A **topological space** is a set $X$ together with a subset $\tau$ of $\mathcal{P}(X)$ which satisfies the following three conditions:

(i) $\emptyset, X \in \tau$,

(ii) $\{A_i\}_{i \in I} \subseteq \tau \Rightarrow \bigcup_{i \in I} A_i \in \tau$, and

(iii) $A_1, \ldots, A_n \in \tau \Rightarrow \bigcap_{i=1}^{n} A_i \in \tau$.

The collection $\tau$ is called the **topology** on $X$, the elements of $\tau$ are called **open sets**, and any subset of $X$ which is the complement of an element of $\tau$ is called a **closed set**. It follows from (ii) and (iii), using de Morgan’s laws from set theory, that an arbitrary intersection or finite union of closed sets, is itself a closed set.

A subset $A \subseteq \tau$ is called a **base** for $(X, \tau)$ if every element of $\tau$ can be written as a union of elements of $A$. In this case we also say that $\tau$ is the **topology generated by** $A$. If there is a countable base $A$ for $\tau$ then we say that $(X, \tau)$ is a **second-countable** space.

Examples:

(1) Every set $X$ with more than one element has at least two topologies. The first is the **discrete topology**, in which we take $\tau = \mathcal{P}(X)$. The second is the **trivial topology**, in which we take $\tau = \{\emptyset, X\}$.

(2) If $(X, d)$ is a metric space then the collection of open balls in $X$ generates a topology called the **metric topology**. As a matter of definition, note that when we say that the collection of open balls generates a topology, we are saying that the collection of all sets which are unions of open balls satisfies the requirements for being a topology.

(3) If $(X, \tau)$ is a topological space and $S \subseteq X$ then the **subspace topology** on $S$ is defined as

$$\{S \cap A : A \in \tau\}.$$
When thinking of a subset $S$ of $X$ as a topological space with the subspace topology, we may refer to $S$ simply as a **subspace** of $X$. The subspace topology is also referred to as the **relative topology**.

To develop this example a little more, if we consider $\mathbb{Q}$ as a subset of $\mathbb{R}$, the latter being taken with the Euclidean metric topology, the collection of rational numbers lying in an open interval will be an open set. However a set consisting of a single rational point will not be open in $\mathbb{Q}$ with respect to this topology. By contrast if we are thinking of $\mathbb{Q}$ with respect to the discrete topology then every set is open. This illustrates the fact that in general there are many choices for the topology on a set $X$, and the natural choice for one problem may not be the right choice for another.

(4) Suppose that $\tau$ and $\tau'$ are two topologies on $X$. We say that $\tau'$ is **coarser** than $\tau$, and that $\tau$ is **finer** than $\tau'$, if $\tau' \subseteq \tau$. In other words, $\tau'$ is coarser than $\tau$ if every open set in $(X, \tau')$ is also open in $(X, \tau)$.

Given a topological space $(X, \tau)$, and any set $A \subseteq X$, the **interior** of $A$ is the union of all open sets contained in $A$. The **closure** of $A$ is the intersection of all closed sets which contain $A$. Equivalently, the interior of $A$ is the largest open set contained in $A$, and the closure of $A$ is the smallest closed set which contains $A$. We will denote the interior of a set $A$ by $A^\circ$ and its closure by $\overline{A}$. The **boundary** of a set $A$, denoted by $\partial A$, is its closure minus its interior, that is

$$\partial A = \overline{A} \setminus A^\circ.$$  

For $A, B \subseteq X$, we say that $A$ is **dense** in $B$ if $\overline{A} = B$. The topological space $X$ is called **separable** if there is a countable set $A$ which is dense in $X$.

For $x \in X$, a set $A \subseteq X$ is called a **neighborhood of** $x$ if there is an open set $U \subseteq A$ with $x \in U$. The topological space $(X, \tau)$ is called a **Hausdorff space** if, for any pair of distinct points $x, y \in X$, there is a neighborhood $U$ of $x$ and a neighborhood $V$ of $y$ with $U \cap V = \emptyset$. Let $\mathcal{U}_x$ denote the collection of all neighborhoods of $x$. We say that a collection of neighborhoods $\mathcal{V}_x \subseteq \mathcal{U}_x$ is a **neighborhood base** for $x$ if, for every $U \in \mathcal{U}_x$, there is a neighborhood $V \in \mathcal{V}_x$ with $V \subseteq U$.

A point $x \in X$ is called a **limit point** of a set $A \subseteq X$ is, for every neighborhood $U$ of $x$, we have that

$$\left(U \cap A\right) \setminus \{x\} \neq \emptyset.$$  

It is not difficult to verify that the closure of a set $A$ is equal to the union of $A$ with the set of all points $x \in X$ which are limit points of $A$.

If $(X, \tau_X)$ and $(Y, \tau_Y)$ are two topological spaces then we say that a function $f : X \to Y$ is **continuous** if $f^{-1}(A) \in \tau_X$ for all $A \in \tau_Y$. If there is a continuous bijective
map \( f : X \to Y \), for which \( f^{-1} \) is also continuous, then we say that \((X, \tau_X)\) and \((Y, \tau_Y)\) are **homeomorphic**.

**Examples:**

(5) Suppose that \( X \) is a set and \( \{Y_i\}_{i \in I} \) is a collection of topological spaces, and for each \( i \in I \) let \( f_i \) be a function from \( X \) to \( Y_i \). The **initial topology** on \( X \) with respect to \( \{f_i\} \) is the coarsest topology on \( X \) with respect to which all of the functions \( f_i \) are continuous.

