \le k_s$. Then define

$$\triangle_u := Y_{1,u}^2 + ... + Y_{k_u,u}^2, \triangle_c := Y_{1,c}^2 + ... + Y_{k_c,c}^2, \triangle_s := Y_{1,s}^2 + ... + Y_{k_s,s}^2. \tag{4.4}$$

There is no guarantee that such basis ever exists, in general, it might involve some Jordan blocks. Again, separate the basis to $\{Y_{1,u}, ..., Y_{k_u,u}\}, \{Y_{1,c}, ..., Y_{k_c,c}\}, \{Y_{1,s}, ..., Y_{k_s,s}\}$ by using the generalized eigenvector of A corresponding to λ_i if $(A - \lambda_i I)^{p_i} = 0$ for some positive integer p_i, etc. for $|\lambda_i| > 1, |\mu_j| = 1, |\nu_\ell| < 1$. Now we define partial norm with more geometrical flavor that can be generalized to manifolds

$$\|f\|_r^{(u)} = \||\triangle_u|^{r/2} f\|_{L^2}, \|f\|_r^{(c)} = \||\triangle_c|^{r/2} f\|_{L^2}, \|f\|_r^{(s)} = \||\triangle_s|^{r/2} f\|_{L^2}. \tag{4.5}$$

To make the idea work for nilmanifolds, one method is to introduce Sobolev norm the same as in [CG90b]. We start with a decomposition of $\mathfrak{n} = E^u \oplus E^c \oplus E^s$ and a particular basis $\{Y_{1,u}, ..., Y_{k_u,u}, Y_{1,c}, ..., Y_{k_c,c}, Y_{1,s}, ..., Y_{k_s,s}\}$ in $\mathfrak{n}$ such that $Y_{1,u}, ..., Y_{k_u,u} \in E^u$, $Y_{1,c}, ..., Y_{k_c,c} \in E^c$, $Y_{1,s}, ..., Y_{k_s,s} \in E^s$. Then impose the Solobev norm as $\|\phi\|_k^2 = \sum_{|\alpha|=k} \|R(Y^\alpha)\phi\|_{L^2}^2$, where $Y^\alpha = Y_1^{\alpha_1}...Y_n^{\alpha_n}, \alpha \in \mathbb{Z}^n$ and the right-invariant vector field is determined by $R(Y)f(g) = \lim_{t\to 0} \frac{1}{t}[f(\exp(tY)g) - f(g)]$. The corresponding partial norms would be

$$\|\phi\|_r^{(u)} = \sum_{|\alpha_u|=k} \|R(Y_u^{\alpha_u})\phi\|_{L^2},$$
$$\|\phi\|_r^{(c)} = \sum_{|\alpha_c|=k} \|R(Y_c^{\alpha_c})\phi\|_{L^2},$$
$$\|\phi\|_r^{(s)} = \sum_{|\alpha_s|=k} \|R(Y_s^{\alpha_s})\phi\|_{L^2}$$

where $\alpha_u \in \mathbb{Z}^{k_u}, \alpha_c \in \mathbb{Z}^{k_c}, \alpha_s \in \mathbb{Z}^{k_s}$ with $k_u + k_c + k_s = n$.

4.1.3 Necessary Conditions are Sufficient

We want to find a smooth solution with tame estimates to the equation

$$\lambda\omega - \omega \circ A = \theta \tag{4.6}$$

under the condition that the obstruction $\vartheta_\pi^A(\theta) = \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)}\theta_{A^{*i}\pi} \circ A^i$ vanishes for every non-trivial irreducible representation $\pi \in (\Gamma\backslash N)^\wedge$. In the dual space this equation has the form

$$\lambda\omega_\pi - \omega_{A^*\pi} \circ A = \theta_\pi$$

where ω_π stands for the projection of ω into the subspace H_π associated to the irreducible representation π. For the one-dimensional representation (that is trivial on the commutator), the problem reduces completely to the torus situation and has been answered in [DK04]. So we will focus on the higher dimensional representations. Let $(\Gamma\backslash N)_0^\wedge$ be the spectrum with the trivial representation π_0 excluded, and for each $\pi \in (\Gamma\backslash N)_0^\wedge$, the projected equation $\lambda\omega_\pi - \omega_{A^*\pi} \circ A = \theta_\pi$ has two formal solutions

$$\omega_{\pi+} = +\sum_{i\geq 0} \lambda^{-(i+1)}\theta_{A^{*i}\pi} \circ A^i, \tag{4.7}$$

$$\omega_{\pi-} = -\sum_{i\leq -1} \lambda^{-(i+1)}\theta_{A^{*i}\pi} \circ A^i. \tag{4.8}$$

Each sum converges absolutely in the L^∞ norm for $\theta \in C^\infty(\Gamma\backslash N)$, and the detailed computation is included in 4.1.1.

When it comes to gluing all the solutions within every H_π together to form the global solution, we refer to the result by Corwin and Greenleaf in [CG90b]: an important estimate about the size of f_π by the distance from origin to coadjoint orbit $\|\pi\|$ and the original function f in the Sobolev norm:

$$\|f_\pi\|_r \leq C\|\pi\|^{-k}\|f\|_{r+k}, \forall f \in C^{r+k}(\Gamma\backslash N), r > 0$$

where C depends only on r and the nilmanifold. In our situation, the estimate about $\|f_{A^{*n}\pi} \circ A^n\|_r$

is handled by using the partial norm techniques.

$$\|f_{A^{*n}\pi}\circ A^n\|_r^V \le \|f_{A^{*n}\pi}\|_r\|A^n|_V\|^r \le C'\|A^n|_V\|^r\|A^{*n}\pi\|^{-k}\|f\|_{r+k}, \forall f\in C^{r+k}(\Gamma\backslash N), r>0.$$

If we pick V as the stable subspace when $n>0$, and unstable space for $n<0$, $\|A^n|_V\|^r$ would not affect our estimate.

Another part of the problem is the estimate of dual orbit growth $\|A^n\pi\|$, which is essentially determined by the automorphism restricted to the center parameter γ, based on the formula $\|A\pi\| = \|A_2^n\gamma\| + |\det(A_1)|\frac{|\vec{n}\cdot\vec{\alpha}|}{\|A_2^n\gamma\|}$. Similar to torus, we decompose γ to expanding, neutral and contracting components with respect to A_2,

$$\mathbb{R}^d = V^u\oplus V^c\oplus V^s, \gamma=\gamma_u+\gamma_c+\gamma_s$$

where $d=\dim[N,N]$.

$$\|A_2^i\gamma_u\|\ge C\rho^i\|\gamma_u\|, \rho>1, i\ge 0$$
$$\|A_2^i\gamma_s\|\ge C\rho^{-i}\|\gamma_s\|, \rho>1, i\le 0$$
$$\|A_2^i\gamma_c\|\ge C(|i|+1)^{-N}\|\gamma_c\|, i\in\mathbb{Z}.$$

Theorem 4.1.4. *Let θ be a C^∞ function on $M=\Gamma\backslash N$, which is the 2-step nilmanifold constructed in 3.4.3, with $dimN = \frac{n(n+1)}{2}$ and $\lambda\in\mathbb{C}, \lambda\ne 1$. Let $n_1=\dim(\Gamma\cdot[N,N])\backslash N = n, n_2 = \dim\Gamma\backslash[N,N] = \frac{n(n-1)}{2}$ and A be an integer matrix in $SL(n_1+n_2,\mathbb{Z})$ with A_1, A_2 the matrix representations of ergodic automorphisms on the quotient of nilmanifold over the center $(\Gamma\cdot[N,N])\backslash N$ and on the center of nilmanifold $\Gamma\backslash[N,N]$ respectively, such that for all non-trivial irreducible representations $\pi\in(\Gamma\backslash N)^\wedge$, denoted by $(\Gamma\backslash N)^\wedge_0$, the following sum (called obstruction) along the dual orbits are zero i.e.*

$$\vartheta_\pi^A(\theta) = \sum_{i=-\infty}^{+\infty}\lambda^{-(i+1)}\theta_{A^{*i}\pi}\circ A^i = 0 \tag{4.9}$$

then the equation

$$\lambda\omega - \omega \circ A = \theta \tag{4.10}$$

has a C^∞ *solution* ω *with the following estimate*

$$\|\omega\|_r \leq C_r \|\theta\|_{r+k} \textit{ for } r \geq 0, k > n_2 + 4 \tag{4.11}$$

in the Sobolev norm and $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$.

