MET Workshop: Exercises

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Notation.

- $\mathbb{R}^d$ is endowed with the standard inner product $(\cdot, \cdot)$ and Euclidean norm $\|\cdot\|$.
- $M_{d\times d}(\mathbb{R})$ denotes the space of $n \times n$ real matrices.
- When $\mathbb{R}^d = E \oplus F$ is a splitting of $\mathbb{R}^d$ into complementary subspaces, we let $\pi_{E//F}$ denote the projection onto E, parallel to F; that is, the range of $\pi_{E//F}$ is E, $\pi_{E//F} \circ \pi_{E//F} = \pi_{E//F}$, and $\ker \pi_{E//F} = F$.
- For a subspace $E \subset \mathbb{R}^d$, let π_E denote the orthogonal projection onto E.
- For a subspace $E \subset \mathbb{R}^d$, let $E^\perp$ denote the orthogonal complement to E.
- For $1 \le k \le d$ and a linear operator $A \in M_{d\times d}(\mathbb{R})$, let $\sigma_k(A)$ denote the k-th singular value of A.
- Here, $\langle w_1, \cdots, w_k\rangle$ denotes the subspace spanned by the vectors $w_1, \cdots, w_k \subset \mathbb{R}^d$.

1 Preliminaries from linear algebra

1.1 The Singular Value Decomposition (SVD)

Definition 1. Let $A \in M_{d\times d}(\mathbb{R})$. The **singular values** $\sigma_1(A), \cdots, \sigma_d(A)$ of A may be defined by

$$\sigma_i(A) = \sqrt{\lambda_i(A^*A)}\,,$$

where $\lambda_i(A^*A)$ denotes the i-th highest eigenvalue of A^*A. Note that A^*A is positive semidefinite, and so $\lambda_1(A^*A) \ge \cdots \ge \lambda_d(A^*A) \ge 0$.

Problem 2. Let $S \in M_{d\times d}(\mathbb{R})$ be a symmetric matrix (i.e. $A^T = A$, where T denotes the transpose). Show that there is an orthogonal basis $\{v_i\}_{1\le i\le d}$ of $\mathbb{R}^d$ consisting of eigenvectors for S.

Problem 3 (Proof of the SVD)**.** Assume that $A \in M_{d\times d}(\mathbb{R})$ is invertible.

(a) Show that $\lambda_k(A^*A) = \lambda_k(AA^*)$ for each $1 \le i \le d$.

(b) Let $\{v_i\}_{1\le i\le d}$ be an orthonormal eigenbasis for A^*A as in Problem 2, ordered so that $A^*Av_i = \lambda_i(A^*A)v_i$ for each $1 \le i \le d$. Let $\{w_i\}_{1\le i\le d}$ be an analogous orthonormal eigenbasis for AA^*, ordered the same way. Prove that $Av_i = \pm\sigma_i(A)w_i$ for each $1 \le i \le d$.

Problem 4. Show that if the invertible matrix A has singular values $\sigma_1 > \ldots > \sigma_d$, then A^{-1} has singular values $\sigma_d^{-1} > \ldots > \sigma_1^{-1}$.

Problem 5. Let $A, B \in M_{d\times d}(\mathbb{R})$ and assume both A, B are invertible. Prove that there exists a basis $\{v_1, \cdots, v_d\}$ of $\mathbb{R}^d$ for which $\{Av_i\}_{1\le i\le d}$ is orthogonal and $\{Bv_i\}_{1\le i\le d}$ is orthogonal.

Problem 6. Let $A \in M_{d\times d}(\mathbb{R})$, and do not assume that A is invertible.

(a) Prove that

$$\sigma_i(A) = \min\{\|A|_F\| : F \subset \mathbb{R}^d, \operatorname{codim} F = i - 1\}$$

for each $1 \le i \le d$. Recall that $\operatorname{codim} F = d - \dim F$ for a subspace $F \subset \mathbb{R}^d$.

(b) Prove that

$$\sigma_i(A) = \max\{m(A|_E) : E \subset \mathbb{R}^d, \dim E = i\},$$

where

$$m(A|_E) := \min\left\{\frac{\|Av\|}{\|v\|} : v \in E \setminus \{0\}\right\} = \left\|(A|_E)^{-1}\right\|^{-1}.$$

Note that $m(A|_E) = 0$ if $A|_E$ is not injective.

