# Using results from dynamical systems to classify algebras and $C^{*}$-algebras. 

Mark Tomforde

UH Dynamics Summer School

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The Landscape of Modern Mathematics
Today I want to tell you about some interactions among the subjects of Dynamical Systems, Algebra, and Functional Analysis,

The common connection among these subjects will be directed graphs.

Dynamical Systems: Shift Spaces
Shifts of finite type may be considered as shift spaces coming from graphs.

Algebra: Algebras over a Field
Leavitt path algebras are algebras constructed from directed graphs.

Functional Analysis: $C^{*}$-algebras
Graph $C^{*}$-algebras are $C^{*}$-algebras constructed from directed graphs.

## Dynamical Systems (Shift Spaces)

Begin with a finite set of symbols $\mathcal{A}:=\{1,2, \ldots, n\}$. Form the set of all infinite sequences

$$
\mathcal{A}^{\mathbb{N}}:=\left\{x_{1} x_{2} x_{3} \ldots \mid x_{i} \in \mathcal{A}\right\}
$$

and all bi-infinite sequences

$$
\mathcal{A}^{\mathbb{Z}}:=\left\{\ldots x_{-2} x_{-1} \cdot x_{0} x_{1} x_{2} \ldots \mid x_{i} \in \mathcal{A}\right\} .
$$

We have a one-sided shift map $\sigma: \mathcal{A}^{\mathbb{N}} \rightarrow \mathcal{A}^{\mathbb{N}}$ given by

$$
\sigma\left(x_{1} x_{2} x_{3} \ldots\right)=x_{2} x_{3} x_{4} \ldots
$$

and a two-sided shift map $\bar{\sigma}: \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ given by

$$
\bar{\sigma}\left(\ldots x_{-2} x_{-1} \cdot x_{0} x_{1} x_{2} \ldots\right)=\ldots x_{-1} x_{0} \cdot x_{1} x_{2} x_{3} \ldots
$$

$\left(\mathcal{A}^{\mathbb{N}}, \sigma\right)$ is the full one-sided shift $\left(\mathcal{A}^{\mathbb{Z}}, \bar{\sigma}\right)$ is the full two-sided shift

Give $\mathcal{A}:=\{1, \ldots, n\}$ the discrete topology.
If we give $\mathcal{A}^{\mathbb{N}}$ the product topology, then $\mathcal{A}^{\mathbb{N}}$ has a basis of cylinder sets of the form

$$
\left[a_{1} \ldots a_{n}\right]:=\left\{x_{1} x_{2} x_{3} \ldots \in \mathcal{A}^{\mathbb{N}}: x_{1}=a_{1}, \ldots, x_{n}=a_{n}\right\}
$$

and $\mathcal{A}^{\mathbb{N}}$ is compact by Tychonoff's theorem. Moreover, $\sigma: \mathcal{A}^{\mathbb{N}} \rightarrow \mathcal{A}^{\mathbb{N}}$ is continuous map (in fact, a local homeomorphism). Thus $\left(\mathcal{A}^{\mathbb{N}}, \sigma\right)$ is a discrete dynamical system.

Similarly, if we give $\mathcal{A}^{\mathbb{Z}}$ the product topology, then $\mathcal{A}^{\mathbb{Z}}$ has a basis of cylinder sets of the form
$\left[a_{1} \ldots a_{n}\right]_{t}:=\left\{\ldots x_{-1} \cdot x_{0} x_{1} \ldots \in \mathcal{A}^{\mathbb{Z}}: x_{t+1}=a_{1}, x_{t+2}=a_{2}, \ldots, x_{t+n}=a_{n}\right\}$
and $\mathcal{A}^{\mathbb{Z}}$ is compact by Tychonoff's theorem. Moreover, $\bar{\sigma}: \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ is homeomorphism. Thus $\left(\mathcal{A}^{\mathbb{Z}}, \bar{\sigma}\right)$ is a discrete dynamical system.

Topology Fun Fact: The cylinder sets are clopen. Both $\mathcal{A}^{\mathbb{N}}$ and $A^{\mathbb{Z}}$ are perfect, compact, Hausdorff, and have countable basis of clopen sets. Thus they are each homeomorphic to the Cantor set.

We seek closed subsets $X \subseteq A^{\mathbb{N}}$ with $\sigma(X)=X$. Then $\left(X,\left.\sigma\right|_{X}\right)$ is a sub-system of $\left(\mathcal{A}^{\mathbb{N}}, \sigma\right)$. We call such $\left(X,\left.\sigma\right|_{X}\right)$ a one-sided shift space.

Likewise, we seek closed subsets $X \subseteq A^{\mathbb{Z}}$ with $\bar{\sigma}(X)=X$. Then $(X, \bar{\sigma} \mid x)$ is a sub-system of $\left(\mathcal{A}^{\mathbb{N}}, \sigma\right)$. We call such a $\left(X,\left.\bar{\sigma}\right|_{X}\right)$ a two-sided shift space.

Let $\mathcal{F}$ be a set of finite sequences of elements from $\{1, \ldots, n\}$. Define $X_{\mathcal{F}}:=\left\{x_{1} x_{2} \ldots \in \mathcal{A}^{\mathbb{N}}:\right.$ no sub-block $x_{k} \ldots x_{k+n}$ is in $\mathcal{F}$ for any $\left.k, n\right\}$ $\bar{X}_{\mathcal{F}}:=\left\{\ldots x_{-1} \cdot x_{0} x_{1} \ldots \in \mathcal{A}^{\mathbb{Z}}:\right.$ no sub-block $x_{k} \ldots x_{k+n}$ is in $\mathcal{F}$ for any $\left.k, n\right\}$ We call $\mathcal{F}$ the forbidden blocks.