We prove the theorem in the rest of this section.

Estimates of ω in C^0

Take $r = 0$.

When $\pi = \pi_0$, we can immediately calculate that $\omega_{\pi_0} = \frac{\theta_{\pi_0}}{\lambda - 1}$ because $\lambda \neq 1$.

For other situations, one has to distinguish between the center parameter $\gamma = 0$ and $\gamma \neq 0$.

(i) If $\gamma = 0$, from last section 3.4.2, the coadjoint orbits are one-point orbits. Thus the maximal integral character is $(M, \pi_\ell) = (N, \chi_\ell)$, where $\chi_\ell(\exp W) = e^{2\pi i \ell(W)}, \ell \in Z^\perp, W \in \mathfrak{n}$. These representations are trivial on the commutator $[N, N] = \exp \mathbb{R}Z$, and hence all the irreducible representations on N can be recovered by lifting the irreducible representations on $[N, N]\backslash N$. The representations on the quotient group $[N, N]\backslash N$ are one-dimensional since it is abelian, and furthermore, the induced automorphism A_1 is ergodic, see [Par69]. So we can reduce the problem to torus $\mathbb{T}^n$, which has been solved in Lemma 4.2 of [DK10].

(ii) If $\gamma \neq 0$, the situation is more complicated. Pick one of the formal solutions in (4.7), say $\omega_{\pi+}$, and we are going to prove the convergence of the solution and tame estimates in the Sobolev norm. First assume that γ_u is the largest term in the decomposition of γ into γ_s, γ_c and γ_u with respect to the automorphism A_2, $\|\gamma_u\| \geq \max\{\|\gamma_s\|, \|\gamma_c\|\}$ in Euclidean norm. Denote all such elements of the representations by $(\Gamma\backslash N)^\wedge_{0,u}$, which is a subset of $(\Gamma\backslash N)^\wedge_0$. This decomposition will have further application in later calculation under the partial norm $\|\omega\|_r^{(c)}$. Without taking the L^2 norm of the derivatives of $\theta_{A^{*i}\pi} \circ A$ into considerations, we can prove the existence of C^0-solution for $|\lambda| \geq 1$.

$$
\begin{aligned}
&\sum_{\pi\in(\Gamma\backslash N)^\wedge_{0,u}} \|\omega_{\pi+}\|_{C^0} \leq \sum_{\pi\in(\Gamma\backslash N)^\wedge_{0,u}} \sum_{i\geq 0} |\lambda|^{-(i+1)} \|\theta_{A^{*i}\pi}\|_{C^0} \\
&\leq \sum_{\pi\in(\Gamma\backslash N)^\wedge_{0,u}} \sum_{i\geq 0} |\lambda|^{-(i+1)} \|A^{*i}\pi\|^{-k} \|\theta\|_k \\
&= \sum_{\ell\in\mathbb{Z},\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_u} \sum_{i\geq 0} |\lambda|^{-(i+1)} (\|A_2^{*i}\gamma\| + \frac{|\ell\cdot \gcd(\gamma_1,\gamma_2,...,\gamma_{n_2})|}{\|A_2^{*i}\gamma\|})^{-k} \|\theta\|_k \\
&= \sum_{\ell=0,\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_u} \sum_{i\geq 0} |\lambda|^{-(i+1)} \|A_2^{*i}\gamma\|^{-k} \|\theta\|_k \\
&+ \sum_{\ell\in(\mathbb{Z}\backslash\{0\}),\gamma\in(\mathbb{Z}^*_{n}\backslash\{0\})_u} \sum_{i\geq 0} |\lambda|^{-(i+1)} (\|A_2^{*i}\gamma\| + \frac{|\ell\cdot \gcd(\gamma_1,\gamma_2,...,\gamma_{n_2})|}{\|A_2^{*i}\gamma\|})^{-k} \|\theta\|_k \\
&\leq \sum_{\gamma\in(\mathbb{Z}^*_3\backslash\{0\})_u} \sum_{i\geq 0} |\lambda|^{-(i+1)} \|A_2^{*i}\gamma\|^{-k} \|\theta\|_k + C' \sum_{\ell\in(\mathbb{Z}\backslash\{0\}),\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_u} \sum_{i\geq 0} |\lambda|^{-(i+1)} \|A_2^{*i}\gamma\|^{-(k-4)} \ell^{-2} \|\theta\|_k \\
&\leq C \sum_{\ell\in(\mathbb{Z}\backslash\{0\}),\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_u} \sum_{i\geq 0} |\lambda|^{-(i+1)} \{\|A_2^{*i}\gamma_u\|^{-k} + \|A_2^{*i}\gamma_u\|^{-(k-4)}\ell^{-2}\} \|\theta\|_k \\
&\leq C \sum_{\ell\in(\mathbb{Z}\backslash\{0\}),\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_u} \sum_{i\geq 0} |\lambda|^{-(i+1)} \{(\rho^i\|\gamma_u\|)^{-k} + (\rho^i\|\gamma_u\|)^{-(k-4)}\ell^{-2}\} \|\theta\|_k \\
&\leq C \sum_{\ell\in(\mathbb{Z}\backslash\{0\}),\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_u} \sum_{i\geq 0} |\lambda|^{-(i+1)} \{(\rho^i\|\gamma\|)^{-k} + (\rho^i\|\gamma\|)^{-(k-4)}\ell^{-2}\} \|\theta\|_k \\
&< C\|\theta\|_k
\end{aligned}
$$

where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_{n_2})$ and $k > n_2 + 4$. In the computation above we parametrize $\pi \in (\Gamma\backslash N)^\wedge_{0,u}$ by $\{\mathbb{Z}^*_{n_2} \setminus \{0\}\}_u$ (a subset of $\{\mathbb{Z}^*_{n_2} \setminus \{0\}\}$) and $\ell \in \mathbb{Z}$, which naturally separates to $\ell \in (\mathbb{Z} \setminus \{0\})$ and $\ell = 0$ due to different approach to the estimate.

At the begining (from the first line to the second line in the computation), we make use of the inequality $\|f_\pi\|_r \leq C\|\pi\|^{-k}\|f\|_{r+k}$ for $r = 0$. And because $\gamma \hookrightarrow V^u$, we can interchange $\|\gamma\|$ and $\|\gamma_u\|$, as $\frac{1}{3}\|\gamma\| \leq \|\gamma_u\| \leq \|\gamma\|$. Moreover, since $\|A_2\gamma_u\| \geq \|\gamma_u\|, \|A_2\gamma_s\| \leq \|\gamma_s\|, \|A_2\gamma_c\| \approx \|\gamma_c\|$, so $\|A_2\gamma_u\| \geq \|A_2\gamma_s\|, \|A_2\gamma_u\| \geq \|A_2\gamma_c\|$, implying $\frac{1}{3}\|A_2\gamma\| \leq \|A_2\gamma_u\| \leq \|A_2\gamma\|$, and in general $\frac{1}{3}\|A_2^i\gamma\| \leq \|A_2^i\gamma_u\| \leq \|A_2^i\gamma\|$ for $i \geq 0$

Here gcd means the greatest common divisor, and $\gcd(\gamma_1, \gamma_2, ..., \gamma_n) \geq 2$ for $\pi \in (\Gamma\backslash G)^\wedge_0$, because

$\gamma_i \in 2\mathbb{Z}, i = 1, ..., n$ and see previous section 3.4.4.1.

The way we handle the estimate of $(\|A_2^{*i}\gamma\| + \frac{|\ell \cdot \gcd(\gamma_1, \gamma_2, ..., \gamma_n)|}{\|A_2^{*i}\gamma\|})^{-k}$ is using Binomial theorem to expand the $(x+y)^n$and keep only one term $C_n^i x^i y^{(n-i)}$ from the expansion.

The last step in our inequality can be verified using the integral test and the polar coordinate system instead of regular Cartesian coordinate system, see the following Lemma 4.1.5 for a detailed explanation. We will use the computation techniques illustrated here quite often.

For $\gamma \hookrightarrow V^s$, or $\|\gamma_s\| \geq \max\{\|\gamma_u\|, \|\gamma_c\|\}$, we have to use the other formal solution $\omega_{\pi-}$, and the computation is almost the same as for $\omega_{\pi+}$.