(c) Prove that

$$\prod_{i=1}^{k} \sigma_i(A) = \max\{\det(A|_W) : W \subset \mathbb{R}^d, \dim W = k\}$$

for each $1 \le k \le d$.

Problem 7. Let $A \in M_{d\times d}$. Let A have singular vectors $v_1, \dots, v_k$. Prove that v_k is a vector in $\langle v_1, \dots, v_{k-1}\rangle^\perp$ that is maximally expanded by A, i.e.,

$$\|Av_k\| = \|A|_{\langle v_1,\cdots,v_{k-1}\rangle^\perp}\|.$$

1.2 Angles

Problem 8. Let $E \subsetneq \mathbb{R}^d$ be a proper subspace. For $v \in \mathbb{R}^d$, define the minimal distance

$$\operatorname{dist}(v, E) := \min\{\|v - e\| : e \in E\}.$$

Prove that

$$\operatorname{dist}(v, E) = \frac{\|\pi_{E^\perp} v\|}{\|v\|}$$

when $v \ne 0$.

Definition 9. The **angle** $\angle(v, w) \in [0, \pi/2]$ between two vectors $v, w \in \mathbb{R}^d$ is defined by

$$\cos\angle(v, w) = \frac{(v, w)}{\|v\|\|w\|}.$$

For a subspace $E \subset \mathbb{R}^d$ and a nonzero vector $v \in \mathbb{R}^d$, we define the **minimal angle** $\angle(v, E) = \min\{\angle(v, e) : e \in E, \|e\| = 1\}$.

Problem 10. Prove that

$$\sin\angle(v, E) = \operatorname{dist}(v, E),$$

where $E \subset \mathbb{R}^d$ is a subspace and $v \in \mathbb{R}^d \setminus \{0\}$.

1.3 Grassmanian on $\mathbb{R}^d$

Definition 11. For $1 \leq k \leq d$, the **Grassmanian** $\mathrm{Gr}(d,k)$ is the set of all k-dimensional subspaces of $\mathbb{R}^d$. We endow $\mathrm{Gr}(d) := \cup_{k=1}^{d} \mathrm{Gr}(d,k)$ with the following metric: for $E_1, E_2 \in \mathrm{Gr}(d)$ we define

$$d_H(E_1, E_2) := \|\pi_{E_1} - \pi_{E_2}\| .$$

Problem 12. The 'H' stands for Hausdorff: the metric d_H is known as the Hausdorff distance, and is in broader generality a metric on the space of compact subsets of $\mathbb{R}^d$: for two compact subsets $A, B \subset \mathbb{R}^d$, we define

$$d_{Haus}(A, B) = \max\{\max_{a \in A} \mathrm{dist}(a, B), \max_{b \in B} \mathrm{dist}(b, A)\} .$$

For $E_1, E_2 \in \mathrm{Gr}(d,k)$, define $B_{E_i} = \{v \in E_i : \|v\| \leq 1\}$. Compare $d_H(E_1, E_2)$ to $d_{Haus}(B_{E_1}, B_{E_2})$.

Problem 13. Prove that d_H is a complete metric on $\mathrm{Gr}(d)$, and that each $\mathrm{Gr}(d,k), 1 \leq k \leq d$, is connected.

Problem 14. Prove that $\mathrm{Gr}(d,2)$ is sequentially compact.

Problem 15. Let $E, E' \in \mathrm{Gr}(d)$ be such that $d_H(E, E') < 1$. Show that E' and $E^{\perp}$ are complements.

Problem 16 (Harder)**.** Let $\mathbb{R}^d = E \oplus F$ be a splitting of $\mathbb{R}^d$ into complementary subspaces, and let $\pi_{E//F}$ denote the projection onto E parallel to F. Show that if $E' \subset \mathbb{R}^d$ is a subspace sufficiently close to E in the d_H metric, then then E', F are complementary in $\mathbb{R}^d$. Can you find an estimate in terms of $\|\pi_{E//F}\|$? Is it optimal?

Problem 17 (Harder - for those that like manifolds)**.** For a fixed k-codimensional subspace, W, of $\mathbb{R}^d$, let $\mathcal{U}$ denote those elements of $\mathrm{Gr}(d,k)$ that have a trivial intersection with W.