## Theorem

$A$ set $X \subseteq \mathcal{A}^{\mathbb{N}}$ is a one-sided shift space iff $X=X_{\mathcal{F}}$ for some set $\mathcal{F}$. (We call $X$ a shift of finite type if $\mathcal{F}$ can be chosen to be a finite set.)

## Theorem

$A$ set $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a two-sided shift space iff $X=\bar{X}_{\mathcal{F}}$ for some set $\mathcal{F}$. (We call $X$ a shift of finite type if $\mathcal{F}$ can be chosen to be a finite set.)

Let $\mathcal{A}=\{0,1\}$.
Example 1: (The Golden Mean Shift)
Let $\mathcal{F}=\{11\}$.
Then $X_{\mathcal{F}}$ is all sequences in $\mathcal{A}^{\mathbb{N}}$ where no consecutive 1 's occur.

Example 2: (The Even Shift)
Let $\mathcal{F}=\{101,10001,1000001, \ldots\}=\left\{10^{2 n+1} 1: n \in \mathbb{N} \cup\{0\}\right\}$.
Then $X_{\mathcal{F}}$ is all sequences in $\mathcal{A}^{\mathbb{N}}$ where there are an even number of 0 's between any two 1 's.

Example 3:
Let $\mathcal{F}=\{10,100,1000, \ldots\}=\left\{10^{n}: n \in \mathbb{N}\right\}$.
Then $X_{\mathcal{F}}$ is all sequences in $\mathcal{A}^{\mathbb{N}}$ where a 0 does not follow a 1 .
Example 1 gives a shift of finite type.
Example 3 also gives shift of finite type (use $\mathcal{F}=\{10\}$ ).
Example 2 is not a shift of finite type.

## Isomorphism of Shift Spaces

## Definition

If $X$ and $Y$ are one-sided shifts of finite type, we say $X$ is conjugate to $Y$ if there is a homeomorphism $\phi: X \rightarrow Y$ such that $\sigma \circ \phi=\phi \circ \sigma$.

## Definition <br> If $X$ and $Y$ are two-sided shifts of finite type, we say $X$ is conjugate to $Y$ if there is a homeomorphism $\phi: X \rightarrow Y$ such that $\bar{\sigma} \circ \phi=\phi \circ \bar{\sigma}$.

Shifts of finite type may be described (up to conjugacy) using graphs.

## Graphs

A (directed) graph $E=\left(E^{0}, E^{1}, r, s\right)$ consists of a set of vertices $E^{0}$, a set of edges $E^{1}$, and maps $r: E^{1} \rightarrow E^{0}$ and $s: E^{1} \rightarrow E^{0}$ identifying the range and source of each edge.


$$
\begin{aligned}
& E^{0}=\{v, w, x\} \\
& E^{1}=\{a, b, c, d, e, f, g, h\} \\
& s(e)=w \text { and } r(e)=x \\
& s(f)=x \text { and } r(f)=x
\end{aligned}
$$

For now, we'll assume our graphs are finite (i.e., $E_{\square}^{0}$ and $E^{1}$ are finite sets),

## Edge Shifts of Graphs

If $E=\left(E^{0}, E^{1}, r, s\right)$ is a graph, we define the one-sided edge shift

$$
X_{E}:=\left\{e_{1} e_{2} e_{3} \ldots: e_{i} \in E^{1} \text { and } r\left(e_{i}\right)=s\left(e_{i+1}\right) \text { for all } i \in \mathbb{N}\right\}
$$

and the two-sided edge shift

$$
\bar{X}_{E}:=\left\{\ldots e_{-1} \cdot e_{0} e_{1} \ldots: e_{i} \in E^{1} \text { and } r\left(e_{i}\right)=s\left(e_{i+1}\right) \text { for all } i \in \mathbb{Z}\right\}
$$

## Theorem

A one-sided shift $X$ is a shift of finite type if and only if there exists a graph $E$ such that $X$ is conjugate to the edge shift $X_{E}$.

## Theorem

A two-sided shift $X$ is a shift of finite type if and only if there exists a graph $E$ such that $X$ is conjugate to the edge shift $\bar{X}_{E}$.

## Algebras of Graphs

If $K$ is a field, a $K$-algebra is a vector space over $K$ with a product that is associative and K-bilinear (i.e., distributive and scalars pull out).

Two $K$-algebras $A$ and $B$ are isomorphic if there is a bijection $\phi: A \rightarrow B$ that is $K$-linear and multiplicative.

## Definition (Leavitt path algebra)

If $E=\left(E^{0}, E^{1}, r, s\right)$ is a finite graph with no sinks and $K$ is a field, we define the Leavitt path algebra $L_{K}(E)$ to be the universal algebra generated by elements $\left\{p_{v}: v \in E^{0}\right\} \cup\left\{s_{e}, s_{e}^{*}: e \in E^{1}\right\}$ satisfying the following relations:
(1) $p_{v} p_{w}=0$ when $v \neq w$, and $p_{v}^{2}=p_{v}$ for all $v \in E^{0}$.
(2) $s_{e}^{*} s_{f}=0$ when $e \neq f$ and $s_{e}^{*} s_{e}=p_{r(e)}$ for all $e \in E^{1}$.
(3) $s_{e}=s_{e} p_{r(e)}=p_{s(e)} s_{e}$ and $s_{e}^{*}=s_{e}^{*} p_{s(e)}=p_{r(e)} s_{e}^{*}$ for all $e \in E^{1}$.
(9) $p_{v}=\sum_{s(e)=v} s_{e} s_{e}^{*}$ for all $v \in E^{0}$.

