University of Houston Summer School on Dynamical Systems: 2014
Convex Cones and Nonequilibrium Dynamical Systems: Exercises

Exercise 1. Give an example of a compact metric space (X,d) and a map $T : X \to X$ such that T admits no invariant Borel probability measures.

Exercise 2. Let (X,d) be a compact metric space and let $(f_i)_{i=1}^{\infty}$ be a sequence of maps on X for which there exists $L < 1$ such that for all $i \in \mathbb{N}$ and for all $p, q \in X$, we have $d(f_i(p), f_i(q)) \leqslant Ld(p,q)$. Find necessary and sufficient conditions under which there exists a unique $z \in X$ such that

$$\lim_{n\to\infty} (f_n \circ f_{n-1} \circ \cdots \circ f_1)(p) = z$$

for all $p \in X$.

Exercise 3. Let V be a real vector space and let $\mathcal{C} \subset V$ be a convex cone. Define $\preccurlyeq$ on V by $\varphi \preccurlyeq \psi$ if and only if $\psi - \varphi \in \mathcal{C} \cup \{0\}$. Let $\varphi, \psi \in \mathcal{C}$. Prove the following.

(a) If $\varphi \preccurlyeq 0 \preccurlyeq \varphi$, then $\varphi = 0$.
(b) For every $a > 0$, $0 \preccurlyeq \varphi$ if and only if $0 \preccurlyeq a\varphi$.
(c) $\varphi \preccurlyeq \psi$ if and only if $0 \preccurlyeq \psi - \varphi$.
(d) For every $a \in \mathbb{R}$ and every sequence $(a_i)_{i=1}^{\infty}$ in $\mathbb{R}$ with $a_i \to a$, if $a_i\varphi \preccurlyeq \psi$ for all $i \in \mathbb{N}$, then $a\varphi \preccurlyeq \psi$.
(e) If $\varphi \succcurlyeq 0$ and $\psi \succcurlyeq 0$, then $\varphi + \psi \succcurlyeq 0$.

Exercise 4. Let $\mathcal{C}$ be the cone in $\mathrm{BV}([0,1],\mathbb{R})$ defined by

$$\mathcal{C} = \{\varphi \in \mathrm{BV}([0,1],\mathbb{R}) : \varphi \geqslant 0, \ \ \varphi \not\equiv 0\}.$$

Give an example of $\varphi, \psi \in \mathcal{C}$ such that $d_{\mathcal{C}}(\varphi, \psi) = \infty$.

Exercise 5. Let V be a real vector space and let $\mathcal{C} \subset V$ be a convex cone. Prove the following.

(a) $d_{\mathcal{C}}(\varphi, a\varphi) = 0$ for every $\varphi \in \mathcal{C}$ and every $a > 0$.
(b) $d_{\mathcal{C}}(\varphi, \psi) = d_{\mathcal{C}}(\psi, \varphi)$ for every $\varphi, \psi \in \mathcal{C}$.