If $\gamma \hookrightarrow V^c$, Lemma 4.1.6 is needed to carry out the estimate, see Lemma 4.1.8 for a detailed elaboration.

Lemma 4.1.5. $\sum_{\gamma \in (\mathbb{Z}_n^* \setminus \{0\})} (\|\gamma\|)^{-k} < \infty$.

We use integral test and change of coordinate system to prove that the infinite sum involved in the estimate above is finite.

Proof.

$$\sum_{\gamma \in (\mathbb{Z}_n^* \setminus \{0\})} (\|\gamma\|)^{-k} \leq \int \cdots \int_{\sum_{i=1}^n x_i^2 \geq 1} \frac{1}{(\sum_{i=1}^n x_i^2)^{k/2}} dx_1 dx_2 ... dx_n$$

$$\leq \int_{r=1}^{\infty} \int_{\varphi_1=0}^{\pi} ... \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} \frac{r^{n-1}}{r^k} \sin^{n-2}(\varphi_1) ... \sin(\varphi_{n-2}) dr d\varphi_1 ... d\varphi_{n-1} < \infty$$

for $k > n$.

$\square$

Estimates of $\|\omega\|_r^{(u)}$ and $\|\omega\|_r^{(s)}$

The next step is to show that the C^0 solution is smooth and satisfies tame estimates. As we want to obtain a better estimate of $\sum_{i \geq 0} |\lambda|^{-(i+1)} \|\theta_{A^{*i}\pi} \circ A^i\|_r$, which might produce exponential growth in the Sobolev norm (L^2 norm of the derivatives of $\theta_{A^{*i}\pi} \circ A^i$ up to centain orders), the partial norms that were introduced earlier in 4.1.2 come into play.

$$\|\omega_\pi\|_r^{(*)} \leq \sum_{i \geq 0, (i \leq 0)} |\lambda|^{-(i+1)} \|\theta_{A^{*i}\pi} \circ A^i\|_r^{(*)} \leq \sum_{i \geq 0, (i \leq 0)} |\lambda|^{-(i+1)} \|A^{*i}\pi\|^{-k} \cdot \|\theta\|_{r+k} \|A^i\|_r^{(*)}$$

where $*$ stands for u, c, s. Roughly speaking, the idea is to apply partial norm as a bridge to achieve the following inequalities

$$\|\omega\|_r \leq \max\{\|\omega\|_r^{(s)}, \|\omega\|_r^{(c)}, \|\omega\|_r^{(u)}, \|\omega\|_{L^2}\} \leq C\|\theta\|_{r+k}.$$

For the stable directions

$$\|f_{A^{*n}\pi} \circ A^n\|_r^{(s)} \leq \|f_{A^{*n}\pi}\|_r \|A^{*n}\|_r^{(s)} \leq C\|A^n\|_r^{(s)} \|A^n\pi\|^{-k} \|f\|_{r+k}, n \geq 0$$

and the unstable directions

$$\|f_{A^{*n}\pi} \circ A^n\|_r^{(u)} \leq \|f_{A^{*n}\pi}\|_r \|A^{*n}\|_r^{(u)} \leq C\|A^{*n}\|_r^{(u)} \|A^{*n}\pi\|^{-k} \|f\|_{r+k}, n < 0$$

where $\|A^n\|_r^{(s)} = \|A^n|_{V^s}\|^r = \sup_{\{v \in V^s, \|v\|=1\}} \|A^n v\|^r < 1$ for $n \geq 0$ and $\|A^{*n}\|_r^{(u)} = \|A^{*n}|_{V^u}\|^r = \sup_{\{v \in V^u, \|v\|=1\}} \|A^n v\|^r < 1$ for $n < 0$. If r is chosen large enough such that $\rho^r |\lambda|^{-1} > 1$ where $\rho = \|A|_{V^u}\| > 1$ and with $\sum_{\pi \in (\Gamma \backslash N)_0^\wedge} \|\pi\|^{-k} < \infty$ for $k \geq (m+1) + ([\frac{m}{2}] + 1)$, $m = \dim N$, we can get the following:

$$
\begin{aligned}
&\|\omega_-\|_r^{(u)} \le \|\omega_{\pi_0}\|_r^{(u)} + \sum_{\pi \in (\Gamma\backslash N)_0^\wedge} \|\omega_{\pi -}\|_r^{(u)} \\
&\le \|\frac{\theta_{\pi_0}}{\lambda - 1}\|_r^{(u)} + \sum_{\pi \in (\Gamma\backslash N)_0^\wedge} \sum_{i<0} |\lambda|^{-(i+1)} \|\theta_{A^{*i}\pi} \circ A^i\|_r^{(u)} \\
&\le \|\frac{\theta_{\pi_0}}{\lambda - 1}\|_r + \sum_{\pi \in (\Gamma\backslash N)_0^\wedge} \sum_{i<0} |\lambda|^{-(i+1)} (\|A^i|_{V^u}\|^r) \|\theta_{A^{*i}\pi}\|_r^{(u)} \\
&\le \|\frac{\theta_{\pi_0}}{\lambda - 1}\|_r + \sum_{\pi \in (\Gamma\backslash N)_0^\wedge} \sum_{i<0} |\lambda|^{-(i+1)} \|A^i|_{V^u}\|^r \|\theta_{\pi}\|_r^{(u)} \\
&\le \|\frac{\theta_{\pi_0}}{\lambda - 1}\|_r + \sum_{\pi \in (\Gamma\backslash N)_0^\wedge} \sum_{i<0} |\lambda|^{-(i+1)} \|\rho^i\|^r \|\theta_{\pi}\|_r^{(u)} \\
&\le \|\frac{\theta_{\pi_0}}{\lambda - 1}\|_r + \sum_{\pi \in (\Gamma\backslash N)_0^\wedge} \sum_{i<0} |\lambda|^{-1} |\lambda^{-1}\rho^r|^i \|\theta_{\pi}\|_r^{(u)} \\
&\le C(\|\theta_{\pi_0}\|_r + \sum_{\pi \in (\Gamma\backslash N)_0^\wedge} \|\theta_\pi\|_r) \le C\|\theta_{\pi_0}\|_r + C' \sum_{\pi \in (\Gamma\backslash N)_0^\wedge} \|\pi\|^{-k} \|\theta\|_{r+k} \\
&< C\|\theta\|_{r+k}
\end{aligned}
$$

where $\omega_- = \sum_{\pi \in (\Gamma\backslash N)^\wedge} \omega_{\pi -}$.

As for $\|\omega\|_r^{(s)}$, we switch to the other formal solution and do the estimate in the partial norm

$\|\cdot\|_r^{(s)}$ to ω_+ with $\rho^{-1} = \|A|_{V^s}\| < 1$

$$
\begin{aligned}
\|\omega_+\|_r^{(s)} &\leq \|\omega_{\pi_0}\|_r^{(s)} + \sum_{\pi\in(\Gamma\backslash N)_0^\wedge} \|\omega_{\pi+}\|_r^{(s)} \\
&\leq \|\frac{\theta_{\pi_0}}{\lambda-1}\|_r^{(s)} + \sum_{\pi\in(\Gamma\backslash N)_0^\wedge}\sum_{i\geq 0} |\lambda|^{-(i+1)}\|\theta_{A^{*i}\pi}\circ A^i\|_r^{(s)} \\
&\leq \|\frac{\theta_{\pi_0}}{\lambda-1}\|_r + \sum_{\pi\in(\Gamma\backslash N)_0^\wedge}\sum_{i\geq 0} |\lambda|^{-(i+1)}(\|A^i|_{V^s}\|^r)\|\theta_{A^{*i}\pi}\|_r^{(s)} \\
&\leq \|\frac{\theta_{\pi_0}}{\lambda-1}\|_r + \sum_{\pi\in(\Gamma\backslash N)_0^\wedge}\sum_{i\geq 0} |\lambda|^{-(i+1)}\|A^i|_{V^s}\|^r\|\theta_{\pi}\|_r^{(s)} \\
&\leq \|\frac{\theta_{\pi_0}}{\lambda-1}\|_r + \sum_{\pi\in(\Gamma\backslash N)_0^\wedge}\sum_{i\geq 0} |\lambda|^{-(i+1)}\|\rho^{-i}\|^r\|\theta_{\pi}\|_r^{(u)} \\
&\leq C(\|\theta_{\pi_0}\|_r + \sum_{\pi\in(\Gamma\backslash N)_0^\wedge} \|\theta_\pi\|_r) \\
&\leq C\|\theta_{\pi_0}\|_r + C'\sum_{\pi\in(\Gamma\backslash N)_0^\wedge} \|\pi\|^{-k}\|\theta\|_{r+k} \\
&< C\|\theta\|_{r+k}
\end{aligned}
$$

where $\omega_+ = \sum_{\pi\in(\Gamma\backslash N)^\wedge}\omega_{\pi+}$.

