Summer School in Dynamical Systems 2017
Applications of dynamics to Diophantine approximation

(Final exam in class on Thursday, May 25)

1. Prove that if $x \in \mathbb{R} \setminus \mathbb{Q}$ then $\{nx \bmod 1 : n \in \mathbb{N}\}$ is dense in $\mathbb{R}/\mathbb{Z}$.

2. Prove that a semigroup $\Sigma \subseteq \mathbb{N}$ is non-lacunary if and only if it contains a pair of multiplicatively independent integers $a, b \geq 2$.

3. Let $\Sigma = \{n_1 < n_2 < \cdots\}$ be the multiplicative subsemigroup of $\mathbb{N}$ generated by 2 and 3. Prove that there exists a constant $C > 0$ with the property that, for all $i \geq 1$,
$$n_{i+1} - n_i \leq \frac{Cn_i}{\log n_i}.$$

4. For $a \in \mathbb{Z},\ a \geq 2$, let $T_a : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by $T_a(x) = ax \bmod 1$. What are the possible values of $h_\mu(T_a)$, as μ runs over all measures in $\mathcal{M}_{T_a}$?

5. Let $\mu \in \mathcal{M}(\mathbb{R}/\mathbb{Z})$. Prove that if $\widehat{\mu}(n) = 0$ for all nonzero integers n, then μ is Lebesgue measure.

6. Show that, for any real number $\alpha \in [0, 1]$, there is a set $X \subseteq \mathbb{R}$ with $\dim(X) = \alpha$.

7. Suppose that p is a prime number, let $\mathbb{Q}_p$ denote the field of p-adic numbers, and let $Y_p = \mathbb{R} \times \mathbb{Q}_p$, with the product topology.

   a) Let $\iota : \mathbb{Z}[1/p] \to Y_p$ be the diagonal embedding, defined by $\iota(\gamma) = (\gamma, \gamma)$. Prove that $\Gamma = \iota(\mathbb{Z}[1/p])$ is a lattice in Y_p. In other words, prove that Γ is discrete and that the quotient $X_p = Y_p/\Gamma$ has a measurable fundamental domain with finite Haar measure.

   b) Show that a fundamental domain for X_p can be identified bijectively with the Cartesian product $[0, 1) \times \mathbb{Z}_p$. As an inoculation against potential confusion, explain why, as a group, X_p is not isomorphic to the direct product of $\mathbb{R}/\mathbb{Z}$ with $\mathbb{Z}_p$.

   c) Let $T_p : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by $T_p(x) = px \bmod 1$. Prove that the invertible extension of T_p is the map $\widetilde{T}_p : X_p \to X_p$ defined by
   $$\widetilde{T}_p((x_\infty, x_p) + \Gamma) = (px_\infty, px_p) + \Gamma.$$