1. Prove that if \( x \in \mathbb{R} \setminus \mathbb{Q} \) then \( \{nx \mod 1 : n \in \mathbb{N}\} \) is dense in \( \mathbb{R}/\mathbb{Z} \).

2. Prove that a semigroup \( \Sigma \subseteq \mathbb{N} \) is non-lacunary if and only if it contains a pair of multiplicatively independent integers \( a, b \geq 2 \).

3. Let \( \Sigma = \{n_1 < n_2 < \cdots \} \) be the multiplicative subsemigroup of \( \mathbb{N} \) generated by 2 and 3. Prove that there exists a constant \( C > 0 \) with the property that, for all \( i \geq 1 \),
\[
n_{i+1} - n_i \leq \frac{Cn_i}{\log n_i}.
\]

4. For \( a \in \mathbb{Z}, a \geq 2 \), let \( T_a : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) be defined by \( T_a(x) = ax \mod 1 \). What are the possible values of \( h(\mu(T_a)) \), as \( \mu \) runs over all measures in \( \mathcal{M}_{T_a} \)?

5. Let \( \mu \in \mathcal{M}(\mathbb{R}/\mathbb{Z}) \). Prove that if \( \hat{\mu}(n) = 0 \) for all nonzero integers \( n \), then \( \mu \) is Lebesgue measure.

6. Show that, for any real number \( \alpha \in [0,1] \), there is a set \( X \subseteq \mathbb{R} \) with \( \dim(X) = \alpha \).

7. Suppose that \( p \) is a prime number, let \( \mathbb{Q}_p \) denote the field of \( p \)-adic numbers, and let \( Y_p = \mathbb{R} \times \mathbb{Q}_p \), with the product topology.
   a) Let \( \iota : \mathbb{Z}[1/p] \to Y_p \) be the diagonal embedding, defined by \( \iota(\gamma) = (\gamma, \gamma) \). Prove that \( \Gamma = \iota(\mathbb{Z}[1/p]) \) is a lattice in \( Y_p \). In other words, prove that \( \Gamma \) is discrete and that the quotient \( X_p = Y_p/\Gamma \) has a measurable fundamental domain with finite Haar measure.
   b) Show that a fundamental domain for \( X_p \) can be identified bijectively with the Cartesian product \( [0,1) \times \mathbb{Z}_p \). As an inoculation against potential confusion, explain why, as a group, \( X_p \) is not isomorphic to the direct product of \( \mathbb{R}/\mathbb{Z} \) with \( \mathbb{Z}_p \).
   c) Let \( T_p : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) be defined by \( T_p(x) = px \mod 1 \). Prove that the invertible extension of \( T_p \) is the map \( \tilde{T}_p : \mathbb{R} \to \mathbb{R} \) defined by
\[
\tilde{T}_p((x_\infty,x_p) + \Gamma) = (px_\infty,px_p) + \Gamma.
\]