## $C^{*}$-algebras of Graphs

$\mathcal{H}$ is separable infinite-dimensional Hilbert space.
$B(\mathcal{H})=\{T: \mathcal{H} \rightarrow \mathcal{H}:\|T\|<\infty\}$
$B(\mathcal{H})$ is a $\mathbb{C}$-algebra, but it also has the operator norm $\|\cdot\|$, and in addition there is an adjoint operation $*$ on $B(\mathcal{H})$ : If $T \in B(\mathcal{H})$ there exists a unique $T^{*} \in B(\mathcal{H})$ such that $\langle T x, y\rangle=\left\langle x, T^{*} y\right\rangle$ for all $x, y \in \mathcal{H}$.

An operator algebra is a subalgebra of $B(\mathcal{H})$ that is closed in the topology coming from $\|\cdot\|$.

A $C^{*}$-algebra is a subalgebra of $B(\mathcal{H})$ that is closed in the topology coming from $\|\cdot\|$ and is closed under the *-operation.

Two $C^{*}$-algebras $A$ and $B$ are $*$-isomorphic if there is a bijection $\phi: A \rightarrow B$ that is $\mathbb{C}$-linear, multiplicative, and $\phi\left(a^{*}\right)=\phi(a)^{*}$ for all $a \in A$.

## C*-algebras of Graphs

## Definition (Graph $C^{*}$-algebra)

If $E=\left(E^{0}, E^{1}, r, s\right)$ is a finite graph with no sinks, we define the graph $C^{*}$-algebra $C^{*}(E)$ to be the universal $C^{*}$-algebra generated by elements $\left\{p_{v}: v \in E^{0}\right\} \cup\left\{s_{e}: e \in E^{1}\right\}$ satisfying the following relations:
(1) $p_{v} p_{w}=0$ when $v \neq w$, and $p_{v}^{*}=p_{v}^{2}=p_{v}$ for all $v \in E^{0}$.
(2) $s_{e}^{*} s_{f}=0$ when $e \neq f$ and $s_{e}^{*} s_{e}=p_{r(e)}$ for all $e \in E^{1}$.
(3) $s_{e}=s_{e} p_{r(e)}=p_{s(e)} s_{e}$ for all $e \in E^{1}$.
(9) $p_{v}=\sum_{s(e)=v} s_{e} s_{e}^{*}$ for all $v \in E^{0}$.

It turns out $L_{\mathbb{C}}(E) \subseteq C^{*}(E)$ and $\overline{L_{\mathbb{C}}(E)}=C^{*}(E)$.

Graph $C^{*}$-algebras are also sometimes called "Cuntz-Krieger algebras" (especially when the graph is finite).

## Sorting It All Out

If $E$ is a finite graph with no sinks, we have various objects and notions of equivalence.

- one-sided edge shift $X_{E}$ (one-sided conjugacy)
- two-sided shifts edge $\bar{X}_{E}$ (two-sided conjugacy)
- Leavitt path algebra $L_{K}(E)$ (isomorphism)
- graph $C^{*}$-algebra $C^{*}(E)$ (*-isomorphism)

Question 1: What are the relationships among these various notions?
It is often difficult to determine when two Leavitt path algebras are isomorphic, or when two graph $C^{*}$-algebras are $*$-isomorphic. However, there are well-known theorems describing conjugacy for one-sided and two-sided shifts of finite type.
Question 2: Can the shift spaces help us to determine when two Leavitt path algebras are isomorphic?
Question 3: Can the shift spaces help us to determine when two graph $C^{*}$-algebras are $*$-isomorphic?

## Irreducible Shifts

A shift space $X$ is irreducible if whenever $u$ and $w$ are allowed blocks, there is a "connecting block" $v$ such that $u v w$ is allowed. Equivalently, there is point $x \in X$ whose forward orbit (i.e., $\left\{x, \sigma(x), \sigma^{2}(x), \ldots\right\}$ in the one-sided case, or $\left\{x, \bar{\sigma}(x), \bar{\sigma}^{2}(x), \ldots\right\}$ in the two-sided case) is dense in $X$.

A finite graph graph with no sinks is called irreducible if there is a path from each vertex to every other vertex and the graph does not consist of a single cycle.

## Theorem

Let $E$ be a finite graph with no sinks. Then the following are equivalent.

- $E$ is irreducible.
- $X_{E}$ is irreducible.
- $\bar{X}_{E}$ is irreducible.
- $L_{K}(E)$ is simple for every field $K$ (simple $=$ no two-sided ideals)
- $C^{*}(E)$ is simple (simple $=$ no closed two-sided ideals)


## Move (O): Outsplitting



The inverse operation is called Outamalgamation.

Theorem (conjugacy for one-sided irreducible shifts of finite type)
Let $E$ and $F$ be finite irreducible graphs. Then the one-sided shifts $X_{E}$ and $X_{F}$ are conjugate if and only if $E$ can be turned into $F$ through a finite number of outsplittings and outamalgamations.

One can prove that if $E$ is a graph and $E^{\prime}$ is formed by performing an outsplitting to $E^{\prime}$, then $L_{K}(E)$ is isomorphic to $L_{K}\left(E^{\prime}\right)$ for all fields $K$ and $C^{*}(E)$ is $*$-isomorphic to $C^{*}\left(E^{\prime}\right)$. Thus, we get the following . . .

## Theorem

Let $E$ and $F$ be finite irreducible graphs. If the one-sided shifts $X_{E}$ and $X_{F}$ are conjugate, then $L_{K}(E)$ is isomorphic to $L_{K}(F)$ for every field $K$.