Estimates of $\|\omega\|_r^{(c)}$

When it comes to $\|\omega\|_r^{(c)}$, we use a different approach by decomposing $(\Gamma\backslash N)_0^\wedge$ into three parts $(\Gamma\backslash N)_{0,u}^\wedge$, $(\Gamma\backslash N)_{0,c}^\wedge$ and $(\Gamma\backslash N)_{0,s}^\wedge$, where $(\Gamma\backslash N)_{0,u}^\wedge$ contains the elements $\pi\in(\Gamma\backslash N)_0^\wedge$ whose center component γ is dominated by γ_u, i.e. $(\|\gamma_u\| \geq \max\{\|\gamma_s\|, \|\gamma_c\|\})$, or write it as $\gamma \hookrightarrow V^u$, similarly $(\Gamma\backslash N)_{0,c}^\wedge$ and $\pi\in(\Gamma\backslash N)_{0,s}^\wedge$ contain the elements $\pi\in(\Gamma\backslash N)_0^\wedge$ whose γ is dominated by γ_c and γ_s respectively.

$$
\begin{aligned}
&\sum_{\pi\in(\Gamma\backslash N)^\wedge_{0,u}} \|\omega_{\pi+}\|^{(c)}_r \le \sum_{\pi\in(\Gamma\backslash N)^\wedge_{0,u}} \sum_{i\ge 0} |\lambda|^{-(i+1)} \|\theta_{A^{*i}\pi}\circ A^i\|^{(c)}_r \\
&\le \sum_{\pi\in(\Gamma\backslash N)^\wedge_{0,u}} \sum_{i\ge 0} |\lambda|^{-(i+1)} \|A^i|_{V^c}\|^r \|A^{*i}\pi\|^{-k}\|\theta\|_{r+k} \\
&\le C \sum_{\gamma\in(\mathbb{Z}^*_n\backslash\{0\})_u} \sum_{i\ge 0,\ell\in\mathbb{Z}} |\lambda|^{-(i+1)} (\|A_2^{*i}\gamma\| + \frac{|\ell\cdot \gcd(\gamma_1,\gamma_2,\ldots,\gamma_n)|}{\|A_2^{*i}\gamma\|})^{-k}\|\theta\|_{r+k} \\
&= \sum_{\ell=0,\gamma\in(\mathbb{Z}^*_n\backslash\{0\})_u} \sum_{i\ge 0} |\lambda|^{-(i+1)} (\|A_2^{*i}\gamma\|)^{-k}\|\theta\|_k \\
&+ \sum_{\ell\in(\mathbb{Z}\backslash\{0\}),\gamma\in(\mathbb{Z}^*_n\backslash\{0\})_u} \sum_{i\ge 0} |\lambda|^{-(i+1)} (\|A_2^{*i}\gamma\| + \frac{|\ell\cdot \gcd(\gamma_1,\gamma_2,\ldots,\gamma_n)|}{\|A_2^{*i}\gamma\|})^{-k}\|\theta\|_k \\
&\le \sum_{\gamma\in(\mathbb{Z}^*_n\backslash\{0\})_u} \sum_{i\ge 0} |\lambda|^{-(i+1)} (\|A_2^{*i}\gamma\|)^{-k}\|\theta\|_k + C' \sum_{\ell\in(\mathbb{Z}\backslash\{0\}),\gamma\in(\mathbb{Z}^*_n\backslash\{0\})_u} \sum_{i\ge 0} |\lambda|^{-(i+1)} (\|A_2^{*i}\gamma\|)^{-(k-4)}\ell^{-2}\|\theta\|_k \\
&\le C \sum_{\ell\in(\mathbb{Z}\backslash\{0\}),\gamma\in(\mathbb{Z}^*_n\backslash\{0\})_u} \sum_{i\ge 0} |\lambda|^{-(i+1)} \{(\|A_2^{*i}\gamma_u\|)^{-k} + (\|A_2^{*i}\gamma_u\|)^{-(k-4)}\ell^{-2}\}\|\theta\|_k \\
&\le C \sum_{\ell\in(\mathbb{Z}\backslash\{0\}),\gamma\in(\mathbb{Z}^*_n\backslash\{0\})_u} \sum_{i\ge 0} |\lambda|^{-(i+1)} \{(\rho^i\|\gamma_u\|)^{-k} + (\rho^i\|\gamma_u\|)^{-(k-4)}\ell^{-2}\}\|\theta\|_k \\
&\le C \sum_{\ell\in(\mathbb{Z}\backslash\{0\}),\gamma\in(\mathbb{Z}^*_n\backslash\{0\})_u} \sum_{i\ge 0} |\lambda|^{-(i+1)} \{(\rho^i\|\gamma\|)^{-k} + (\rho^i\|\gamma\|)^{-(k-4)}\ell^{-2}\}\|\theta\|_k \\
&< C\|\theta\|_k
\end{aligned}
$$

for $k > n_2 + 4$ by Lemma 4.1.5.

If γ_s dominates, we can use the other solution $\omega_{\pi-}$ to attain a similar result and the computation for $\gamma \hookrightarrow V^s$ is of no essential difference with respect to the one we just did.

If γ_c turns out to be the dominated term ($\gamma \in (\Gamma\backslash N)^\wedge_{0,c}$), we need the following Lemma.

Lemma 4.1.6 (Katznelson, [Kat71]). *Let A be a $r \times r$ matrix with integer coefficients. Assume that $\mathbb{R}^r$ splits as $\mathbb{R}^r = V \oplus V'$ with V, V' invariant under A and such that $A|_V, A|_{V'}$ have no common eigenvalues. If $V \cap \mathbb{Z}^r = \{0\}$, then there exists a constant τ such that $d(m, V) \ge \tau\|m\|^{-r}$*

for all $m \in \mathbb{Z}^r$, where $\|.\|$ is Euclidean norm and d is Euclidean distance.

For a detailed proof, see Lemma 4.1 in [DK10].

As a corollary of Katznelson Lemma, we have

Remark 4.1.7. *In particular, if A_2 is ergodic and $V = V^s \oplus V^c$ in Lemma 4.1.6, then $V \cap \mathbb{Z}^r = 0$. Therefore the above Lemma implies for $m \in \mathbb{Z}^r$*

$$\|\pi_1(m)\| \geq \tau \|m\|^{-r}$$

where $\pi_1(m)$ is the projection of m to V^u, the expanding subspace for A, and V^s, V^c are the contracting and neutral subspace respectively.

So no integer can stay mostly in the neutral direction for too long, after the time which is approximately $\ln |m|$, the expanding direction takes over. To be more precise,

$$\|A_2^i \gamma\| \geq \|A_2^i \gamma_u\| \geq C\rho^i \|\gamma_u\| \geq \tau C \rho^i \|\gamma\|^{-r} \geq \tau C \rho^{i-i_0} \|\gamma\|$$

for $i \geq i_0$ and $i_0 = [\frac{(1+r)\log\|\gamma\|}{\log \rho}] + 1$.

