2018 Houston Summer School on Dynamical Systems

Problem set: Review of Ergodic Theory

1. Define the doubling map $T(x) = 2x \mod 1$ on the unit interval with the Borel σ-algebra and Lebesgue measure.
(a) Show that T preserves Lebesgue measure.
(b) Show that T is isomorphic to the full one-sided shift on 2 symbols with the corresponding Borel σ-algebra and Bernoulli measure with $p_1 = p_2 = 1/2$.

2. Chacón's transformation will be defined in terms of a cutting-and-stacking construction during lecture. Use the construction to argue the following:
(a) The transformation is invertible.
(b) The transformation preserves Lebesgue measure.
(c) The transformation has continuous spectrum (which is equivalent to weak-mixing).
(d) The transformation is not strong-mixing.

3. Let $n \in \mathbb{N}$ and let X be the topological space with n elements, taken with the discrete topology. Any permutation $\sigma \in S_n$ is then a continuous map from X to itself. Describe, in terms of σ, the space $\mathcal{M}^\sigma(X)$ and the subset $\mathcal{E}^\sigma(X)$.

4. Let $X = [0, 1)$ and let $T : X \to X$ be the doubling map, defined above. Find a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of distinct elements of $\mathcal{E}^T(X)$ which converges in the weak-$\star$ topology to Lebesgue measure on X.

5. Let T be a continuous transformation on a compact metric space X. Prove that any two distinct measures in $\mathcal{E}^T(X)$ are mutually singular.

6. Let X be a compact metric space and $T : X \to X$ a continuous map. Prove that, for any $f \in C(X)$ and for any $x \in X$,

$$\inf_{\mu\in\mathcal{E}^T(X)} \int f \, \mathrm{d}\mu \;\leq\; \liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x).$$

7. (Harder) Give an example of a compact metric space X and a continuous map $T : X \to X$, for which the set $\mathcal{E}^T(X)$ is not a closed subset of $\mathcal{M}^T(X)$.