(a) Prove that $\mathcal{U}$ is an open subset of $\mathrm{Gr}(d,k)$.

(b) Fix an element $V_0 \in \mathcal{U}$, a basis $e_1, \dots, e_k$ for V_0 and a basis $f_1, \dots, f_{d-k}$ for W. For an element $V \in \mathcal{U}$, since $V \oplus W = \mathbb{R}^d$, each e_i may be expressed uniquely as $v_i + w_i$ with $v_i \in V$ and $w_i \in W$. Form a matrix, $\Phi(V)$ whose ith column is the coefficients of w_i in the (f_j) basis.

(c) Prove that Φ is a bijection from $\mathcal{U}$ to $\mathbb{R}^{(d-k)\times k}$.

(d) Prove that the collection of $(\mathcal{U}, \Phi)$ (as W varies over $\mathrm{Gr}(d, d-k)$, V_0 varies over $\mathcal{U}$ and the bases vary over bases for W and V_0) forms a smooth manifold structure on $\mathrm{Gr}(d,k)$.

1.4 Volumes and wedges

Problem 18. Verify that:

- $v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k$; and
- $v_1 \wedge \cdots \wedge (cv + c'v') \wedge \cdots \wedge v_k = v_1 \wedge \cdots \wedge v \wedge \cdots \wedge v_k + c' v_1 \wedge \cdots \wedge v' \wedge \cdots \wedge v_k$
- If $e_1, \dots, e_d$ is a basis for V, then $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : i_1 < \ldots < i_k\}$ spans $\bigwedge^k V$.

Problem 19. Let V be a k-dimensional subspace of $\mathbb{R}^d$. Let two bases for V be $e_1, \dots, e_k$ and $f_1, \dots, f_k$.

Prove that $f_1 \wedge \cdots \wedge f_k = c\, e_1 \wedge \cdots \wedge e_k$, where c is the determinant of the matrix of coefficients of the f vectors in terms of the e vectors.

Problem 20. Prove that there is a linear map $\bigwedge^k A$ from $\bigwedge^k V$ to itself such that $\bigwedge^k A(v_1 \wedge \cdots \wedge v_k) = (Av_1) \wedge \cdots \wedge (Av_k)$ for all $v_1, \ldots, v_k \in V$.

Problem 21. Let u, v and w be three orthonormal vectors in $\mathbb{R}^3$. Prove that $u \wedge v$, $u \wedge w$ and $v \wedge w$ are orthonormal.

Problem 22. Let $v_1, \cdots, v_d \subset \mathbb{R}^d$ be a basis. Write $\mathcal{P} = \{\sum_{i=1}^d \alpha_i v_i : \alpha_i \in [0,1], 1 \leq i \leq d\}$. Show that

$$\mathrm{Leb}(\mathcal{P}) = \|v_d\| \cdot \prod_{i=1}^{d-1} \mathrm{dist}(v_i, \langle v_{i+1}, \cdots, v_d \rangle)\,.$$

Problem 23. Let $\mathbb{R}^d = E \oplus F$ be a splitting into complementary subspaces E, F, and let $A \in M_{d\times d}(\mathbb{R})$ be an invertible matrix. Using Problem 22, estimate $\det(A|_E)\det(A|_F)$ in terms of $\det(A)$ and the quantities $\|\pi_{E//F}\|, \|\pi_{AE//AF}\|$. What happens if E, F are subspaces spanned by singular vectors for A?

2 Subadditivity and the Kingman Subadditive Ergodic Theorem

2.1 Subadditive sequences

Problem 24. Let $\{a_n\}_{n\geq 1}$ be a subadditive sequence of reals, i.e., for any $m, n \geq 1$, we have that

$$a_{m+n} \leq a_m + a_n\,.$$

Prove that $\lim_{n\to\infty} n^{-1} a_n$ converges (perhaps to $-\infty$), and prove that it the limiting value coincides with $\inf_{n\geq 1} n^{-1} a_n$.

Problem 25. Let $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$ be sequences of reals for which

$$a_{m+n} \leq a_m + a_n + b_n$$

for each $m, n \geq 1$. Formulate conditions on $\{b_n\}$ under which $\lim_n n^{-1} a_n$ converges.

2.2 Kingman Subadditive Ergodic Theorem (KSET)

For the ensuing exercises, let us assume the KSET in the following form.