Lemma 4.1.8. *If γ_c is the dominated term $(\gamma \in (\Gamma \backslash N)^\wedge_{0,c})$, then*

$$\sum_{\pi \in (\Gamma \backslash G)^\wedge_{0,c}} \|\omega_{\pi+}\|_r^{(c)} \leq C \|\theta\|_{r+k}.$$

Proof.

$$\|\omega_{\pi+}\|_r^{(c)} \leq \sum_{i \geq 0}^{i_0 - 1} |\lambda|^{-(i+1)} \|A^i \pi\|^{-k} \|\theta\|_{r+k} + C \sum_{i=i_0}^{\infty} |\lambda|^{-(i+1)} \rho^{-(k-4)(i-i_0)} \|\gamma\|^{-(k-4)} \|\theta\|_{r+k}$$

and due to γ_c dominated and $|\lambda| \geq 1$, we have

$$
\begin{aligned}
&\sum_{\pi\in(\Gamma\backslash G)^\wedge_{0,c}} \|\omega_{\pi+}\|_r^{(c)} \leq \sum_{\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_c,\ell\in\mathbb{Z}\backslash\{0\}} \sum_{i\geq 0}^{i_0-1} |\lambda|^{-(i+1)}(C_1\|\gamma_c\| + \frac{|\ell\cdot\gcd(\gamma_1,\gamma_2,...,\gamma_{n_2})|}{C_2\|\gamma_c\|})^{-k}\|\theta\|_{r+r} \\
&+ \sum_{\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_c,\ell\in\mathbb{Z}\backslash\{0\}} \sum_{i=i_0}^{\infty} |\lambda|^{-(i+1)}(C_1\|A_2^i\gamma\| + \frac{|\ell\cdot\gcd(\gamma_1,\gamma_2,...,\gamma_{n_2})|}{C_2\|A_2^i\gamma\|})^{-k}\|\theta\|_{r+k} \\
&+ \sum_{\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_c,\ell=0} \sum_{i=0}^{\infty} |\lambda|^{-(i+1)}(C_1\|A_2^i\gamma\| + \frac{|\ell\cdot\gcd(\gamma_1,\gamma_2,...,\gamma_{n_2})|}{C_2\|A_2^i\gamma\|})^{-k}\|\theta\|_{r+k} \\
&\leq C \sum_{\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_c,\ell\in\mathbb{Z}\backslash\{0\}} |i_0|\|\gamma\|^{-(k-4)}(\ell\cdot\gcd|(\gamma_1,\gamma_2,...,\gamma_{n_2})|)^{-2}\|\theta\|_{k+r} \\
&+ C' \sum_{\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_c,\ell\in\mathbb{Z}\backslash\{0\}} \sum_{i=i_0}^{\infty} |\lambda|^{-(i+1)}\rho^{-(k-4)(i-i_0)}\|\gamma\|^{-(k-4)}(\ell\cdot\gcd|(\gamma_1,\gamma_2,...,\gamma_{n_2})|)^{-2}\|\theta\|_{r+k} \\
&+ C'' \sum_{\gamma\in(\mathbb{Z}^*_{n_2}\backslash\{0\})_c} (\sum_{i=0}^{i_0-1} |\lambda|^{-(i+1)}\|\gamma\|^{-k} + \sum_{i=i_0}^{\infty} |\lambda|^{-(i+1)}\rho^{-k(i-i_0)}\|\gamma\|^{-k})\|\theta\|_{r+k} \\
&< C\|\theta\|_{r+k}
\end{aligned}
$$

where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_{n_2})$ and $k > \frac{\log|\lambda|^{-1}}{\log\rho} + n_2 + 4$.

The first sum is bounded by the choice of i_0, $i_0 \sim \log\|\gamma\|$; the second part is finite due to the geometric series and integral test; the third one is convergent because of $i_0 \sim \log\|\gamma\|$ and the geometric series.

□

After these preparations, we are ready to estimate $\|\omega\|_r^{(c)}$

$$
\begin{aligned}
\|\omega\|_r^{(c)} &\leq \sum_{\pi\in(\Gamma\backslash N)^\wedge} \|\omega\|_r^{(c)} \\
&\leq \|\omega_{\pi_0}\|_r^{(c)} + \sum_{\pi\in(\Gamma\backslash N)^\wedge_{0,u}} \|\omega_\pi\|_r^{(c)} + \sum_{\pi\in(\Gamma\backslash N)^\wedge_{0,c}} \|\omega_\pi\|_r^{(c)} + \sum_{\pi\in(\Gamma\backslash N)^\wedge_{0,s}} \|\omega_\pi\|_r^{(c)} \\
&< C\|\theta\|_{r+k}.
\end{aligned}
$$

We have already pointed out that for $|\lambda| < 1$, the equation $\lambda\omega - \omega\circ A = \theta$ can be transformed to $\omega \circ A^{-1} - \lambda^{-1}\omega = \theta \circ A^{-1}$ for $|\lambda^{-1}| \geq 1$. Therefore, the estimate for $|\lambda| < 1$ follows using the other formal solution $\omega_{\pi-} = -\sum_{i=-1}^{-\infty} \lambda^{-(i+1)}\theta_{(A^*)^i\pi} \circ A^i$ and the fact that A^{-1} is an ergodic automorphism, thus going backward for sufficient time, the contracting direction takes over. We can use Lemma 4.1.6 for A^{-1} to attain a similar estimate.

In summary, we decompose Sobolev norm to L^2-norm, unstable, central and stable partial norms

$$\|\omega\|_r \approx \|\omega\|_{L^2} + \|\omega\|_r^{(u)} + \|\omega\|_r^{(c)} + \|\omega\|_r^{(s)}$$

where $\approx$ means two norms are equivalent. For the partial norm $\|\omega\|_r^{(u)}$ and $\|\omega\|_r^{(s)}$, the growth of A^n can be well controlled directly. As for the $\|\omega\|_r^{(c)}$, we need to decompose representations into unstable, central and stable parts and estimate each of them separately.

So we have shown obstructions vanishing is a necessary and sufficient condition for the corresponding twisted coboundary equation to have a smooth solution with tame estimates. The next stage is to use the higher rank condition to kill all the obstructions, we call this the higher rank trick.

4.2 Higher Rank Trick

If λ, μ are simple eigenvalues of A, B respectively, then we consider solving the following system

$$\begin{aligned} \lambda\omega - \omega \circ A &= f \\ \mu\omega - \omega \circ B &= g \end{aligned} \tag{4.12}$$

where f and g are Γ-periodic functions on N.

It is not difficult to see that if A and B commute and there exists a common solution to (4.12), then $(\lambda g - g \circ A) - (\mu f - f \circ B) = 0$.

Define

$$\triangle_A^\lambda(g) := \lambda g - g \circ A, \triangle_B^\mu(f) := \mu f - f \circ B$$

$$L(f, g) := \triangle_B^\mu(f) - \triangle_A^\lambda(g) = (\mu f - f \circ B) - (\lambda g - g \circ A).$$

We will see that $L(f,g) = 0$ is not only necessary but also sufficient for the existence of a solution to (4.12) under the higher rank condition that $A^\ell B^k$ is ergodic on $\Gamma\backslash N$ and the induced automorphism on $\Gamma\backslash[N,N]$ is ergodic for every non-trivial $(\ell,k) \neq (0,0)$.

Lemma 4.2.1. *If α is a higher-rank action and $L(f,g) = 0$, where f and g are Γ-periodic functions then the equations* (4.12)

$$\begin{aligned} \lambda\omega - \omega \circ A &= f \\ \mu\omega - \omega \circ B &= g \end{aligned} \tag{4.13}$$

have a common smooth solution satisfying

$$\|\omega\|_r \leq C_r \max\{\|f\|_{r+k}, \|g\|_{r+k}\} \text{ for } r \geq 0, k > n_2 + 4 \tag{4.14}$$

where $n_2 = \dim(\Gamma\backslash[N,N])$, the center dimension of the nilmanifold.

Proof. From $L^2(\Gamma\backslash N) = \sum_{\pi\in(\Gamma\backslash N)^\wedge} H_\pi$ and any $f \in L^2(\Gamma\backslash N)$, we have $f = \sum_{\pi\in(\Gamma\backslash N)^\wedge} f_\pi$ in the L^2 sense. If we put enough smoothness on f, the sum converges uniformly and absolutely, see [CG90b]. As discussed before, the obstruction $\vartheta^A_\pi(f) = \sum_{k=-\infty}^{+\infty} \lambda^{-(k+1)} f_{A^{*k}\pi} \circ A^k$ vanishing is equivalent to the coboundary equation having a solution with tame estimates, and we will see how does the condition $L(f,g) = 0$ kill the obstructions. An important observation in the proof is that $\triangle^\lambda_A(\vartheta^A_\pi(f)) = 0$.

