Homogeneous Dynamics Exercises

1 Some Hyperbolic Geometry and Mautner's phenomena

In this tutorial we review some hyperbolic geometry and guide the readers through a proof of the Mautner phenomena.

Let $\mathbb{H} := \{z = x+iy : y > 0\}$ denote the upper half plane and equip each tangent plane $T_z\mathbb{H} = \mathbb{C}$ with the inner product $\langle v, w\rangle_z = (v \cdot w)\frac{1}{y^2}$ where $z = x + iy \in \mathbb{H}$ and $v, w \in T_z\mathbb{H}$ (by $v \cdot w$ we mean the usual dot product on $\mathbb{C}$).

For a continuous piecewise differentiable curve $\phi : [0, 1] \to \mathbb{H}$ we define its hyperbolic arc length as

$$L(\phi) := \int_0^1 \|D\phi(t)\|_{\phi(t)} dt = \int_0^1 \frac{1}{Im(\phi(t))} \sqrt{\phi'(t) \cdot \phi'(t)} dt$$

and we use the arc length to define a distance on $\mathbb{H}$ as $d_{\mathbb{H}}(z, z') = \inf_\phi L(\phi)$ for points $z, z' \in \mathbb{H}$ where the infimum is taken over paths ϕ that start at z and end at z'.

This metric on $\mathbb{H}$ has a huge group of isometries that we can identify with $SL_2(\mathbb{R})$. We do this by recalling the action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ by Mobius transformations. Namely, for a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathbb{H}$ define $g \cdot z = \frac{az+b}{cz+d}$. (Notice that there is some duplication with this action, by which we mean we have that $g \cdot z = (-g) \cdot z$. For this reason, we consider $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$.)

Exercise 1.

1. Show that $PSL_2(\mathbb{R})$ acting on $\mathbb{H}$ by Mobius transformations is an isometric and transitive action. (To show it is isometric it suffices to show that Dg preserves the inner product.)

2. Consider the tangent bundle $T\mathbb{H} := \cup_{z\in\mathbb{H}} T_z\mathbb{H} = \{(z, v) : z \in \mathbb{H}, v \in T_z\mathbb{H}\}$. Show $PSL_2(\mathbb{R})$ extends to an action on the tangent bundle by the derivative action. That is, $PSL_2(\mathbb{R}) \curvearrowright T\mathbb{H}$ by $Dg \cdot (z, v) = (g \cdot z, g'(z)v) = \left(\frac{az+b}{cz+d}, \frac{1}{(cz+d)^2}v\right)$.

3. Consider the unit tangent bundle $T^1\mathbb{H} := \{(z, v) \in T\mathbb{H} : \|v\| = 1\}$. Show that $PSL_2(\mathbb{R}) \curvearrowright T^1\mathbb{H}$ is a simply transitive action. Conclude that $T^1\mathbb{H} = PSL_2(\mathbb{R})$ by the Orbit-Stabilizer Theorem. This correspondence allows us to turn questions of geometry in $T^1\mathbb{H}$ into questions of linear algebra of $PSL_2(\mathbb{R})$ and vice-versa.

Exercise 2. In this exercise we consider some useful generating sets for $SL_2(\mathbb{R})$ that come in handy.

1. Consider $u_s^+ = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and $u_s^- = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ for $s \in \mathbb{R}$.

   Show that $\langle u_s^+, u_s^-\rangle = SL_2(\mathbb{R})$. That is, that every $g \in SL_2(\mathbb{R})$ can be written as a product of of upper and lower unipotent matrices. (Hint: think of multiplying by u^+ and u^- as row reduction. For example, multiplying on the left by u^+ corresponds to row reducing on the top row and multiplying on the right by u^- corresponds to row reducing the left column.)

2. Show $\langle u^+, w\rangle = SL_2(\mathbb{R}) = \langle u^-, w\rangle$ where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3. Show that $SL_2(\mathbb{R}) = [SL_2(\mathbb{R}), SL_2(\mathbb{R})]$ where the right-hand side is the subgroup generated by elements of the form $ghg^{-1}h^{-1}$. (Hint: What is $g_t \cdot u_s^+ \cdot (g_t)^{-1}$ where $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$?)

4. (Optional) Show that $SL_2(\mathbb{R})$ is unimodular using the last characterization of $SL_2(\mathbb{R})$.

Exercise 3. In this exercise we outline how to prove the Mautner phenomena. There are various formulations of this and we state the following special case of it.