Theorem 26 (KSET)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $T : \Omega \to \Omega$ be a μ-invariant measurable transformation. Let $\{f_n : \Omega \to \mathbb{R}\}$ be a subadditive sequence of measurable functions for the mpt $(\Omega, \mathcal{F}, \mu, T)$, i.e., for each $m, n \geq 1$ and for μ-almost every $x \in \Omega$,*

$$f_{m+n}(x) \leq f_m(x) + f_n(x)\,.$$

Assume that $f_1^+ \in L^1(\mu)$. Then, the limits

$$f^*(x) = \lim_{n\to\infty} \frac{1}{n} f_n(x) \in [-\infty, \infty)$$

exist for μ-almost every $x \in \Omega$, and

$$\int f^*(x) d\mu(x) = \lim_{n\to\infty} \int \frac{f_n(x)}{n} d\mu(x) = \inf_{n\to\infty} \int \frac{f_n(x)}{n} d\mu(x)\,.$$

For the next two problems, assume the setting and conclusions of the KSET as posed above.

Problem 27. Prove that f^* is μ-almost surely T-invariant.

Problem 28. Let T be an invertible ergodic measure-preserving transformation. Let (f_n) be a sub-additive sequence of functions over T: $f_{n+m}(\omega) \leq f_n(\omega) + f_m(T^n\omega)$.

- Let $g_n(\omega) = f_n(T^{-n}\omega)$. Prove that (g_n) is sub-additive over σ^{-1}: $g_{n+m}(\omega) \leq g_n(\omega) + g_m(T^{-n}\omega)$.
- Prove that $\lim_{n\to\infty} \frac{1}{n} g_n(\omega)$ exists a.e. and has the same limit as f^* a.e.
- What if T is not ergodic?

2.3 First passage percolation

Let $\mathcal{E}$ denote the collection of edges in the $\mathbb{Z}^2$ lattice. Let ν be a probability measure on $(0,\infty)$ (with $\int x\,d\nu(x) < \infty$).

Let $\Omega = (0,\infty)^{\mathcal{E}}$ be the collection of all weightings of $\mathcal{E}$ and equip X with the probability measure $\nu^{\mathcal{E}}$ (so that in a realization $\omega \in \Omega$, each edge is assigned a weight from the distribution ν independently of all other edges). Define a $\mathbb{Z}^2$ action, $\tau\mathbf{v}$ on Ω that translates the pattern of edge weightings through $-\mathbf{v}$.

Now for $\mathbf{v} \in \mathbb{Z}^2$, define $F_{\mathbf{v}}(\omega)$ to be the length of the shortest path from $\mathbf{0}$ to $\mathbf{v}$ (where the length of a path is the sum of the weights of the edges).

Problem 29. In this problem, we develop the application of the Kingman subbadditive ergodic theorem to the problem of first passage percolation on $\mathbb{Z}^2$.

(a) For $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$, $\omega \in \Omega$, prove that

$$F_{\mathbf{u}+\mathbf{v}}(\omega) \leq F_{\mathbf{u}}(\omega) + F_{\mathbf{v}}(\tau_{\mathbf{u}}(\omega)).$$

In particular, if $\mathbf{v}$ is any non-zero integer vector, then defining $\sigma = \tau_{\mathbf{v}}$ and $f_n(\omega) = F_{n\mathbf{v}}(\omega)$, (f_n) is a sub-additive sequence for the ergodic dynamical system $\sigma\colon \Omega \to \Omega$.

(b) Show that $\int (f_1(\omega))^+ d\nu(\omega) < \infty$.

(c) Show that ν is an ergodic invariant measure for $\sigma : \Omega \to \Omega$.

(d) Applying (b) and (c), verify that the Kingman subadditive ergodic theorem implies that the limit

$$\lim_{n\to\infty} \frac{f_n(\omega)}{n}$$

exists and converges ν-almost surely to the constant $g(\mathbf{v}) \in \mathbb{R} \cup \{-\infty\}$, where

$$g(\mathbf{v}) := \lim_{n\to\infty} \frac{1}{n} \int_{\Omega} F_{n\mathbf{v}}(\omega) d\nu(\omega)\,.$$

(e) Check that $g(\mathbf{v}) > 0$ for any $\mathbf{v} \in \mathbb{Z}^2 \setminus \{0\}$.

