Houston Summer School on Dynamical Systems

Problem set: Dynamical systems with hyperbolic behaviour

1. Let X be a separable metric space and $T\colon X \to X$ be continuous. Given $x \in X$ and $n \in \mathbb{N}$, let $\mathcal{E}_{x,n} = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k x}$ be the nth *empirical measure* for x. That is, the measure $\mathcal{E}_{x,n}$ is defined by

$$\int \phi(y)\, d\mathcal{E}_{x,n}(y) = \frac{1}{n}\sum_{k=0}^{n-1} \phi(T^k x)$$

for every continuous $\phi\colon X \to \mathbb{R}$. Suppose $n_j \to \infty$ and $\mu \in \mathcal{M}(X)$ are such that $\mathcal{E}_{x,n_j} \to \mu$.

(a) Show that μ is T-invariant.

(b) Give an example to show that this may fail if T is not continuous.

(c) Say that x is *generic* for μ if $\mathcal{E}_{x,n} \to \mu$ (without passing to a subsequence). Birkhoff's ergodic theorem says that if μ is ergodic and G_μ is the set of generic points for μ, then $\mu(G_\mu) = 1$. Give an example showing that G_μ may be empty if μ is invariant but not ergodic.

(d) Let Σ be the full shift on two symbols and let μ be any σ-invariant measure (not necessarily ergodic). Show that μ has a generic point.

2. Let A be a finite alphabet and $\Sigma \subset A^{\mathbb{Z}}$ (or $A^{\mathbb{N}}$) be a closed σ-invariant subset. Given $n \in \mathbb{N}$, let

$$L_n = \{w = w_1 \cdots w_n \mid w_i \in A \text{ for each } 1 \le i \le n, \\ \text{and } w \text{ appears as a subword of some } x \in \Sigma\}.$$

(a) Let $a_n = \log \#L_n$ and show that

$$a_{n+m} \le a_n + a_m \text{ for every } n, m \in \mathbb{N}. \qquad (\star)$$

(b) A sequence satisfying $(\star)$ is called *subadditive*. Prove *Fekete's lemma*: if a_n is a subadditive sequence, then $\lim \frac{1}{n} a_n$ exists and is equal to $\inf \frac{1}{n} a_n$ (which may be $-\infty$).

(c) Deduce that $h(\Sigma) = \lim_{n\to\infty} \frac{1}{n} \log \#L_n(\Sigma)$ exists for every shift Σ. This limit is called the *topological entropy* of the shift Σ.

(d) Let Σ be the SFT on two symbols $\{0, 1\}$ where the only forbidden word is 11. Show that $\#L_n(\Sigma)$ is the Fibonacci sequence.

(e) Let Σ be a topological Markov chain with transition matrix M – that is, a sequence x is in Σ if and only if $M_{x_n,x_{n+1}} = 1$ for every n. Show that $\#L_n$ is the sum of all the entries of M^{n-1}.

(f) Let λ be a positive real eigenvalue of M with the property that $|\chi| \leq \lambda$ for all eigenvalues χ. (Existence of such an eigenvalue is part of the Perron–Frobenius theorem.) Prove that $h(\Sigma) = \log \lambda$.

3. The *Dyck shift* Σ is the two-sided shift on the four-symbol alphabet A whose letters are the brackets (, [,], and), and whose allowable sequences are precisely those in which the brackets are opened and closed in the right order. That is, the word ([ ]) is allowable, but the word ([ )] is not. Similarly, ((([is allowable, but ( ] ((is not. Say two symbols $a, b \in A$ are a matched pair if $a =$ (and $b =$), or if $a =$ [and $b =$], and write $a \multimap\!\!\circ\, b$. Given $x \in \Sigma$, use the same notation for the following binary relation on $\mathbb{N}$:

◇ $n \multimap\!\!\circ\, n+1$ if $x_n \multimap\!\!\circ\, x_{n+1}$;
◇ $m \multimap\!\!\circ\, n$ if there is $m < i < n$ such that $m \multimap\!\!\circ\, i$ and $i+1 \multimap\!\!\circ\, n$;
◇ $m \multimap\!\!\circ\, n$ if $x_m \multimap\!\!\circ\, x_n$ and $m+1 \multimap\!\!\circ\, n-1$.

