# EXERCISES FOR PARTIALLY HYPERBOLIC SYSTEMS

(1) Let $f : M \to M$ be a diffeomorphisms of a compact manifold and $\Lambda \subset M$ be a compact partially hyperbolic set for $f$. If $U$ is a sufficiently small neighborhood of $\Lambda$, then the set

$$\Lambda_U = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U})$$

is partially hyperbolic.

(2) Let $M$ be a compact manifold and $f : M \to M$ be a partially hyperbolic diffeomorphism. If $g : M \to M$ is a diffeomorphism that is sufficiently $C^1$ close to $f$, then $g$ is partially hyperbolic.

(3) Let $f : M \to M$ be a partially hyperbolic diffeomorphism with 1-dimensional center and $M$ be compact. In this case we know that locally the center direction can be integrated to give a not necessarily unique local center manifold. The $\epsilon$-Bowen ball at $x \in M$ for $f$ is

$$B_\epsilon(x, f) = \{y \in M \; : d(f^n x, f^n y) < \epsilon \forall n \in \mathbb{Z}.$$

For $\epsilon$ sufficiently small if there exist 2 local center manifolds at $x$, then they agree on the Bowen ball.

(4) For a partially hyperbolic diffeomorphism $f : M \to M$ of a compact manifold. Suppose there is an adapted metric and constants $\lambda_1 < \mu_1 \le \mu_2 < \lambda_2$ for the splitting $E^s \oplus E^c \oplus E^u$. Then the following are equivalent:
- $W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(W^s_{\text{loc}}(f^n x))$
- $W^s(x) = \{y \in M \; : \; d_s(f^n x, f^n y) \le (\lambda_1 + \epsilon)^n d_s(x, y)$ where $\epsilon > 0$ is sufficiently small and $d_s(\cdot, \cdot)$ is the induced distance on the immersed submanifold $W^s(x)$.

(5) Assume that $\Lambda$ is a partially hyperbolic topological attractor. Show that the set $\Lambda$ is saturated by strong unstable manifolds $W^u$ and that for each $x \in \Lambda$ we have $W^u(x) \subset \Lambda$.

(6) Let $f : M \to M$ be a $C^1$-diffeomorphsim and $\Lambda \subset M$ be a compact partially hyperbolic set with a splitting $E^s \oplus E^c \oplus E^u$. Show that if $\mu$ is an ergodic measure that has a simple spectrum (so all Lyapunov exponents have multiplicity one), then the Oseledet's splitting respects the splitting $E^s \oplus E^c \oplus E^u$ (so every bundle is contained either on $E^s$, $E^c$, or $E^u$). Show more generally that if there is a continuous invariant splitting $E \oplus F$, then $E$ is uniformly contracted if and only if for every ergodic measure $\mu$ supported on $\Lambda$ the larges Lyapunov exponent of $\mu$ along $E$ is negative.

(7) Consider the Heisenberg group of $3 \times 3$ matrices

$$\mathcal{H} := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

This group is closed under matrix products and therefore is a Lie group with Lie algebra $\mathfrak{h}$ that is generated by the matrices

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the only non-trivial bracket relation is $[X, Y] = XY - YX = Z$. Consider the lattice $\mathcal{H}_2$ consisting of the matrices

$$\mathcal{H}_2 := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y \in \mathbb{Z}, z \in \frac{1}{2}\mathbb{Z} \right\}.$$

Show that $N_k = \mathcal{H}/\mathcal{H}_2$ is compact. Prove that $(x, y, z) \mapsto (2x + y, x + y, z + x^2 + \frac{y^2}{2} + xy)$ is a partially hyperbolic diffeomorphism of $N_2$. What are the invariant bundles? Show that the center direction integrates into a foliation by circles.

(8) Let $f$ be a partially hyperbolic diffeomorphism of the 3-torus that preserves Lebesgue measure. Suppose that $f$ is ergodic and that the center direction has a foliation by $C^2$ circles. Suppose that the Lyapunov exponent in the center direction is positive. Then there exists a set $S$ of full measure on the 3-torus that meets every leaf of the center foliation in a set of leave-measure 0. In particular, the center foliation is not absolutely continuous.