Some questions.

(1) (a) Suppose that (X_i) is a sequence of bounded functions on a probability space (X, μ) and let

$$S_k(x) = X_1(x) + X_2(x) + \ldots + X_k(x)$$

Suppose that

$$\lim_{n\to\infty} \frac{S_{n^2}(x)}{n^2} = 0$$

for μ a.e. $x \in X$.

Show that this implies

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = 0$$

for μ a.e. $x \in X$. (b) Now let ϕ be a bounded function (or observation) on a dynamical system (T, X, μ) with $\int \phi d\mu = 0$.

Suppose for all $n > 0$

$$|\int \phi(x)\phi(T^n x) d\mu| \leq p(n)$$

where $\sum_n p(n) < \infty$. Let $S_n(x) = \sum_{j=0}^{n-1} \phi \circ T^j(x)$.

Using the fact that if $j \geq i$, $\int \phi(T^j x)\phi(T^i x) d\mu = \int \phi(T^{j-i}x)\phi(x) d\mu$ if $j > i$ show that

$$\int (S_n(x))^2 d\mu \leq Cn$$

for some constant C.

Use Chebychev

$$\mu(x : |S_n(x)/n| > r) = \mu(x : |S_n(x)|^2 > n^2 r) \leq \frac{1}{n^2 r} \int (S_n(x))^2 d\mu$$

to show that

$$|\int \phi(x)\phi(T^n x) d\mu| \leq p(n)$$

where $\sum_n p(n) < \infty$ implies that

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = 0$$

(2) Let $T : [0,1) \to [0,1)$ be the doubling map which preserves Lebesgue measure. Define, for continuous functions ϕ,

$$(P\phi)(x) = \sum_{y \in T^{-1}x} \frac{\phi(y)}{2}$$

Show that if ψ and ϕ are continuous then

$$\int \phi\psi \circ T dx = \int (P\phi)\psi dx$$

and

$$P(\phi \circ T) = \phi$$

(Continuity is not needed but it simplifies the statement).

(3) Suppose that p is a periodic orbit of period k for the doubling map $(T, [0,1), m)$ so that $T^k(p) = p$. Show that for any $\epsilon > 0$ $m[T^k(B_\epsilon(p)) \cap (B_\epsilon(p)] > \frac{\epsilon}{2^k}$ where m is Lebesgue measure and $B_\epsilon(p)$ is a ball of radius ϵ about p.

Use this to show that a periodic point p of periodic k for the doubling maps does not have exponential return time statistics i.e. if $\tau_{B_\epsilon(p)}(x) := \min\{j > 1 : T^j(x) \in B_\epsilon(p)\}$ then

$$\lim_{\epsilon \to 0} \frac{1}{m(B_\epsilon(p)} \{x \in B_\epsilon(p) : \tau_{B_\epsilon(p)}(x) < \frac{t}{\mu(B_\epsilon(p)}\} \not\to 1 - e^{-t}$$