In the next problem, we prove some properties of the function g.

Problem 30.

(a) For $k \in \mathbb{Z}, \mathbf{v} \in \mathbb{Z}^2$, show that $g(k\mathbf{v}) = |k| g(\mathbf{v})$.

(b) Extend g to $\mathbb{Q}^2$ by defining $g(\mathbf{v}) = n^{-1} g(n\mathbf{v})$ when $n\mathbf{v} \in \mathbb{Z}^2$. Conclude from part (a) that this extension of g is well-defined, and that $g(c\mathbf{v}) = |c| g(\mathbf{v})$ for $c \in \mathbb{Q}, \mathbf{v} \in \mathbb{Q}^2$.

(c) Prove that $g(\mathbf{v} + \mathbf{w}) \le g(\mathbf{v}) + g(\mathbf{w})$.

(d) Conclude that g is continuous on $\mathbb{Q}^2$, hence admits a unique continuous continuation to all of $\mathbb{R}^2$. Check that g is positive-valued on all of $\mathbb{R}^2$.

2.4 Lyapunov exponents

Problem 31. Check using Problem 4 and the Kingman sub-additive ergodic theorem that the Lyapunov exponents for the inverse cocycle $B(\omega) = A(\sigma^{-1}\omega)^{-1}$ over σ^{-1} are $-\lambda_k, \dots, -\lambda_1$, with multiplicities $m_k, \dots, m_1$.

Problem 32. Check from the previous exercise and the Kingman sub-additive ergodic theorem that the Lyapunov exponents for the dual cocycle $C^{(n)}(\omega)$ over σ^{-1}, where $C(\omega) = A(\sigma^{-1}\omega)^T$ over σ^{-1} are the same as those for the cocycle $A_\omega^{(n)}$ over σ.

3 Guided proof of MET for a single 2×2 matrix.

Problem 33 (A warmup)**.** Why is the unstable manifold defined as $W^u(p) = \{x \colon d(T^{-n}x, p) \to 0 \text{ as } n \to \infty\}$? (that is: why are inverse powers of T used?) (Think about the map $T(x) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} x$ for a concrete example).

Problem 34. In this exercise, we will prove, in a roundabout way, the 'one-sided' Multiplicative Ergodic Theorem for a matrix $A \in M_{2\times 2}(\mathbb{R})$.

(a) Show that the limits

$$L_k = \lim_n \frac{1}{n} \log \sigma_k(A^n) \tag{1}$$

exist for each $1 \le k \le 2$. *Hint: use Problem 6, part (c) and Fekete's Lemma.*

(b) Assume that $L_1 > L_2$. For each n, let F_n denote the singular subspace corresponding to $\sigma_2(A^n)$, i.e., F_n is the (unique) 1-dimensional subspace of $\mathbb{R}^2$ for which

$$\|A^n|_{F_n}\| = \sigma_2(A^n)\,.$$

Show that the sequence of subspaces $\{F_n\}$ is Cauchy by obtaining a bound on $\|(\mathrm{Id} - \pi_{F_{n+1}})|_{F_n}\|$.

(c) Let F denote the limiting subspace as in (b). Show the following:

$$\text{For any } v \in F \setminus \{0\},\ \lim_{n\to\infty} \frac{1}{n} \log \|A^n v\| = L_2\,, \text{ and}$$
$$\text{for any } v \in \mathbb{R}^2 \setminus F,\ \lim_{n\to\infty} \frac{1}{n} \log \|A^n v\| = L_1\,.$$

In particular, observe that F is invariant, i.e., $AF \subset F$.

(d) Identify the collection of limits L_1, L_2 in terms of the eigenvalues of A. Identify F in terms of the eigenvectors of A.

At this point, we have proved the following.

Proposition 35 (One-sided MET for a single 2×2 matrix)**.** *Let $A \in M_{2\times 2}(\mathbb{R})$, and let L_1, L_2 be as in Problem 34. If $L_1 > L_2$, then there exists a subspace $F \subset \mathbb{R}^2$ for which (i) $AF \subset F$, (ii) for any $v \in F \setminus \{0\}$, we have $\lim_{n\to\infty} \frac{1}{n} \log \|A^n v\| = L_2$, and for any $v \in \mathbb{R}^2 \setminus F$, we have $\lim_{n\to\infty} \frac{1}{n} \log \|A^n v\| = L_1$.*

Problem 36. Let $A \in M_{2\times 2}(\mathbb{R})$ (i.e., A is a two-by-two matrix with real entries), and assume that A has two distinct eigenvalues λ_1, λ_2 for which $|\lambda_1| > |\lambda_2| > 0$.