## Theorem

Let $E$ and $F$ be finite irreducible graphs. If the one-sided shifts $X_{E}$ and $X_{F}$ are conjugate, then $C^{*}(E)$ is *-isomorphic to $C^{*}(F)$.

Unfortunately, neither converse holds.

When we perform outsplittings on a graph, the isomorphism of the Leavitt path algebras and $*$-isomorphism of the graph $C^{*}$-algebras is of a particular type.

If $E$ is a graph, and $K$ is a field, then inside $L_{K}(E)$ we have a subalgebra

$$
D_{E}:=\operatorname{span}_{K}\left\{s_{e_{1}} \ldots s_{e_{n}} s_{e_{n}}^{*} \ldots s_{e_{1}}^{*}: n \in \mathbb{N} \text { and } e_{1}, \ldots, e_{n} \in E^{1}\right\}
$$

called the Cartan subalgebra of $L_{K}(E)$.
Also, inside $C^{*}(E)$ we have a closed subalgebra

$$
\bar{D}_{E}:=\overline{\operatorname{span}}_{\mathbb{C}}\left\{s_{e_{1}} \ldots s_{e_{n}} s_{e_{n}}^{*} \ldots s_{e_{1}}^{*}: n \in \mathbb{N} \text { and } e_{1}, \ldots, e_{n} \in E^{1}\right\}
$$

called the Cartan subalgebra of $C^{*}(E)$.
When we outsplit a graph the isomorphism (respectively *-isomorphism) obtained between the associated Leavitt path algebras (respectively, graph $C^{*}$-algebra) preserves the Cartan subalgebras.

Theorem (conjugacy for one-sided irreducible shifts of finite type)
Let $E$ and $F$ be finite irreducible graphs. Then the one-sided shifts $X_{E}$ and $X_{F}$ are conjugate if and only if $E$ can be turned into $F$ through a finite number of outsplittings and outamalgamations.

Some better theorems for algebras and $C^{*}$-algebras . . .

## Theorem

Let $E$ and $F$ be finite irreducible graphs. If the one-sided shifts $X_{E}$ and $X_{F}$ are conjugate, then for any field $K$ there exists an isomorphism
$\phi: L_{K}(E) \rightarrow L_{K}(F)$ with $\phi\left(D_{E}\right)=D_{F}$.

## Theorem

Let $E$ and $F$ be finite irreducible graphs. If the one-sided shifts $X_{E}$ and $X_{F}$ are conjugate, then there exists a *-isomorphism $\phi: C^{*}(E) \rightarrow C^{*}(F)$ with $\phi\left(\bar{D}_{E}\right)=\bar{D}_{F}$.

Unfortunately, it is still the case that neither converse holds.

However, if we work backward from isomorphism (or $*$-isomorphism) we can get a weaker notion of equivalence of one-sided shift spaces.

If $x=e_{1} e_{2} \ldots \in X_{E}$, we define the orbit of $x$ to the the set

$$
\operatorname{orb}(x):=\bigcup_{k=0}^{\infty} \bigcup_{l=0}^{\infty} \sigma^{-k}\left(\sigma^{\prime}(x)\right)
$$

## Definition

Let $X_{E}$ and $X_{F}$ be two one-sided edge shifts. If there is a homeomorphism $h: X_{E} \rightarrow X_{F}$ such that $h(\operatorname{orb}(x))=\operatorname{orb}(h(x))$ for all $x \in X_{E}$, then $X_{E}$ and $X_{F}$ are said to be topologically orbit equivalent. In this case, there exists $k_{1}, l_{1}: X_{E} \rightarrow \mathbb{N} \cup\{0\}$ such that

$$
\sigma^{k_{1}(x)}(h(\sigma(x)))=\sigma^{l_{1}(x)}(h(x)) \quad \text { for all } x \in X_{E}
$$

Similarly, there exists $k_{2}, I_{2}: X_{F} \rightarrow \mathbb{N} \cup\{0\}$ such that

$$
\sigma^{k_{2}(x)}\left(h^{-1}(\sigma(x))\right)=\sigma^{l_{2}(x)}\left(h^{-1}(x)\right) \quad \text { for all } x \in X_{F} .
$$

If we can choose $k_{1}, l_{1}: X_{E} \rightarrow \mathbb{N} \cup\{0\}$ and $k_{2}, l_{2}: X_{F} \rightarrow \mathbb{N} \cup\{0\}$ continuous, we say $X_{E}$ and $X_{F}$ are continuously orbit equivalent.

## Theorem

Let $E$ and $F$ be finite irreducible graphs. Then the following are equivalent
(1) The one-sided shifts $X_{E}$ and $X_{F}$ are continuously orbit equivalent.
(0) For any field $K$ there exists an isomorphism $\phi: L_{K}(E) \rightarrow L_{K}(F)$ with $\phi\left(D_{E}\right)=D_{F}$.
(0 There exists a *-isomorphism $\phi: C^{*}(E) \rightarrow C^{*}(F)$ with $\phi\left(\bar{D}_{E}\right)=\bar{D}_{F}$.

Furthermore, there is an algebraic characterization of when two edge shifts are continuously orbit equivalent.