We start from the condition $L(f,g) = 0$

$$\begin{aligned} \triangle^\lambda_A g &= \triangle^\mu_B f \\ \lambda g - g \circ A &= \mu f - f \circ B, \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \triangle^\lambda_A \omega &= f \\ \triangle^\mu_B \omega &= g. \end{aligned} \tag{4.16}$$

In order to show that the obstruction $\vartheta_\pi^B(g) = 0$ and obtain a C^∞ solution ω for the second coboundary equation $\triangle_B^\mu \omega = g$, we pass the equation $\triangle_A^\lambda g = \triangle_B^\mu f$ to the dual space

$$\begin{aligned}\sum^B (\triangle_B^\mu f)_\pi &= \sum^B (\triangle_A^\lambda g)_\pi \\ \sum^A (\triangle_A^\lambda g)_\pi &= \sum^A (\triangle_B^\mu f)_\pi\end{aligned} \tag{4.17}$$

where $\sum^A f_\pi := \vartheta_\pi^A(f) = \sum_{i=-\infty}^{+\infty} \lambda^{-(i+1)} f_{A^{*i}\pi} \circ A^i$, and similarly $\sum^B g_\pi := \vartheta_\pi^B(g) = \sum_{i=-\infty}^{+\infty} \mu^{-(i+1)} g_{B^{*i}\pi} \circ B^i$.

For the first equation in (4.17), we can switch the order of $\sum^B$ and $\triangle_B^\mu$ since both sums converge absolutely because f has enough smoothness.

$$\sum^B (\triangle_B^\mu f)_\pi = \triangle_B^\mu \sum^B f_\pi = \sum_{i\in\mathbb{Z}} \mu^{-i} f_{B^{*i}\pi} \circ B^i - \sum_{i\in\mathbb{Z}} \mu^{-(i+1)} f_{B^{*(i+1)}\pi} \circ B^i = 0$$

which implies

$$\sum^B (\triangle_A^\lambda g)_\pi = 0.$$

So, the obstruction for g is not only multiplied by μ under the action of B, but also multiplied by λ under the action of A, i.e. $\lambda \sum^B g_\pi = \sum^B g_{A^*\pi} \circ A$. Now apply iteration to the equation

$$\lambda g - g \circ A = \mu f - f \circ B$$

by composing A on both sides and again passing the equation to the dual space, we have

$$\sum^B (\lambda g \circ A - g \circ A^2)_\pi = \sum^B (\lambda g_{A^*\pi} \circ A - g_{A^{*2}\pi} \circ A^2) = 0$$

implying

$$\lambda^2 \sum^B g_\pi = \lambda \sum^B g_{A^*\pi} \circ A = \sum^B g_{A^{*2}\pi} \circ A^2.$$

Follow the same iterative procedure,

$$\lambda^k \sum^B g_\pi = \sum^B g_{A^{*k}\pi} \circ A^k$$

for any $k \in \mathbb{Z}$. Summing up all these equations together,

$$\sum_{k\in\mathbb{Z}} \lambda^k \sum^B g_\pi = \sum_{k\in\mathbb{Z}} \sum^B g_{A^{*k}\pi} \circ A^k. \tag{4.18}$$

The double sum on the left-hand side of (4.18) does not converge unless $\sum^B g_\pi = 0$ while the right-hand side of (4.18) converges absolutely because of the following computation when $|\mu| < 1$.

$$\sum_{\ell\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} \|\mu^{-(\ell+1)} g_{A^{*i}B^{*\ell}\pi} \circ (A^i B^\ell)\|_{L^2} \le \sum_{\ell\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} |\mu|^{-(\ell+1)} \|(A^*)^i (B^*)^\ell \pi\|^{-k} \|g\|_k$$

$$\le \sum_{i\in\mathbb{Z}} \sum_{\ell\ge 0} |\mu|^{-(\ell+1)} \|(A^*)^i (B^*)^\ell \pi\|^{-k} \|g\|_k + \sum_{i\in\mathbb{Z}} \sum_{\ell< 0} |\mu|^{-(\ell+1)} \|(A^*)^i (B^*)^\ell \pi\|^{-k} \|g\|_k.$$

The second part converges because of a result from [CG90b] that we can choose k_0 such that for $k > k_0$, $\sum_{\pi\in(\Gamma\backslash G)^\wedge_0} \|\pi\|^{-k} < \infty$. The first part is finite because of the following argument

$$\sum_{i\in\mathbb{Z}} \sum_{\ell\ge 0} |\mu|^{-(\ell+1)} \|(A^*)^i (B^*)^\ell \pi\|^{-k} \|g\|_k \le$$

$$\sum_{i\ge 0} \sum_{\ell\ge 0} |\mu|^{-(\ell+1)} (\rho_A^{-ik})(\rho_B^{-lk}) \|\gamma_{uu}\|^{-k} + \sum_{i\le 0} \sum_{\ell\ge 0} |\mu|^{-(\ell+1)} (\rho_A^{ik})(\rho_B^{-lk}) \|\gamma_{su}\|^{-k}$$

where $\gamma_{uu} \in V_A^u \cap V_B^u, \gamma_{su} \in V_A^s \cap V_B^u$, and k is properly selected such that $|\mu \rho_B^k| > 1$. We need that both $V_A^u \cap V_B^u$ and $V_A^s \cap V_B^u$ be non-trivial, and this can be achieved by a good choice of ergodic generators A and B, see Remark 4.2.3.

Therefore $\sum^B g_\pi = 0, \forall \pi \in (\Gamma\backslash N)^\wedge_0$. Similarly, $\sum^A f_\pi = 0, \forall \pi \in (\Gamma\backslash N)^\wedge_0$.

By Lemma 4.1.4, two formal solutions ω_- and ω_+ of each equation in (4.16) are C^∞ functions and they coincide.

If ω solves the second equation $\triangle_B^\mu \omega = g$, then

$$\triangle_A^\lambda \triangle_B^\mu = \triangle_A^\lambda g = \triangle_B^\mu f$$

$$\triangle_B^\mu(\triangle_A^\lambda \omega - f) = 0$$

because of the commutativity of operators $\triangle_A^\lambda$ and $\triangle_B^\mu$.

The ergodicity of A, B implies that $\triangle_A^\lambda, \triangle_B^\mu$ are injective operators on C^∞ functions, see Remark 4.2.2. Therefore $\triangle_A^\lambda \omega = f$, i.e. ω solves the first equation as well. $\square$

Remark 4.2.2. *To prove $\triangle_A^\lambda$ is injective on $C^\infty(\Gamma\backslash N)$, we show $\lambda f - f \circ A = 0$ implies $f = 0$. From $\lambda f = f \circ A$, we have $\lambda f_\pi = f_{A^*\pi} \circ A$. Apply iteration, $\lambda^i f = f_{A^{*i}\pi} \circ A^i, i \in \mathbb{Z}$. By summing up all these equations together, we obtain $\sum_{i\in\mathbb{Z}} \lambda^i f_\pi = \sum_{i\in\mathbb{Z}} f_{A^{*i}\pi} \circ A^i$. Since A is ergodic and f is C^∞, the right hand side converges in L^2 norm. On the other hand, the left hand site converges only if $f_\pi = 0$ for any $\pi \in (\Gamma\backslash N)_0^\wedge$. Since $\lambda \neq 1$, we also have $f_{\pi_0} = 0$ and conclude that $f = 0$. Therefore, $\triangle_A^\lambda$ is injective.*

In order to estimate the decay of $f_{A^{*i}B^{*\ell}\pi} \circ (A^i B^\ell)$, see the following Remark regarding the Lyapunov directions and a good choice of ergodic generators.

Remark 4.2.3. *We would like to point out that there is always a good choice of ergodic generators A, B that fit the computation purpose, that is, $V_A^u \cap V_B^u \neq \{0\}$ and $V_A^s \cap V_B^u \neq \{0\}$. Here we only consider the case of A, B being semisimple, i.e. no Jordan blocks will be involved, and A, B share the common neutral invariant subspaces, otherwise the action generated by A, B would be hyperbolic. The procedure can be broken down into two steps. Step 1: Choose A and B in distinct Weyl Chambers, such that neither $V_A^u = V_B^u, V_A^s = V_B^s$ nor $V_A^u = V_B^s, V_A^s = V_B^u$. Step 2: We will argue that either $V_A^u \cap V_B^u \neq \{0\}, V_A^s \cap V_B^u \neq \{0\}$ or $V_A^u \cap V_{B^{-1}}^u \neq \{0\}, V_A^s \cap V_{B^{-1}}^u \neq \{0\}$. Suppose $V_A^u \cap V_B^u = \{0\}$, then $V_A^u \subset V_B^s$ and at the same time, $V_A^s \cap V_B^s \neq \{0\}$, that is why $V_A^u \cap V_{B^{-1}}^u \neq \{0\}, V_A^s \cap V_{B^{-1}}^u \neq \{0\}$, and we can replace $\{A, B\}$ by $\{A, B^{-1}\}$.*

4.3 Proof of the Main Theorem

4.3.1 Infinitesimal Rigidity

We are ready to prove the infinitesimal rigidity and explain its connection with studying coboundary equations on nilmanifolds.