Heuristically, $m \multimap\!\!\circ\, n$ in x if every bracket that is opened at or after position m has been closed by position n. Consider the subsets

$$\Sigma^R = \{x \in \Sigma \mid \text{ for every } n \in \mathbb{N} \text{ there is } m < n \text{ with } m \multimap\!\!\circ\, n\},$$
$$\Sigma^L = \{x \in \Sigma \mid \text{ for every } m \in \mathbb{N} \text{ there is } n > m \text{ with } m \multimap\!\!\circ\, n\}.$$

That is, Σ^R is the set of sequences where every right bracket has a matching left bracket, and Σ^L is the set of sequences where every left bracket has a matching right bracket.

(a) Let $X = \{0, 1, 2\}^{\mathbb{Z}}$ be the full shift on three symbols and define $h\colon \Sigma \to X$ by $h(x)_n = H(x_n)$, where $H\colon A \to \{0, 1, 2\}$ maps the symbol (to 1, the symbol [to 2, and both symbols),] to 0. Show that h is 1-1 on Σ^R.

(b) Let μ be an σ-invariant probability measure on Σ and show that $\mu(\Sigma^R \cup \Sigma^L) = 1$.

(c) Consider the directed graph G whose vertices are non-negative integers and which has the following edges:

◇ 2 edges from 0 to 0;

◇ 2 edges from n to $n+1$ for every $n \geq 0$;

◇ 1 edge from n to $n-1$ for every $n \geq 1$.

Let a_n be the number of paths of length n on this graph that start at 0. Show that $\#L_n(\Sigma) = a_n$.

Hint: it may help to label the two edges from 0 to 0 with the right brackets) and], the two edges from n to $n+1$ with (and [, and the edge from n to $n-1$ with ") or]".

(d) Show that $h(\Sigma) = \log 3$.

(e) If you know about measure-theoretic entropy and the variational principle, show that Σ has two ergodic measures of maximal entropy, and that both are fully supported on Σ.

4. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T\colon X \to X$ a measure-preserving map. Given measurable sets $A, B \subset X$, let

$$C_n(A,B) := |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)|$$

be the nth correlation function of A, B. Similarly, given L^2 test functions ϕ, ψ, let

$$C_n(\phi,\psi) := \left| \int \phi \cdot (\psi \circ T^n)\, d\mu - \int \phi\, d\mu \int \psi\, d\mu \right|.$$

(a) Show that $C_n(A,B) \to 0$ for every A, B if and only if $C_n(\phi,\psi) \to 0$ for every ϕ, ψ. In this case the measure is called *mixing*.

(b) Our results on decay of correlations all involve $C_n(\phi,\psi)$ for sufficiently regular test functions, instead of $C_n(A,B)$, or $C_n(\phi,\psi)$ for arbitrary L^2 functions. This is because even when $C_n(\phi,\psi)$ decays exponentially for Hölder continuous functions (or some other nice class), we may have very slow decay for $C_n(A,B)$, or for arbitrary measurable functions.

Demonstrate this phenomenon as follows: let $T\colon [0,1] \to [0,1]$ be the doubling map $T(x) = 2x \pmod 1$, and let μ be Lebesgue measure on $[0,1]$. Find measurable sets $A, B \subset [0,1]$ such that $C_n(A,B)$ only decays polynomially – that is, there are $\gamma, c > 0$ such that $C_n(A,B) \geq cn^\gamma$ for all n.

Note that this is equivalent to answering the same question where X is the full shift on two symbols and μ is $(\frac{1}{2}, \frac{1}{2})$-Bernoulli.