If $E$ is a graph, define the vertex matrix to be the $E^{0} \times E^{0}$ matrix with

$$
A_{E}(v, w):=\text { the number of edges from } v \text { to } w \text {. }
$$

The Bowen Franks group of an $n \times n$ matrix $M$ is

$$
\operatorname{BF}(M):=\operatorname{coker}\left\{I-M: \mathbb{Z}^{n} \rightarrow \mathbb{Z}^{n}\right\}=\mathbb{Z}^{n} /(I-M) \mathbb{Z}^{n}
$$

For a graph $E$, we will be concerned with

$$
B F\left(A_{E}^{t}\right)=\mathbb{Z}^{E^{0}} /\left(I-A_{E}^{t}\right) \mathbb{Z}^{E^{0}} \quad \text { and } \quad u_{E}=\left[(1,1, \ldots, 1)^{t}\right]
$$

## Theorem

Let $E$ and $F$ be finite irreducible graphs. The one-sided shifts $X_{E}$ and $X_{F}$ are continuously orbit equivalent if and only if there is an isomorphism $\psi: \operatorname{BF}\left(A_{E}^{t}\right) \rightarrow B F\left(A_{F}^{t}\right)$ such that $\psi\left(u_{E}\right)=u_{F}$ and $\operatorname{sgn} \operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn} \operatorname{det}\left(I-A_{F}^{t}\right)$.

## Theorem

Let $E$ and $F$ be finite irreducible graphs. The following are equivalent.
(1) The one-sided shifts $X_{E}$ and $X_{F}$ are continuously orbit equivalent.
(2) For any field $K$ there exists an isomorphism $\phi: L_{K}(E) \rightarrow L_{K}(F)$ with $\phi\left(D_{E}\right)=D_{F}$.
(3) There exists a *-isomorphism $\phi: C^{*}(E) \rightarrow C^{*}(F)$ with $\phi\left(\bar{D}_{E}\right)=\bar{D}_{F}$.
(9) There is an isomorphism $\psi: B F\left(A_{E}^{t}\right) \rightarrow B F\left(A_{F}^{t}\right)$ such that $\psi\left(u_{E}\right)=u_{F}$ and $\operatorname{sgn} \operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn} \operatorname{det}\left(I-A_{F}^{t}\right)$.

Fact: In operator algebra $K$-theory $\left(K_{0}\left(C^{*}(E)\right),[1]\right) \cong\left(B F\left(A_{E}^{t}\right), u_{E}\right)$, and in algebraic $K$-theory $\left(K_{0}\left(L_{K}(E)\right),[1]\right) \cong\left(B F\left(A_{E}^{t}\right), u_{E}\right)$.

Good: Relates notion of equivalence for one-sided shifts to a kind of isomorphism for Leavitt path algebras and a kind of $*$-isomorphism for graph $C^{*}$-algebras. It also describes this in terms of an algebraic invariant that can be computed.

Bad: Doesn't characterize isomorphism / *-isomorphism,

We will come back to these issues later with a better answer. But first, we need to consider two-sided edge shifts.

Move (O): Outsplitting


Outsplitting

$s^{-1}(v)=\{e, f\} \cup\{g\} \cup\{h\}$

Move (I): Insplitting


Insplitting
$\Longrightarrow$

$$
r^{-1}(v)=\{a\} \cup\{b\}
$$

## Theorem (Williams) (conjugacy for two-sided shifts of finite type)

 Let $E$ and $F$ be finite irreducible graphs. Then the two-sided shifts $\bar{X}_{E}$ and $\bar{X}_{F}$ are conjugate if and only if $E$ can be turned into $F$ through a finite number of outsplittings, insplittings, outamalgamations, and inamalgamations.Outsplitting preserves isomorphism (respectively, *-isomorphism) of the Leavitt path algebra (respectively, graph $C^{*}$-algebra).

However, insplitting does not.
Rather than isomorphism / *-isomorphism, we need to consider a weaker notion of equivalence for algebras and $C^{*}$-algebras:

Morita equivalence.

Two rings $R$ and $S$ are defined to be Morita equivalent if their categories of left modules are equivalent. There are various ways to characterize this. One conceptually useful way is . . .

If $R$ is a ring, let $M_{\infty}(R)$ denote the set of countably infinite matrices with only a finite number of nonzero entries. Note: $M_{\infty}(R)=\bigcup_{n=1}^{\infty} M_{n}(R)$.

Two rings $R$ and $S$ are Morita equivalent if and only if $M_{\infty}(R) \cong M_{\infty}(S)$.
Two $C^{*}$-algebras $A$ and $B$ are Morita equivalent if $\overline{M_{\infty}(A)} \cong \overline{M_{\infty}(B)}$.
Note that $M_{n}(A) \cong A \otimes M_{n}(\mathbb{C})$, and so $M_{\infty}(A) \cong A \otimes M_{\infty}(\mathbb{C})$. Thus

$$
\overline{M_{\infty}(A)} \cong \overline{A \otimes M_{\infty}(\mathbb{C})} \cong \bar{A} \otimes \overline{M_{\infty}(\mathbb{C})}=A \otimes \mathcal{K}
$$

Two $C^{*}$-algebras $A$ and $B$ are Morita equivalent if and only if

$$
A \otimes \mathcal{K} \cong B \otimes \mathcal{K}
$$

Two algebras / C*-algebras that are Morita equivalent have the same ideal structure, same representation theory, and many of the same properties and invariants.

In particular, if $A$ and $B$ are both algebras or $C^{*}$-algebras that are Morita equivalent, then $K_{0}(A) \cong K_{0}(B)$.

If $E$ and $F$ are graphs and $L_{K}(E)$ is Morita equivalent to $L_{K}(F)$, then $B F\left(A_{E}^{t}\right)=B F\left(A_{F}^{t}\right)$.

If $E$ and $F$ are graphs and $C^{*}(E)$ is Morita equivalent to $C^{*}(F)$, then $B F\left(A_{E}^{t}\right)=B F\left(A_{F}^{t}\right)$.