Let Λ be a finitely generated group, M a compact manifold and $\varphi : \Lambda \times M \to M$ a C^∞-action of Λ on M. There is a natural "formal tangent space" at the point $[\varphi]$ determined by the action φ, which is given by the 1-cocycles over φ with coefficients in the smooth vector fields on M. The 1-coboundaries form a closed subspace of the formal tangent space, and when these two spaces are equal, the action is said to be *infinitesimally rigid*. Let $H^1(\Lambda, \mathrm{Vect}^\infty(M)) = Z^1(\Lambda, \mathrm{Vect}^\infty(M))/B^1(\Lambda, \mathrm{Vect}^\infty(M))$, in other words, φ is said to be C^∞-infinitesimally rigid if $H^1(\Lambda, \mathrm{Vect}^\infty(M))$ is trivial. See 1.1 and [Hur95].

For the action $\alpha : \mathbb{Z}^2 \times (\Gamma\backslash N) \to (\Gamma\backslash N)$ and $R \in Z^1(\mathbb{Z}^2, \mathrm{Vect}^\infty(\Gamma\backslash N))$. Define

$$R_A := R((1,0)), R_B := R((0,1)). \tag{4.19}$$

If $\delta_v^2 R = 0$, then R_A, R_B satisfy the equation

$$L(R_A, R_B) \stackrel{def}{=} (BR_A - R_A \circ B) - (AR_B - R_B \circ A) = 0. \tag{4.20}$$

If R belongs to $B^1(\mathbb{Z}^2, \mathrm{Vect}^\infty(\Gamma\backslash N))$, then R_A, R_B satisfy

$$\begin{aligned} A\Omega - \Omega \circ A &= -R_A \\ B\Omega - \Omega \circ B &= -R_B \end{aligned} \tag{4.21}$$

where A and B are ergodic generators, $A = \alpha((1,0)), B = \alpha((0,1))$ such that $A^l B^k$ is ergodic for any non-zero $(\ell, k) \in \mathbb{Z}^2$.

If A and B commute and there exists a common solution to equations (4.21), it is immediate that $L(R_A, R_B) = \triangle^B R_A - \triangle^A R_B = 0$, where $\triangle^B R_A = BR_A - R_A \circ B, \triangle^A R_B = AR_B - R_B \circ A$.

So H^1 is trivial if $L(R_A, R_B) = 0$ implies the existence of Ω such that (4.21) hold.

Therefore the condition $L(R_A, R_B) = 0$ is not only necessary but also sufficient for the existence of a solution to (4.21) under the higher rank condition.

The system (4.21) splits further into several simpler systems through appropriate basis and the fact that A and B commute.

$$\begin{aligned} J_A\Omega - \Omega \circ A &= \Theta \\ J_B\Omega - \Omega \circ B &= \Psi \end{aligned} \tag{4.22}$$

where J_A is a matrix consisting of Jordan block of A corresponding to an eigenvalue of A, J_B is the corresponding block of B, and Θ and Ψ are vector valued Γ-periodic maps given by the maps R_A and R_B. $L(R_A, R_B) = 0$ splits as

$$J_A\Psi - \Psi \circ A = J_B\Theta - \Theta \circ B. \tag{4.23}$$

We have proved infinitesimal rigidity for semisimple case in Lemma 4.2.1, and the elaboration for general Jordan block will be explained in the end of this chapter.

4.3.2 Twisted Cohomology over a $\mathbb{Z}^k$ Action with Coefficients in $C^\infty(M)$

For a $\mathbb{Z}^2$ action

$$\rho : \mathbb{Z}^2 \to \mathrm{Diff}^\infty(M),$$

acting on $C^\infty(M)$ by $f \mapsto f \circ \rho(g)$.

We can introduce the cohomology sequence

$$C^0(\mathbb{Z}^2, C^\infty(M)) \xrightarrow{\delta^1} C^1(\mathbb{Z}^2, C^\infty(M)) \xrightarrow{\delta^2} C^2(\mathbb{Z}^2, C^\infty(M)) \tag{4.24}$$

where

$$\delta^1\omega \quad := \quad (\triangle_A^\lambda \omega, \triangle_B^\mu \omega) \tag{4.25}$$

$$\delta^2(f,h) \quad := \quad \triangle_A^\lambda h - \triangle_B^\mu \omega f \tag{4.26}$$

for $\lambda, \mu \in \mathbb{R}$ and $\triangle_A^\lambda \omega = \lambda\omega - \omega \circ A, \triangle_B^\mu \omega = \mu\omega - \omega \circ B$.

We have shown that $H_\varrho^1(\mathbb{Z}^2, C^\infty(M)) = 0$ in Lemma 4.2.1, and what remains to be proved is $H_\varrho^1(\mathbb{Z}^2, \mathrm{Vect}^\infty(M)) = 0$, and more generally, $H_\varrho^1(\mathbb{Z}^k, \mathrm{Vect}^\infty(M)) = 0$ for $k > 2$.

Recall some notations in $H_\varrho^1(\mathbb{Z}^k, \mathrm{Vect}^\infty(M))$. Let $\phi \in C^0(\mathbb{Z}^k, \mathrm{Vect}^\infty(M)) = \mathrm{Vect}^\infty(M), \beta \in C^1(\mathbb{Z}^k, \mathrm{Vect}^\infty(M))$ (maps from $\mathbb{Z}^k$ to $\mathrm{Vect}^\infty(M)$), and $\gamma \in C^2(\mathbb{Z}^k, \mathrm{Vect}^\infty(M))$ (maps from $\mathbb{Z}^k \times \mathbb{Z}^k$ to $\mathrm{Vect}^\infty(M)$). Coboundary operators are defined as

$$\delta_v^1\phi(g) \quad := \quad \varrho(g)_*\phi - \phi \tag{4.27}$$

$$\delta_v^2\beta(g_1, g_2) \quad := \quad (\varrho(g_2)_*\beta(g_1) - \beta(g_1)) - (\varrho(g_1)_*\beta(g_2) - \beta(g_2)). \tag{4.28}$$

For the first equation we have:

$$(DA) \circ \phi \circ A^{-1} - \phi = \Theta$$

$$(DB) \circ \phi \circ B^{-1} - \phi = \Psi$$

this can be converted to

$$(DA) \circ \phi - \phi \circ A = \Theta \circ A$$

$$(DB) \circ \phi - \phi \circ B = \Psi \circ B.$$

For a frame $Y_1, ..., Y_n \in \mathrm{Vect}^\infty(\Gamma \backslash N)$ and $\phi(x) = \sum_{i=1}^n \omega_i(x)Y_i$, the first coboundary equation $\delta_v^1\phi(g) = \varrho(g)_*\alpha - \alpha$ can be written as

$$(DA_g)[\sum_{i=1}^n \omega_i(\varrho(g^{-1})(x))Y_i] - \sum_{i=1}^n \omega_i(x)Y_i = \sum_{i=1}^n \theta_i(x)Y_i \tag{4.29}$$

where DA_g is the matrix representation of $D(\varrho(g))$ under the basis $Y_1, ..., Y_n \in \mathfrak{n}$. If $D(\varrho(g))$ is diagonalizable, and for convenience, we assume it is the same basis $Y_i \in \mathrm{Vect}^\infty(\Gamma\backslash N), i = 1, ..., n$ such that $(DA_g)Y_k = \lambda_k Y_k, k = 1, ..., n$. Then the system (4.29) can be reduced to a family of coboundary equations, which have been solved in Theorem 4.1.4. That is why we are interested in solving a single coboundary equation

$$\lambda\omega - \omega \circ A_g = \theta$$

where $A_g = \varrho(g)$ for one $g \in \mathbb{Z}^k$. Similarly, we can interpret the second coboundary equation $\delta_v^2\beta(g_1, g_2) = (\varrho(g_2)_*\beta(g_1) - \beta(g_1)) - (\varrho(g_1)_*\beta(g_2) - \beta(g_2))$ in the same basis $Y_1, ..., Y_n \in \mathfrak{n}$ such that $\beta(g_1) = \sum_{i=1}^n f_i Y_i, \beta(g_2) = \sum_{i=1}^n h_i Y_i$ for C^∞-functions $f_i, h_i : \Gamma\backslash N \to \mathbb{R}$.