Theorem 1. *(Mautner phenomena) Suppose $SL_2(\mathbb{R})$ acts in a measure preserving way on a probability space (X, μ). Notice then that $SL_2(\mathbb{R})$ acts on $L^2(X, \mu)$ via $(g \cdot \phi)(x) := (\phi \circ g^{-1})(x) = \phi(g^{-1}x)$ where $g \in SL_2(\mathbb{R})$, $\phi \in L^2(X, \mu)$, and $x \in X$. Then any function ϕ invariant under the upper unipotent matrix group $U = \{u_s^+ : s \in \mathbb{R}\}$ is invariant under $SL_2(\mathbb{R})$.*

Invariant under U means $u \cdot \phi = \phi$ for every $u \in U$. We prove this by consider the continuous function $\rho : SL_2(\mathbb{R}) \to \mathbb{C}$ given by $\rho(g) = \langle g \cdot \phi, \phi \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(X, \mu)$.

1. Show that $\rho(ugu') = \rho(g)$ for all $g \in SL_2(\mathbb{R})$ and $u, u' \in U$.

2. Show that ρ is invariant under g_t for all $t \in \mathbb{R}$. (Hint: What is the product $u_r^+ \cdot u_\varepsilon^- \cdot u_r^+$ where $r, s, \varepsilon \in \mathbb{R}$? Calculate this product and the substitute $r = \frac{e^t - 1}{\varepsilon}$ and $s = \frac{-r}{1 + r\varepsilon}$.)

3. Show that ρ is invariant under u_s^- for all $s \in \mathbb{R}$. Conclude that ρ is invariant under $SL_2(\mathbb{R})$.

4. Show in this case that if μ is ergodic with respect to $SL_2(\mathbb{R})$, then it must be ergodic with respect to U.

2 $SL_2(\mathbb{Z})$ is a lattice in $SL_2(\mathbb{R})$

In this tutorial we show $SL_2(\mathbb{Z})$ is a lattice in $SL_2(\mathbb{R})$. Recall, a lattice $\Gamma < SL_2(\mathbb{R})$ is a discrete subgroup such that $SL_2(\mathbb{R})/\Gamma$ carries a finite $SL_2(\mathbb{R})$-invariant measure. We prove this by finding a fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ with finite volume and then lifting this to a finite volume fundamental domain on $T^1\mathbb{H}$ which we can identify with $SL_2(\mathbb{R})$ by virtue of exercise 1 from the last section. For completeness, we recall these definitions.

Definition 1. *We call a subset D of $SL_2(\mathbb{R})$ discrete if for every $x \in D$ has that the only sequences $(x_n) \subset D$ with $x_n \to x$ are the sequences that are eventually constant.*

Definition 2. *Let X be a locally compact space and G act on X by homeomorphisms. An open subset F of X is a fundamental domain for the action of G if*

1. *$\cup_{\gamma \in G} \gamma \overline{F}$, where $\overline{F}$ is the closure of F, and*

2. *$\gamma_1 F \cap \gamma_2 F = \emptyset$, for all $\gamma_1, \gamma_2 \in G, \gamma_1 \neq \gamma_2$.*

Rougly speaking, a fundamental domain is how we 'see" a quotient X/G. Think about the case $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ by translations. Then we often 'see" the resulting quotient $\mathbb{R}^2/\mathbb{Z}^2$ by instead drawing the open unit square in $\mathbb{R}^2$ and remembering that we identify the top edge with the bottom edge and the left edge with the right edge. This works because when we move the unit square around by $\mathbb{Z}^2$ we tile $\mathbb{R}^2$ and these translates never overlap, which is to say the open unit square in $\mathbb{R}^2$ is a fundamental domain for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$!

Exercise 1. Show that $SL_2(\mathbb{Z})$ is discrete in $SL_2(\mathbb{R})$. (Hint: consider the point $p = ni \in \mathbb{H}$ where $n > 1$. Show $\gamma \cdot p \neq p$ for all $\gamma \in SL_2(\mathbb{Z})$ with $\gamma \neq Id$. Why is this enough?)

Recall that we can equip $\mathbb{H}$ with a notion of hyperbolic area via the hyperbolic metric. That is, for $A \subseteq \mathbb{H}$, define

$$m(A) = \int_A \frac{1}{y^2} dxdy.$$

Exercise 2. Consider the subset $F' = \{z = x + iy : |z| > 1, |x| < 1/2\}$. Show that F' has finite area with respect to this area form.

Let us take for granted that F' is a fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$. Then by "throwing in all unit vectors" to each point $x \in F'$, we get a subset of $T^1\mathbb{H}$. That is, consider $F = \{(z, v) : z \in F', v \in T_z\mathbb{H}\}$.

Exercise 3. Show $F = \{g \in SL_2(\mathbb{R}) : g \cdot i \in F'\}$ and that this forms a fundamental domain for the action of $SL_2(\mathbb{Z})$ on $SL_2(\mathbb{R})$.

This shows $SL_2(\mathbb{Z})$ is a lattice in $SL_2(\mathbb{R})$.

3 Mahler's Compactness Criterion

In this tutorial we give a proof of Mahler's compactness and get acquainted with the space of unimodular lattices which we denote by X_d.

Exercise 1.