However: The sign of $\operatorname{det}\left(I-A_{E}^{t}\right)$ need not be preserved!

Theorem (Williams)(conjugacy for two-sided shifts of finite type)
Let $E$ and $F$ be finite irreducible graphs. Then the two-sided shifts $X_{E}$ and $X_{F}$ are conjugate if and only if $E$ can be turned into $F$ through a finite number of outsplittings, insplittings, outamalgamations, and inamalgamations.

Outsplitting and insplitting both preserve Morita equivalence.

## Theorem

Let $E$ and $F$ be finite irreducible graphs. If the two-sided shifts $\bar{X}_{E}$ and $\bar{X}_{F}$ are conjugate, then $L_{K}(E)$ is Morita equivalent to $L_{K}(F)$ for every field $K$.

## Theorem

Let $E$ and $F$ be finite irreducible graphs. If the two-sided shifts $\bar{X}_{E}$ and $\bar{X}_{F}$ are conjugate, then $C^{*}(E)$ is Morita equivalent to $C^{*}(F)$.

Unfortunately, neither converse holds.

Again, we need a weaker notion of equivalence for two-sided shifts.

## Definition

If $X_{E}$ is a shift space, the suspension flow is the quotient space

$$
S X_{E}:=\left(X_{E} \times \mathbb{R}\right) /\left\{(x, t) \sim\left(\sigma_{E}(x), t-1\right)\right\}
$$

There is a flow on $S X_{E}$ induced by the flow $\phi_{t}$ on $X_{E} \times \mathbb{R}$ given by $\phi_{t}(x, s)=(x, s+t)$. The shift spaces $\left(X_{E}, \sigma_{E}\right)$ and $\left(X_{F}, \sigma_{F}\right)$ are said to be flow equivalent if there is a homeomorphism $h: S X_{E} \rightarrow S X_{F}$ carrying orbits of the flow on $S X_{E}$ to orbits of the flow on $S X_{F}$ and preserving the orientation.

Parry and Sullivan have given a characterization of flow equivalence in terms of moves on the graph. In addition to outsplitting and insplitting, we need one more move.

## Move (R): Reduction



Reduction
$\Longrightarrow$
$s^{-1}(w)$ is a single edge $f$
$s\left(r^{-1}(w)\right)$ is a single vertex $v$

Move (R) is also sometimes called the "Parry-Sullivan Move".
The Inverse of Reduction is called Delay.

## Theorem (Parry and Sullivan)

Let $E$ and $F$ be finite irreducible graphs. The following are equivalent
(1) The two-sided shifts $\bar{X}_{E}$ and $\bar{X}_{F}$ are flow equivalent.
(2) $E$ can be transformed into $F$ via moves ( $O$ ), ( $I$ ), ( $R$ ), and their inverses
(Franks) The two above statements are also equivalent to
(3) $\operatorname{BF}\left(A_{E}^{t}\right) \cong \operatorname{BF}\left(A_{F}^{t}\right)$ and $\operatorname{sgn}\left(\operatorname{det}\left(I-A_{E}^{t}\right)\right)=\operatorname{sgn}\left(\operatorname{det}\left(I-A_{\digamma}^{t}\right)\right)$

Thus we have a characterization of flow equivalence in terms of moves, and in terms of a (computable) algebraic invariant.

Since moves (O), (I), and (R) preserve Morita equivalence, we obtain

## Theorem

Let $E$ and $F$ be finite irreducible graphs. If the two-sided shifts $\bar{X}_{E}$ and $\bar{X}_{F}$ are flow equivalent, then for every field $K$ the Leavitt path algebra $L_{K}(E)$ is Morita equivalent to $L_{K}(F)$.

## Theorem

Let $E$ and $F$ be finite irreducible graphs. If the two-sided shifts $\bar{X}_{E}$ and $\bar{X}_{F}$ are flow equivalent, then $C^{*}(E)$ is Morita equivalent to $C^{*}(F)$.

The converse of the second theorem does not hold. There are $E$ and $F$ such that $C^{*}(E)$ is Morita equivalent to $C^{*}(F)$ but $\operatorname{sgn}\left(\operatorname{det}\left(I-A_{E}^{t}\right)\right) \neq \operatorname{sgn}\left(\operatorname{det}\left(I-A_{F}^{t}\right)\right)$. Thus $\bar{X}_{E}$ and $\bar{X}_{F}$ are not flow equivalent.

No one knows whether or not there are converses to the second theorem. This is a major open question in the subject of Leavitt path algebras.

For now, let's focus our attention on graph $C^{*}$-algebras.

Work of Parry and Sullivan together with work of Franks shows that if $E$ and $F$ are irreducible, then

Parry-Sullivan
$\bar{X}_{E}$ is flow equivalent to $\bar{X}_{F} \Longleftrightarrow E$ can be transformed into $F$ via Moves (O), (I), (R), and their inverses
Franks
$\Longleftrightarrow \operatorname{coker}\left(I-A_{E}\right) \cong \operatorname{coker}\left(I-A_{F}\right)$ and
$\operatorname{sgn}\left(\operatorname{det}\left(I-A_{E}^{t}\right)\right)=\operatorname{sgn}\left(\operatorname{det}\left(I-A_{F}^{t}\right)\right)$
$\underline{\text { Move (CS): Cuntz Splice }}$


## Cuntz Splice



## Theorem (Rørdam)

Theorem: For irreducible graphs, the Cuntz splice preserves Morita equivalence of the associated graph $C^{*}$-algebra.

