Quadratic forms

1. (Classification of quadratic forms)
Recall that a real quadratic form \( Q : \mathbb{R}^n \to \mathbb{R} \) can be expressed as
\[
Q(x_1, \ldots, x_n) = (x_1, \ldots, x_n)^t A_Q(x_1, \ldots, x_n),
\]
for some symmetric matrix \( A_Q \).

(a) Use the symmetry of \( A_Q \) to conclude that there is a coordinate system for which
\[
Q(x_1, \ldots, x_n) = \sum c_i y_i^2,
\]
where \( c_i \) are non-increasing.

(b) Find the matrix corresponding to \( Q(x_1, x_2) = x_1 x_2 \) and express the quadratic form in the standard form (i.e. as in part (a)).

2. (a) Let \( Q_0(x_1, \ldots, x_n) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2 \). Describe the isometry group
\[
SO(p, n-p) := SO(Q_0) = \{ A \in SL_n(\mathbb{R}) : Q_0(Ax) = Q_0(x) \}.
\]

(b) Let \( p, q \) be the number of positive and negative \( c_i \)'s, respectively. We call \( (p, q) \) the signature of the quadratic form. Suppose that \( Q \) is indefinite and non-degenerate, i.e. \( p, q \geq 1 \) and \( p + q = n \). Describe the isometry group \( SO(Q) \) in terms of \( SO(p, q) \).

3. Show that \( SO(2,1) \) is locally isomorphic to \( SL_2(\mathbb{R}) \). Are they isomorphic?

4. Let \( Q(x, y) = y^2 - a^2 x^2 \) where \( a \) is a badly approximable number, say, there exists \( C > 0 \) such that
\[
\inf_{p, q \in \mathbb{Z}, q \neq 0} |q||qa - p| > C.
\]
Show that
\[
\inf_{(x,y) \in \mathbb{Z}^2 - \{(0,0)\}} |Q(x, y)| > 0.
\]
(This shows that the condition “\( Q \) is of \( n \geq 3 \) variables” cannot be removed from the Oppenheim conjecture. Note that an algebraic number of degree 2 is badly approximable.)

5. Let \( Q_0 = 2x_1 x_3 - x_2^2 \) and \( H = SO(Q_0) \).

(a) Compute the Lie algebra \( \mathfrak{h} \) of \( H \) explicitly.

(b) Show that the connected component \( H^0 \) containing identity of \( H \) is generated by unipotent one-parameter subgroups. (Hint: Work with the Lie algebra \( \mathfrak{h} \).)

(a) Show that $G = KAK$.
(b) Find a subgroup of matrices $N$ of $G = SL_n(\mathbb{R})$ such that $G = KAN$.

(Remark. The standard reference for these decompositions is Knapp, "Lie groups beyond introduction". It uses Lie group theory and the proofs are highly non-trivial for general $n$.)

7. (Decomposition of the isometry group associated to quadratic forms) Let $H = SO(p,q)$ and let $A$ be the group of diagonal elements in $H$.

(a) Describe a maximal compact subgroup $K$ of $SO(p,q)$.
(b) Describe the group $N$ of unipotent matrices.
(c) Show that $H = KAK$.
(d) Show that $H = KAN$.

Geodesic flows on hyperbolic spaces

8. Let $G = SL_2(\mathbb{R})$ acting on the upper half plane $H^2 = \{z = x + iy : y > 0\}$ by Mobius transformations. Let $K = SO(2)$.

(a) Show that $G$ acts transitively on $H^2$ with stabilizer $\text{Stab}(i)$ of $i$ equal to $K$.
(b) Show that $PSL_2(\mathbb{R}) = G/\{\pm \text{Id}\}$ acts simply transitively on $T^1H^2$.

9. Define the hyperbolic space (in hyperboloid model) by $H^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}$.

The Riemannian metric is defined by

$$ds^2 = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.$$ 

Recall that a Riemannian metric (an infinitesimal object) on a manifold $M$ induces a (global) distance function on $M$ as follows, by integrating the metric along paths (like in Calculus). The length of a path $\gamma : [0,1] \to M$ (relative to $ds$) is:

$$l(\gamma) = \int_0^1 ds(\dot{\gamma}(t))dt.$$ 

(a) Show that $G = SO(n,1)$ acts by isometries transitively $\mathbb{H}^n$.
(b) Show that $G$ acts transitively on $\mathbb{H}^n$ with the stabilizer of any point isomorphic to $O(n)$.
(c) Show that $G$ acts transitively on the unit tangent bundle $T^1\mathbb{H}^n$.

(d) Does $G$ act simply transitively on $T^1\mathbb{H}^n$?

10. (a) Let $\Gamma_1 = SL_2(\mathbb{Z})$. Compute the volume of the quotient manifold $M = \Gamma_1 \backslash \mathbb{H}^2$.

(b) Construct a lattice subgroup $\Gamma_2 \subset G$ such that the quotient manifold $M$ is compact.

(c) Construct a discrete subgroup $\Gamma_3 \subset G$ such that the quotient manifold $M$ has infinite volume. (Hint: choose some elements of $\Gamma_1$ and consider the subgroup of $\Gamma_1$ generated by these elements.)

Geodesic flows on Riemannian locally symmetric spaces

11. Let $G = SL_n(\mathbb{R})$ and denote its maximal compact subgroup by $K = SO(n)$. Consider a lattice subgroup $\Gamma \subset G$ acting freely on $\tilde{M} = G/K$. For example, $\Gamma = SL_n(\mathbb{Z})$. Let $M = \Gamma \backslash G/K$ be the quotient Riemannian manifold which has finite volume.

(a) On the Lie algebra $\mathfrak{sl}(n,\mathbb{R})$, define $\theta(X) = (-X)^t$. It is called a Cartan involution. The eigenspace corresponding to the eigenvalue $-1$ is $p = \{X \in \mathfrak{sl}(n,\mathbb{R}) : X^t = X\}$.

(b) We define the geodesic flow by

$$\varphi_t(\Gamma g \cdot v) = \Gamma g \exp(tv) \cdot v,$$

where $v \in p_1$ (the unit sphere of $p$) and $g \in G$. Let $d\mu$ be locally defined by $d\mu = dp \cdot dv$. Show that $d\mu$ is invariant under the geodesic flow.

(c) What is the condition on $G/K$ which ensures the ergodicity of the geodesic flow on $M$? Justify your answer.