2018 Houston Summer School on Dynamical Systems

Problem set: Homogeneous dynamics and number theory

Quadratic forms

1. (Classification of quadratic forms)
   Recall that a real quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ can be expressed as

   $$Q(x_1, \dots, x_n) = (x_1, \dots, x_n)A_Q(x_1, \dots, x_n)^t,$$

   for some symmetric matrix A_Q.

   (a) Use the symmetry of A_Q to conclude that there is a coordinate system for which $Q(x_1, \dots, x_n) = \sum c_i y_i^2$, where c_i are non-increasing.

   (b) Find the matrix corresponding to $Q(x_1, x_2) = x_1 x_2$ and express the quadratic form in the standard form (i.e. as in part (a)).

2. (a) Let $Q_0(x_1, \dots, x_n) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$. Describe the isometry group

   $$SO(p, n-p) := SO(Q_0) = \{A \in SL_n(\mathbb{R}) : Q_0(Ax) = Q_0(x)\}.$$

   (b) Let p, q be the number of positive and negative c_i's, respectively. We call (p, q) the signature of the quadratic form. Suppose that Q is indefinite and non-degenerate, i.e. $p, q \geq 1$ and $p + q = n$. Describe the isometry group $SO(Q)$ in terms of $SO(p, q)$.

3. Show that $SO(2, 1)$ is locally isomorphic to $SL_2(\mathbb{R})$. Are they isomorphic?

4. Let $Q(x, y) = y^2 - a^2x^2$ where a is a badly approximable number, say, there exists $C > 0$ such that

   $$\inf_{p,q \in \mathbb{Z}, q \neq 0)} |q||qa - p| > C.$$

   Show that

   $$\inf_{(x,y) \in \mathbb{Z}^2 - \{(0,0)\}} |Q(x, y)| > 0.$$

   (This shows that the condition "Q is of $n \geq 3$ variables" cannot be removed from the Oppenheim conjecture. Note that an algebraic number of degree 2 is badly approximable.)

5. Let $Q_0 = 2x_1x_3 - x_2^2$ and $H = SO(Q_0)$.

   (a) Compute the Lie algebra $\mathfrak{h}$ of H explicitly.

   (b) Show that the connected component H^0 containing identity of H is generated by unipotent one-parameter subgroups. (Hint: Work with the Lie algebra $\mathfrak{h}$.)

6. Let $G = SL_n(\mathbb{R})$, $A = \{$ diagonal matrices $\}$ and $K = SO(n)$.

   (a) Show that $G = KAK$.

   (b) Find a subgroup of matrices N of $G = SL_n(\mathbb{R})$ such that $G = KAN$.

   (Remark. The standard reference for these decompositions is Knapp, "Lie groups beyond introduction". It uses Lie group theory and the proofs are highly non-trivial for general n.)

7. (Decomposition of the isometry group associated to quadratic forms) Let $H = SO(p, q)$ and let A be the group of diagonal elements in H.

   (a) Describe a maximal compact subgroup K of $SO(p, q)$.

   (b) Describe the group N of unipotent matrices.

   (c) Show that $H = KAK$.

   (d) Show that $H = KAN$.

Geodesic flows on hyperbolic spaces

8. Let $G = SL_2(\mathbb{R})$ acting on the upper half plane $\mathbb{H}^2 = \{z = x + iy : y > 0\}$ by Mobius transformations. Let $K = SO(2)$.

   (a) Show that G acts transitively on $\mathbb{H}^2$ with stabilizer $Stab(i)$ of i equal to K.

   (b) Show that $PSL_2(\mathbb{R}) = G/\{\pm Id\}$ acts simply transitively on $T^1\mathbb{H}^2$.

9. Define the hyperbolic space (in hyperboloid model) by

$$\mathbb{H}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \sum_{i=1,\ldots,n} x_i^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}.$$

   The Riemmanian metric is defined by

$$ds^2 = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.$$

   Recall that a Riemannian metric (an infinitesimal object) on a manifold M induces a (global) distance function on M as follows, by integrating the metric along paths (like in Calculus). The length of a path $\gamma : [0, 1] \to M$ (relative to ds) is:

$$l(\gamma) = \int_0^1 ds(\dot{\gamma}(t))dt.$$

   (a) Show that $G = SO(n, 1)$ acts by isometries transitively $\mathbb{H}^n$.

   (b) Show that G acts transitively on $\mathbb{H}^n$ with the stabilizer of any point isomorphic to $O(n)$.

(c) Show that G acts transitively on the unit tangent bundle $T^1\mathbb{H}^n$.

(d) Does G act simply transitively on $T^1\mathbb{H}^n$?

10. (a) Let $\Gamma_1 = SL_2(\mathbb{Z})$. Compute the volume of the quotient manifold $M = \Gamma_1 \backslash \mathbb{H}^2$.

(b) Construct a lattice subgroup $\Gamma_2 \subset G$ such that the quotient manifold M is compact.

(c) Construct a discrete subgroup $\Gamma_3 \subset G$ such that the quotient manifold M has infinite volume. (Hint : choose some elements of Γ_1 and consider the subgroup of Γ_1 generated by these elements.)

Geodesic flows on Riemannian locally symmetric spaces

11. Let $G = SL_n(\mathbb{R})$ and denote its maximal compact subgroup by $K = SO(n)$. Consider a lattice subgroup $\Gamma \subset G$ acting freely on $\widetilde{M} = G/K$. For example, $\Gamma = SL_n(\mathbb{Z})$. Let $M = \Gamma\backslash G/K$ be the quotient Riemannian manifold which has finite volume.

(a) On the Lie algebra $sl(n,\mathbb{R})$, define $\theta(X) = (-X)^t$. It is called a Cartan involution. The eigenspace corresponding to the eigenvalue -1 is $\mathfrak{p} = \{X \in sl(n,\mathbb{R}) : X^t = X\}$.

(b) We define the geodesic flow by

$$\varphi_t(\Gamma g \cdot v) = \Gamma g \exp(tv) \cdot v,$$

where $v \in \mathfrak{p}_1$ (the unit sphere of $\mathfrak{p}$) and $g \in G$. Let $d\mu$ be locally defined by $d\mu = dp \cdot dv$. Show that $d\mu$ is invariant under the geodesic flow.

(c) What is the condition on G/K which ensures the ergodicity of the geodesic flow on M? Justify your answer.