Let $E$ be a graph, and perform the Cuntz splice to obtain $F$.

$$
A_{F}=\left(\begin{array}{cc|ccc}
1 & 1 & 0 & 0 & \cdots \\
1 & 1 & 1 & 0 & \cdots \\
\hline 0 & 1 & & \\
0 & 0 & & A_{E} \\
\vdots & \vdots & &
\end{array}\right)
$$

Then $\operatorname{BF}\left(A_{E}^{t}\right) \cong \operatorname{BF}\left(A_{F}^{t}\right)$, but $\operatorname{det}\left(I-A_{F}^{t}\right)=-\operatorname{det}\left(I-A_{E}^{t}\right)$.

## Theorem (Cuntz and Krieger)

Suppose $E$ and $F$ are finite irreducible graphs. Then $C^{*}(E)$ is Morita equivalent to $C^{*}(F)$ if and only if $\operatorname{BF}\left(A_{E}^{t}\right) \cong \operatorname{BF}\left(A_{F}^{t}\right)$.

Moreover, in this case one can transform $E$ into $F$ using Moves ( $O$ ), (I), $(R)$, their inverse moves, and Move (CS).

Proof:
$\operatorname{BF}\left(A_{E}^{t}\right) \cong \mathrm{BF}\left(A_{F}^{t}\right) \Longrightarrow \operatorname{BF}\left(A_{E}^{t}\right) \cong B F\left(A_{F}^{t}\right)$
(If sgn $\operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn}\left(\operatorname{det}\left(I-A_{F}^{t}\right)\right)$, great.
If not, apply Cuntz Splice.)
$\Longrightarrow \operatorname{BF}\left(A_{E}^{t}\right) \cong \operatorname{BF}\left(A_{F}^{t}\right)$ and $\operatorname{sgn} \operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn}\left(\operatorname{det}\left(I-A_{F}^{t}\right)\right)$
$\Longrightarrow$ (Franks) $\bar{X}_{E}$ flow equivalent to $\bar{X}_{F}$
$\Longrightarrow$ (Parry and Sullivan) $E$ can be turned into $F$ via
Moves(O), (I), (R) and their inverses
$\Longrightarrow C^{*}(E)$ Morita equivalent to $C^{*}(F)$.

For Leavitt path algebras, we cannot determine if the Cuntz splice affects the Morita equivalence class of the associated Leavitt path algebra.

We cannot even answer this in the simplest case:


Is $L_{K}\left(E_{2}\right)$ Morita equivalent to $L_{K}\left(E_{2}^{-}\right)$? No one knows.

Therefore the best we can do is the following.

## Theorem (Abrams, Louly, Pardo, and Smith)

Suppose $E$ and $F$ are finite irreducible graphs. If $\operatorname{BF}\left(A_{E}^{t}\right) \cong \operatorname{BF}\left(A_{F}^{t}\right)$ and $\operatorname{sgn}\left(\operatorname{det}\left(I-A_{E}^{t}\right)\right)=\operatorname{sgn}\left(\operatorname{det}\left(I-A_{F}^{t}\right)\right)$, then for any field $K$ we have that $L_{K}(E)$ is Morita equivalent $L_{K}(F)$.

Let's return to our isomorphism theorems from one-sided shifts.

## Theorem

Let $E$ and $F$ be finite irreducible graphs. The following are equivalent:
(1) The one-sided shifts $X_{E}$ and $X_{F}$ are continuously orbit equivalent.
(2) For any field $K$ there exists an isomorphism $\phi: L_{K}(E) \rightarrow L_{K}(F)$ with $\phi\left(D_{E}\right)=D_{F}$.
(3) There exists a *-isomorphism $\phi: C^{*}(E) \rightarrow C^{*}(F)$ with $\phi\left(\bar{D}_{E}\right)=\bar{D}_{F}$.
(9) There is an isomorphism $\psi: B F\left(A_{E}^{t}\right) \rightarrow B F\left(A_{F}^{t}\right)$ such that $\psi\left(u_{E}\right)=u_{F}$ and $\operatorname{sgn} \operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn} \operatorname{det}\left(I-A_{F}^{t}\right)$.

If there is an isomorphism $\psi: B F\left(A_{E}^{t}\right) \rightarrow B F\left(A_{F}^{t}\right)$ such that $\psi\left(u_{E}\right)=u_{F}$, then $C^{*}(E)$ is Morita equivalent to $C^{*}(F)$, and one can use the fact that $\psi\left(u_{E}\right)=u_{F}$ to prove that $C^{*}(E)$ and $C^{*}(F)$ are actually $*$-isomorphic.

Thus we have .

## Theorem

Let $E$ and $F$ be finite irreducible graphs. Then $C^{*}(E)$ is *-isomorphic to $C^{*}(F)$ if and only if there is an isomorphism $\psi: B F\left(A_{E}^{t}\right) \rightarrow B F\left(A_{F}^{t}\right)$ such that $\psi\left(u_{E}\right)=u_{F}$.
or in $C^{*}$-algebra terms . . .

## Theorem

Let $E$ and $F$ be finite graphs with no sinks, and suppose $C^{*}(E)$ and $C^{*}(F)$ are simple. Then $C^{*}(E)$ is *-isomorphic to $C^{*}(F)$ if and only if there is an isomorphism $\psi: K_{0}\left(C^{*}(E)\right) \rightarrow K_{0}\left(C^{*}(F)\right)$ such that $\psi\left(\left[1_{C^{*}(E)}\right]\right)=\left[1_{C^{*}(F)}\right]$.

We can obtain a similar result for Leavitt path algebras - but again, our uncertainty about whether the sign of the determinant is necessary causes some problems.