(a) Determine $\lim_{n\to\infty} \frac{1}{n} \log \|A^n v\|$ for each $v \in \mathbb{R}^2$. Does your result change if $\| \cdot \|$ is replaced with any norm $| \cdot |$ on $\mathbb{R}^n$?

(b) Let E_1, E_2 denote the eigenspaces corresponding to λ_1, λ_2, respectively, and assume that $v \in \mathbb{R}^2 \setminus E_2$. Show that

$$\lim_{n\to\infty} \angle(A^n v, E_1) = 0 \,.$$

(c) Determine the value of

$$\lim_{n\to\infty} \frac{1}{n} \log \angle(A^n v, E_1) \,.$$

Problem 37. In this exercise, we prove, again in a roundabout way, the 'two-sided' MET for a matrix $A \in M_{2\times 2}(\mathbb{R})$. Let us assume that $L_1 > L_2$, where L_1, L_2 are as in Problem 34. For now, let us assume that A is invertible.

(a) Let $\hat{L}_1, \hat{L}_2$ denote the limits as in (1) with A^{-1} replacing A, and show that $\hat{L}_1 = -L_2$ and $\hat{L}_2 = -L_1$.

(b) Apply Problem 34, item (b) with A^{-1} replacing A, and let E denote the limiting subspace. Show that

$$\text{For any } v \in E \setminus \{0\}, \ \lim_{n\to\infty} \frac{1}{n} \log \|A^{-n} v\| = -L_1 \,, \text{ and}$$
$$\text{for any } v \in \mathbb{R}^2 \setminus E, \ \lim_{n\to\infty} \frac{1}{n} \log \|A^{-n} v\| = -L_2 \,.$$

In particular, observe that E is invariant, i.e., $AE = E$.

(c) Identify the subspace E in terms of the eigenvectors of A.

Problem 38. In this exercise, we prove in an alternative way the two-sided MET for a matrix $A \in M_{2\times 2}(\mathbb{R})$, this time without explicitly using the fact that A is invertible.

(a) Let $\bar{L}_1, \bar{L}_2$ denote the (possibly $-\infty$) limits as in (1) with A^T, the transpose of A, replacing A. Show that $\bar{L}_1 = L_1, \bar{L}_2 = L_2$.

(b) Apply Problem 34, item (b) with A^T replacing A, and let $\bar{F}$ denote the limiting subspace. Note that automatically, $\bar{F}^\perp$ is invariant under A. Show that $\bar{F}^\perp$ complements F (as in Problem 34).

(c) In the case when A is invertible, show that $E = \bar{F}^{\perp}$ with E as in Problem 37.

Problem 39. In this exercise, we show yet another way of obtaining the two-sided MET for a matrix $A \in M_{2\times 2}(\mathbb{R})$, again not explicitly using the fact that A is invertible.

(a) Let $L_1 > L_2$ be as in Problem 34, and for each n let E_n denote the one-dimensional subspace corresponding to $\sigma_1(A^n)$– i.e., E_n is the unique one-dimenisonal subspace with the property that $m(A^n|_{E_n}) = \min\{|A^n e| : e \in E_n, \|e\| = 1\} = \sigma_1(A^n)$. Letting $E'_n = A^n E_n$, show that $\{E'_n\}$ is a Cauchy sequence.

(b) Let $\bar{E}$ denote the limiting subspace from part (a). Show that $\bar{E}, F$ are complements, and that $A\bar{E} = \bar{E}$.

(c) Show that $\bar{E}$ coincides with the subspace E as in Problem 37 when A is invertible.

4 The projective Markov chain and Lyapunov exponents

Throughout these exercises we refer to the following construction.