$$\{(DB)[\sum_{i=1}^n f_i(\varrho(g_2^{-1}))Y_i] - \sum_{i=1}^n f_i Y_i\} - \{(DA)[\sum_{i=1}^n h_i(\varrho(g_1^{-1}))Y_i] - \sum_{i=1}^n h_i Y_i\} = \sum_{i=1}^n \varphi_i Y_i \qquad (4.30)$$

where DA is the matrix representation of $D(\varrho(g_1))$, DB is the matrix representation of $D(\varrho(g_2))$ in the basis $Y_1, ..., Y_n \in \mathfrak{g}$, $A = \varrho(g_1), B = \varrho(g_2)$ and $\beta(g_1)(x) = \sum_{i=1}^n f_i(x)Y_i(x), \beta(g_2)(x) = \sum_{i=1}^n h_i(x)Y_i(x)$. If DA, DB are semisimple, then there exist $Y_k \in \mathrm{Vect}^\infty(\Gamma\backslash N), k = 1, ..., n$ such that $(DA)Y_k = \lambda_k Y_k, (DB)Y_k = \mu_k Y_k, k = 1, ..., n$, and the question can be reduced to

$$(\mu f \circ B^{-1} - f) - (\lambda h \circ A^{-1} - h) = \varphi$$

which is equivalent to

$$(\mu f \circ A - f \circ A \circ B) - (\lambda h \circ B - h \circ B \circ A) = \varphi \circ A \circ B.$$

If we set $f_A = f \circ A, h_B = h \circ B, \varphi' = \varphi \circ A \circ B$, then the equation becomes $(\mu f_A - f_A \circ B) - (\lambda h_B - h_B \circ A) = \varphi'$. Here we consider $\varphi = 0$, so the equation is

$$(\mu f_A - f_A \circ B) - (\lambda h_B - h_B \circ A) = 0.$$

We can refer to Lemma 4.2.1 and conclude that $H^1_\varrho(\mathbb{Z}^k, \mathrm{Vect}^\infty(M)) = 0$.

Suppose there are some Jordan blocks in the matrix representations DA and DB, i.e., DA and DB are not simultaneously diagonalizable (We repeat the calculation with Jordan blocks here for the sake of completeness, see [DK10]). Choose a basis in which DA is in its Jordan form, then in the same basis DB has block diagonal form. We can take $m \times m$ blocks J_A and J_B corresponding to eigenvalues λ and μ of DA and DB, respectively and (4.30) splits into equation

$$(J_B F \circ B^{-1} - F) - (J_A H \circ A^{-1} - H) = 0$$

which is equivalent to

$$(J_B F' - F' \circ B) - (J_A H' - H' \circ A) = 0 \tag{4.31}$$

where $F' = F \circ A, H' = H \circ B$. Let $J_A = (a_{ij}), a_{ii} = \lambda$ for $i = 1, ..., m$ and $a_{i,i+1} = *_i \in \{0, 1\}$ for $i = 1, ..., m-1$ and $J_B = (b_{ij})$ where $b_{ii} = \mu$ for $i = 1, ..., m$ and $b_{i,i+1} = *_i \in \{0, 1\}$ (λ is an eigenvalue of A, μ is an eigenvalue of B). Equation (4.31) splits into m equations and we call them $(EQ)_k$.

$$(\triangle_B^\mu f_k + \sum_{i=k+1}^{m} b_{ki} f_i) - (\triangle_A^\lambda h_k + *_k h_{k+1}) = 0 \tag{4.32}$$

where $k = 1, ..., m$ and f_i, h_i are coordinate functions of F, H respectively. For $k = m$ the equation $(EQ)_m$ becomes

$$\triangle_B^\mu f_m - \triangle_A^\lambda h_m = 0.$$

Since $L(f_m, h_m) = 0$, there exists ω_m which solves simultaneously the last of m pairs of equations in (4.22), namely the equations $\triangle_A^\lambda \omega_m = f_m$ and $\triangle_B^\mu \omega_m = h_m$. Now the $(m-1)$-st part of equations in (4.22) is

$$\begin{aligned} \triangle_A^\lambda \omega_{m-1} + *_{m-1} \omega_m &= f_{m-1} \\ \triangle_B^\mu \omega + b_{m-1,m} \omega_m &= h_{m-1}, \end{aligned} \tag{4.33}$$

while the cocycle condition for f_{m-1} and h_{m-1} is

$$\triangle_B^\mu f_{m-1} + b_{m-1,m} f_{m-1} = \triangle_A^\lambda h_{m-1} + *_{m-1} h_{m-1}. \tag{4.34}$$

By substituting $f_m = \triangle_A^\lambda \omega_m$ and $h_m = \triangle_B^\mu \omega_m$ into (4.34), we obtain that

$$L(f_{m-1} - *_{m-1}\omega_m, h_{m-1} - b_{m-1,m}\omega_m) = 0$$

which allows us to conclude that there exists some ω_{m-1} solving the system (4.33).

Now we proceed by induction. Fix k between 1 and $m-2$ and assume that for all $i \geq k$, we have obtained a solution ω_i, i.e., for every $i = k+1, ..., m$ we have a C^∞ function ω_i which solves the i-th pair of equations of (4.22):

$$\begin{aligned} \triangle_A^\lambda \omega_i + *_i \omega_{i+1} &= f_i \\ \triangle_B^\mu \omega_i + \sum_{\ell=i+1}^{m} b_{i\ell}\omega_\ell &= h_i. \end{aligned} \tag{4.35}$$

We wish to find ω_k that solves the k-th pair of equations in (4.22):

$$\begin{aligned} \triangle_A^\lambda \omega_i + *_i \omega_{i+1} &= f_i \\ \triangle_B^\mu \omega_i + \sum_{\ell=i+1}^{m} b_{i\ell}\omega_\ell &= h_i \end{aligned} \tag{4.36}$$

providing that the k-th equation in (4.23) is satisfied by f_k and h_k; i.e.,

$$\triangle_B^\mu f_k + \sum_{i=k+1}^{m} b_{ki} f_i = \triangle_A^\lambda h_k + *_k h_{k+1}. \tag{4.37}$$

Now we use the fact that all the subsequent pairs of equations are sloved; i.e., we substitute all f_i for $i = k+1, ..., m$ and the h_{k+1} into (4.37) using their expression as in (4.35). This implies

$$\triangle_B^\mu f_k + \sum_{i=k+1}^{m} (b_{ki}\triangle_A^\lambda \omega_i + *_i b_{ki}\omega_{i+1}) = \triangle_A^\lambda h_k + *_k \triangle_B^\mu \omega_{k+1} + \sum_{i=k+1}^{m} b_{k+1,i+1}\omega_{i+1}.$$

Since A and B commute, by simply comparing coefficients, we have

$$*_i b_{ki} = *_k b_{k+1,i+1}$$

for any fixed k between 1 and $m-1$ and for all $i = k+1, ..., m-1$. Together with the linearity of operators $\triangle^\lambda$ and $\triangle^\mu$ to simplify the above expression to

$$\triangle_B^\mu (f_k - *_k \omega_{k+1}) = \triangle_A^\lambda (h_k - \sum_{i=k+1}^{m} b_{ki}\omega_i).$$

Thus the functions $f_k - *_k\omega_{k+1}$ and $h_k - \sum_{i=k+1}^m b_{ki}\omega_i$ satisfy the solvability condition $L(f_k - *_k\omega_{k+1}, h_k - \sum_{i=k+1}^m b_{ki}\omega_i) = 0$ and again we can conclude that pair of equations (4.37) has a common C^∞ solution ω_k.

Since k is an arbitrary integer between 1 and $m-1$, it follows that there exists a solution Ω to (4.22) providing the condition (4.22) is satisfied. This can be repeated for all corresponding blocks of A and B. Therefore, we prove that the infinitesimal rigidity for higher rank partially hyperbolic actions on a family of 2-step nilmanifolds. When it comes to tame estimates for vector fields, there may be more loss if Jordan blocks are involved.

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