## Theorem

Let $E$ and $F$ be finite irreducible graphs, and let $K$ be a field. If $\operatorname{sgn} \operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn} \operatorname{det}\left(I-A_{F}^{t}\right)$ and there is an isomorphism $\psi: B F\left(A_{E}^{t}\right) \rightarrow B F\left(A_{F}^{t}\right)$ such that $\psi\left(u_{E}\right)=u_{F}$, then $L_{K}(E)$ is isomorphic to $L_{K}(F)$.
or in algebra terms . . .

## Theorem

Let $E$ and $F$ be finite graphs with no sinks, let $K$ be a field, and suppose that $L_{K}(E)$ and $L_{K}(F)$ are simple. If $\operatorname{sgn} \operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn} \operatorname{det}\left(I-A_{F}^{t}\right)$ and there is an isomorphism $\psi: K_{0}\left(L_{K}(E)\right) \rightarrow K_{0}\left(L_{K}(F)\right)$ such that $\psi\left(\left[1_{L_{K}(E)}\right]\right)=\left[1_{L_{K}(F)}\right]$, then $L_{K}(E)$ is isomorphic to $L_{K}(F)$.

## One-Sided Shifts - Summary of Results

Let $E$ and $F$ be finite graphs with no sinks.
Theorem: If $E$ and $F$ are irreducible
$X_{E}$ conjugate to $X_{F} \Longleftrightarrow E \leftrightarrow F$ using Move (O) and its inverse
$\Longrightarrow X_{E}$ and $X_{F}$ are continuously orbit equivalent
Theorem: If $E$ and $F$ are irreducible, TFAE:
(1) $X_{E}$ and $X_{F}$ are continuously orbit equivalent.
(2) $\forall$ fields $K, \exists$ isomorphism $\phi: L_{K}(E) \rightarrow L_{K}(F)$ with $\phi\left(D_{E}\right)=D_{F}$.
(3) $\exists$-isomorphism $\phi: C^{*}(E) \rightarrow C^{*}(F)$ with $\phi\left(\bar{D}_{E}\right)=\bar{D}_{F}$.
(9) $\left.\operatorname{BF}\left(A_{E}^{t}\right), u_{E}\right) \cong\left(\operatorname{BF}\left(A_{F}^{t}\right), u_{F}\right)$ and $\operatorname{sgn} \operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn} \operatorname{det}\left(I-A_{F}^{t}\right)$.

Theorem: If $C^{*}(E)$ and $C^{*}(F)$ are simple,

$$
C^{*}(E) \cong_{*} C^{*}(F) \Longleftrightarrow\left(K_{0}\left(C^{*}(E)\right),\left[1_{C^{*}(E)}\right]\right) \cong\left(K_{0}\left(C^{*}(F)\right),\left[1_{C^{*}(F)}\right]\right)
$$

Theorem: If $L_{K}(E)$ and $L_{K}(F)$ are simple,

$$
\left(K_{0}\left(L_{K}(E)\right),\left[1_{L_{K}(E)}\right]\right) \cong\left(K_{0}\left(L_{K}(F)\right),\left[1_{L_{K}(F)}\right]\right) \Longrightarrow L_{K}(E) \cong L_{K}(F)
$$ and $\operatorname{sgn} \operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn} \operatorname{det}\left(I-A_{F}^{t}\right)$

Note: $\left(K_{0}\left(C^{*}(E)\right),\left[1_{C^{*}(E)}\right]\right) \cong\left(K_{0}\left(L_{K}(E)\right),\left[1_{L_{K}(E)}\right]\right) \cong\left(\operatorname{BF}_{( }\left(A_{E}^{t}\right), u_{E}\right)^{E}$

## Two-Sided Shifts - Summary of Results

Let $E$ and $F$ be finite graphs with no sinks.
Theorem: If $E$ and $F$ are irreducible
$\bar{X}_{E}$ conjugate to $\bar{X}_{F} \Longleftrightarrow E \leftrightarrow F$ using Moves (O), (I) and their inverses
$\Longrightarrow \bar{X}_{E}$ and $\bar{X}_{F}$ are flow equivalent
Theorem: If $E$ and $F$ are irreducible, TFAE:
(1) $\bar{X}_{E}$ and $\bar{X}_{F}$ are flow equivalent.
(2) $E \leftrightarrow F$ using Moves (O), (I), (R) and their inverses.
(3) $\operatorname{BF}\left(A_{E}^{t}\right) \cong \operatorname{BF}\left(A_{F}^{t}\right)$ and $\operatorname{sgn} \operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn} \operatorname{det}\left(I-A_{F}^{t}\right)$.

Theorem: If $C^{*}(E)$ and $C^{*}(F)$ are simple, TFAE
(1) $C^{*}(E)$ is Morita equivalent to $C^{*}(F)$.
(2) $E \leftrightarrow F$ using Moves (O), (I), (R) their inverses, and Move (CS).
(3) $K_{0}\left(C^{*}(E)\right) \cong K_{0}\left(C^{*}(F)\right)$.

Theorem: If $L_{K}(E)$ and $L_{K}(F)$ are simple,
$K_{0}\left(L_{K}(E)\right) \cong K_{0}\left(L_{K}(F)\right)$ and $\Longrightarrow L_{K}(E)$ Morita equivalent to $L_{K}(F)$ sgn $\operatorname{det}\left(I-A_{E}^{t}\right)=\operatorname{sgn} \operatorname{det}\left(I-A_{F}^{t}\right)$
Note: $K_{0}\left(C^{*}(E)\right) \cong K_{0}\left(L_{K}(E)\right) \cong \operatorname{BF}\left(A_{E}^{t}\right)$.