Definition 40. Let T be an almost-surely invertible random variable on $M_{2\times 2}(\mathbb{R})$ and consider the IID matrix product $\{T_n\}$ distributed like T. Let $\{\psi_n\}_n$ denote the Markov chain on the projective space $P^1 \cong [0, \pi)$ of $\mathbb{R}^2$ for which

$$\overline{u}_{\psi_{n+1}} = \overline{T_n u_{\psi_n}}\,;$$

Here, we write $u_\psi = (\cos\psi, \sin\psi)$, and for $v \in \mathbb{R}^2 \setminus \{0\}$ we write $\bar{v} \in P^1$ for the equivalence class of v. The Markov chain $\{\psi_n\}_n$ is referred to as the *projective Markov chain* for the matrix product $\{T_n\}$.

The transition probabilities $P(\psi, \cdot)$ for $\psi \in P^1 \cong [0, \pi)$ for the projective Markov chain are given by

$$P(\psi, A) = \mathbb{P}(\overline{T u_\psi} \in A)\,.$$

for Borel $A \subset P^1$. Recall that a probability measure ν on P^1 is *stationary* when

$$\nu(A) = \int_{P^1} d\nu(\psi) P(\psi, A)$$

for any Borel $A \subset P^1$.

Problem 41. Let $\{\psi_n\}$ be the projective Markov chain for an IID matrix product $\{T_n\}$. Prove the following Lemma.

Lemma 42.

(a) Let ν be any stationary measure for the Markov chain $\{\psi_n\}$ on P^1, which we regard as a measure on $[0, \pi)$. Then,

$$\lambda_1 \geq \int \log \|T u_\psi\| d\nu(\psi) d\mathbb{P}(T)\,.$$

(b) If ν is absolutely continuous with respect to Lebesgue measure on P^1, then

$$\lambda_1 = \int \log \|T u_\psi\| d\nu(\psi) d\mathbb{P}(T)\,.$$

Hint: for (a), use the Birkhoff ergodic theorem. For (b), use the Multiplicative Ergodic Theorem.

5 Computing LE of simple matrix cocycles

Below, for $L > 0$, we set

$$H_L = \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix}$$

and for $\theta \in [0, 2\pi)$,

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Problem 43. Define the random matrix T by setting $T = R_{\pi/2}$ with probability $p \in (0, 1]$, and $T = H_2$ with probability $1 - p$. Let $\{T_n\}$ be an IID sequence with the same law as T. Show that the Lyapunov exponents of this random matrix product are both zero.

Problem 44. Define the random matrix T by setting $T = R_{2\pi/3}$ with probability $p \in (0, 1]$, and $T = H_4$ with probability $1 - p$. Let $\{T_n\}$ be an IID sequence with the same law as T. Show that the top Lyapunov exponent of this random matrix product is positive when p is sufficiently small. *Hint: look for an invariant subset of projective space under the actions of both* $R_{2\pi/3}$ *and* H_4.

Problem 45. Define the random matrix T by setting $T = R_\theta$ with probability $p \in (0, 1]$, where θ is distributed uniformly in $[0, 2\pi)$, and $T = H_2$ with probability $1 - p$. Let $\{T_n\}$ be an IID sequence with the same law as T. Show that for p sufficiently small, the top Lyapunov exponent of this random matrix product is positive. *Hint: consider the orientation of the vector* $T^n(1, 0)$ *after each time a rotation is sampled.*

Problem 46. Consider the random matrix product of the form $T = R_\theta H_L$, where $L > 0$ is a large fixed constant and θ is distributed according to an absolutely continuous law on $[0, 2\pi)$ with a bounded density ρ.

(a) Let ν be any stationary measure for the associated projective Markov chain. Show that ν is absolutely continuous, and bound the density of ν in terms of the bound on the density of ρ.

(b) Apply Lemma 42 to obtain a lower bound for λ_1 in terms of L. Conclude that $\lambda_1 > 0$ for L sufficiently large. *Hint:* $P^1 = \{|\psi - \pi/2| > \epsilon\} \cup \{|\psi - \pi/2| \leq \epsilon\}$.

Problem 47. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ denote rotation by an irrational angle $\alpha/2\pi$; parametrizing $\mathbb{S}^1 = [0, 1)$, this means that $f(x) = x + \alpha (\mathrm{mod}\, 1)$. Define the cocycle $A : \mathbb{S}^1 \to M_{2\times 2}(\mathbb{R})$ by $A_x = R_{2\pi x}$ for $x \in [0, 1)$. Does there exist an equivariant subspace of $\mathbb{R}^2$ for this cocycle?