Problem 1. If $X$ is a topological space, a vector bundle over $X$ is a function $V : X \to \text{Gr}(d, k)^1$. The fiber over $x \in X$ is the space $V_x = V(x)$. A section of a vector bundle is a map $v : X \to \mathbb{R}^d$ such that $v(x) \in V_x$ for every $x \in X$.

(i) Show that if at each point $x \in X$, $\mathbb{R}^d = E^s_x \oplus E^u_x$, then any $\mathbb{R}^d$-valued function $\delta : X \to \mathbb{R}^d$ can be written as $\delta = \delta^s + \delta^u$, where $\delta^s$ is section of the vector bundle $E^s$.

(ii) Show that if $V$ is continuous, then $\mathcal{B} = \{(x, v) : v \in V(x)\}$ is a closed subspace of $X \times \mathbb{R}^d$ ($\mathcal{B}$ is called the total space) and the projection $\pi : \mathcal{B} \to X$ defined by $\pi(x, v) = x$ has $\pi^{-1}(x) = V_x$.

(iii) A bundle $V$ is called invariant under a dynamical system $f : X \to X$ and a cocycle $A_x$ if $A_x V(x) \subset V(f(x))$. If $T : \mathcal{B} \to \mathcal{B}$ is defined by $T(x, v) = (f(x), A_x v)$, show that $T$ is well-defined, continuous, and that $T^n(x, v) = (f^n(x), A_x^n(v))$.

(iv) Show that if $V = V^1 \oplus V^2$ is a sum of two subbundles, $V^1$ and $V^2$ are invariant under $f$ and $A$, and $\pi^i_x : V_x \to V^i_x$ is the projection of $V$ onto $V^i$, then $A_x \circ \pi^i_x = \pi^i_{f(x)} \circ A_x$.

(v) Assume that $\mathbb{R}^d = E^s \oplus E^u$, $A_x$ is invertible and $\|A_x|_{E^s}\| < \lambda < 1$ and $\|A_x^{-1}|_{E^u}\| < \lambda < 1$. Let $B_x$ be a cocycle such that $\|A_x - B_x\| < \varepsilon$. If $V$ is a vector bundle, let $C(V, \delta) = \{v : \angle(v, V) < \delta\}$.

(a) Show that for sufficiently small $\varepsilon, \delta B_x(C(E^s_x, \delta)) \subset C(E^s_{f(x)}, \delta)$

(b) Show that if $d_{C^0}(f, g)$ is sufficiently small, then the same property holds.

(c) Show that $\bigcap_{n \leq 0} B^{-n}_x(C(E^s_{g^n(x)}, \delta))$ is a subspace of dimension $\dim E^s$.

Problem 2. Let $\Sigma_d$ be the space of (2-sided) sequences on the alphabet $\{1, \ldots, d\}$ equipped with the metric:

$$d((x_n), (y_n)) = \sum_{n=-\infty}^{\infty} 2^{-|n|}\delta(x_n, y_n)$$

where $\delta(a, b) = 1$ if $a = b$ and 0 otherwise. Show that the shift $\sigma : \Sigma_d \to \Sigma_d$ satisfies the same shadowing property and Anosov closing lemma proved in lecture for hyperbolic sets.
Problem 3 (Invertible Extensions). Given \( f : Y \to Y \) a continuous dynamical system on a compact metric space \( Y \), we construct a compact metric space \( X_f \) as:

\[
X_f = \{ (\ldots, x_{-1}, x_0, x_1, \ldots) : x_{i+1} = f(x_i) \} \subset Y^\mathbb{Z}
\]

(i) Show that \( X_f \) is a closed, \( \sigma \)-invariant (and hence compact) subset of \( Y^\mathbb{Z} \)

(ii) Show that \( f \) is a factor \( \sigma : X_f \to X_f \) (ie, there is a semiconjugacy \( h : X_f \to Y \))

(iii) Show that \( f \) and \( \sigma \) are topologically conjugate if \( f \) is invertible

(iv) (*) Let \( f = E_2 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \). Show that \( \sigma \) is topologically conjugate to the Smale-Williams solenoid. An exact formula for a Smale-Williams Solenoid system \( f : D^2 \times S^1 \to D^2 \times S^1 \) is:

\[
f(x, y) = \left( \frac{1}{10} x + \frac{1}{2} i(y), 2y \right)
\]

Where \( i : S^1 \to D^2 \) is the inclusion of \( S^1 \) to the boundary. \([Hint: \text{Construct a homeomorphism } h_n \text{ from the space of sequences starting from index } -n \text{ to } f^n(\{0\} \times S^1) \text{ such that in the second coordinate, } h_{n-1} \circ \sigma = f \circ h_{n-1}. \text{ Then take a limit in } C^0(X_f, D^2 \times S^1)]\)