1. Show that the action of $SL_d(\mathbb{R})$ on X_d is transitive.
2. Compute the stabilizer of $\mathbb{Z}^d$. Conclude that we can identify X_d with $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$.

Hence, we can equip X_d with the quotient topology of $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$. The next exercise gives a more concrete description of this topology.

Exercise 2.

Consider X_d with the following topology: Say $\Lambda_n \to \Lambda$ in X_d if there is a basis $\{b_1^{(n)}, \cdots, b_d^{(n)}\}$ of Λ_n and $\{b_1, \cdots, b_d\}$ of Λ such that $\lim_n b_i^{(n)} \to b_i$ as $n \to \infty$ for $i = 1, \cdots, d$.

Show that this topology coincides with the topology inherited from $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$. (Hint: It suffices to show that $f : SL_d(\mathbb{R})/SL_d(\mathbb{Z}) \to X_d$ given by $gSL_d(\mathbb{Z}) \mapsto g\mathbb{Z}^d$ is continuous and X_d with the new topology is locally compact and separable. This shows f is a homeomorphism.)

Now we state Mahler's Compactness Criterion which gives a characterization of what (relatively) compact subsets of X_d "look like". It should be no surprise that this criterion is based on properties of the points of the lattice.

Theorem 2. *(Mahler's Compactness Criterion) A subset K of X_d is relatively compact if and only if non-zero vectors of Λ are uniformly bounded away from 0 for all $\Lambda \in K$. That is, if there is $s > 0$ such that $\Lambda \cap B_s(0) = \{0\}$ for every $\Lambda \in K$.*

Exercise 3. In this exercise we prove the forward direction of Mahler's Compactness Criterion as follows. Suppose for the sake of contradiction that for every $n \geq 1$, there is $\Lambda_n \in K$ with $\Lambda_n \cap B_{1/n}(0) \neq \{0\}$. That is, there is $x_n \in \Lambda_n \setminus \{0\}$ such that $\lim_n x_n = 0$. By calling $\Lambda_n = g_n\mathbb{Z}^d$, this last condition says $x_n = g_n y_n \to 0$ with $y_n \in \mathbb{Z}^d \setminus \{0\}$.

1. Show there is a sequence $(\gamma_n) \in SL_d(\mathbb{Z})$ and $g \in SL_d(\mathbb{R})$ such that $\lim_n g_n\gamma_n = g$.
2. Show $\lim_n \gamma_n^{-1} g_n = 0$ and hence we have a contradiction since $\gamma_n^{-1} g_n \in \mathbb{Z}^d \setminus \{0\}$ for all $n \in \mathbb{N}$.

Exercise 4.. In this exercise we prove the reverse direction.

Suppose that there is $s > 0$ such that $\Lambda \cap B_s(0) \neq \{0\}$ for every $\Lambda \in K$. Let $b_1 \in \Lambda$ be such that $\|b_1\| = \min\{\|b\| : b \in \Lambda \setminus \{0\}\}$. Note that $s \leq \|b_1\|$ by assumption and by Minkowski's convex body theorem we have $\|b_1\| \leq 2^d + 1$ which is a constant depending only on the dimension. Let $W = (\mathbb{R}b_1)^\perp$ and $\pi_W : \mathbb{R}^d \to W$ be orthogonal projection onto W. Consider the $(d-1)$-dimensional lattice $\Lambda_W := \pi_W(\Lambda)$ (which after rescaling we can assume is also unimodular.)

1. Prove by way of contradiction that all non-zero vectors in Λ_W are uniformly bounded away from 0. Call this uniform bound s'. How does s' relate to s?

2. Suppose by induction that for Λ_W we have basis $\{y_2, \ldots, y_d\}$ such that $s' \leq \|y_i\| \leq \kappa_{d-1}$. Prove that we can choose $\{b_2, \ldots, b_d\}$ (to complete a basis for Λ) such that the b_i satisfy $\|b_i\| \leq \sqrt{\|\pi_W(b_i)\|^2 + \|b_1\|^2}$ for $i \geq 2$ and $b_i \in \pi_W^{-1}(y_i)$. Note that for each i we have a uniform bound $\sqrt{|\pi_W(b_i)\|^2 + \|b_1\|^2} \leq C_d$, where C_d is only dependent on the dimension d.

Now, for any sequence $\Lambda_n \in K$, which we write as $\Lambda_n = g_n\mathbb{Z}^d$, we have the columns of g_n are uniformly bounded from below (by s) and above (by a constant depending only on the dimension d.) That is, if the columns of g_n are $b_1^{(n)}, b_2^{(n)}, \ldots, b_d^{(n)}$, we have shown

$$s \leq \|b_i^{(n)}\| \leq C_d$$

where $i = 1, \cdots, d$ for *every* $n \in \mathbb{N}$.

3. Finish the proof by showing the g_n belong to a fixed compact set of $SL_d(\mathbb{R})$ and hence K is relatively compact.