(6) A ‘dual’ notion to the topology defined in the previous example is the following. Suppose \( X \) is a set and \( \{Y_i\}_{i \in I} \) is a collection of topological spaces, and for each \( i \in I \) let \( f_i \) be a function from \( Y_i \) to \( X \). Then the **final topology** on \( X \) with respect to \( \{f_i\} \) is the finest topology on \( X \) with respect to which all of the functions \( f_i \) are continuous.

(7) Suppose \( \{(X_i, \tau_i)\}_{i \in I} \) is a collection of topological spaces, let \( X \) be the Cartesian product \( \prod_{i \in I} X_i \) and for each \( j \in I \) let \( \pi_j : X \to X_j \) be the **projection map** (i.e. the map onto the \( j \)th coordinate). The **product topology** on \( X \) is defined to be the initial topology with respect to \( \{\pi_j\}_{j \in I} \). A base for this topology is

\[
\left\{ \prod_{i \in I} A_i : A_i \in \tau_i, \ A_i = X_i \text{ for all but finitely many } i \right\}.
\]

(8) With the same notation as in the previous example, the **box topology** on \( X \) is defined to be the topology generated by

\[
\left\{ \prod_{i \in I} A_i : A_i \in \tau_i \right\}.
\]

When \( I \) is finite this topology is the same as the product topology. However in general the two topologies are not the same. One strong argument in favor of using the product topology is that Tychonoff’s Theorem (see below) is not true in general for the box topology.

(9) Let \( \{(X_i, d_i)\}_{i=1}^n \) be metric spaces and consider the Cartesian product \( X = \prod_{i=1}^n X_i \). This product is also a metric space with respect to the metric \( d : X \times X \to [0, \infty) \) defined by

\[
d(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq n} \{d_i(x_i, y_i)\},
\]
and it is not difficult to show that $X$ with the induced metric topology is homeomorphic to $X$ with the product topology (i.e. with each $X_i$ taken with the metric topology).

It is also true that the product of countably many metric spaces $\{(X_i, d_i)\}_{i=1}^{\infty}$, taken with the product topology, is metrizable (i.e. there is a metric on the Cartesian product for which the induced metric topology is the product topology). One metric which realizes the product topology is given by

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i(1 + d_i(x_i, y_i))}.$$ 

In general it is not always true that a product of metric spaces with respect to the product topology is metrizable.

(10) Let $(X, \tau)$ be a topological space and suppose that $X = \bigcup_{y \in Y} X_y$ is a partition of the set $X$. Let $\pi : X \to Y$ be the map which takes the constant value $y$ on $X_y$, for each $y \in Y$. The identification topology on $Y$ is defined to be the finest topology for which the map $\pi$ is continuous. In this topology a set $A \subseteq Y$ is open if and only if $\pi^{-1}(A) \in \tau$. The topological space $Y$ constructed in this way is called an identification space.

(11) Suppose $G$ is a group which is also a topological space, and let $H$ be a subgroup of $G$. Then there is a partition of $G$ into distinct left cosets of $H$, which allows us to view the collection of cosets $G/H$ as an identification space.

For example consider $\mathbb{R}$ under addition, taken with the usual metric topology. Then $\mathbb{Z}$ is a subgroup and the identification space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the unit circle in the complex plane with the subspace topology.

An open cover of a set $K$ in a topological space $(X, \tau)$ is a collection of open sets whose union contains $K$. The set $K$ is compact if every open cover of $K$ can be replaced by a finite subcover. The set $K$ is locally compact if every point in $K$ has a compact neighborhood.

Examples:

(12) Closed subsets of compact sets are compact. To see this suppose that $C \subseteq K$ with $K$ compact and $C$ closed. If $\{A_i\}_{i \in I}$ is an open cover of $C$ then, since $A = X \setminus C$ is open, we have that $\{A_i\}_{i \in I} \cup A$ is an open cover of $K$. By compactness this cover has a finite subcover $\{A_i'\}_{i=1}^{n}$. After removing $A$ if necessary this gives a finite cover of $C$ which is a subset of $\{A_i\}_{i \in I}$. 

(13) A continuous image of a compact set is compact. Let \( f : X \to Y \) be a continuous map between topological spaces and suppose that \( K \subseteq X \) is compact. If \( \{A_i\}_{i \in I} \) is an open cover of \( f(K) \) then, by continuity of \( f \) we have that \( \{f^{-1}(A_i)\}_{i \in I} \) is an open cover of \( K \). By compactness this cover has a finite subcover \( \{f^{-1}(A'_i)\}_{i=1}^n \), where each of the sets \( A'_i \) taken from the original cover, and then \( \{A'_i\}_{i=1}^n \) gives a finite cover of \( f(K) \).

An important theorem due to Tychonoff says that the product of any collection of compact spaces, taken with the product topology, is compact.

A topological space \((X, \tau)\) is disconnected if it can be written as a disjoint union of two nonempty open sets. In other words \( X \) is disconnected if there exist nonempty \( U, V \in \tau \) with \( U \cap V = \emptyset \) and \( X = U \cup V \). If \( X \) is not disconnected then we say that it is connected. A subset \( S \) of \( X \) is connected (resp. disconnected) if it is connected (resp. disconnected) as a topological space with the subspace topology.

Examples:

(14) Suppose \( \{S_i\}_{i \in I} \) is a nonempty collection of connected subsets of \((X, \tau)\) with the property that, for any \( i, j \in I \), the intersection \( S_i \cap S_j \) is non-empty. Then the union \( S = \bigcup_{i \in I} S_i \) is also connected.

(15) If \( S \) is a connected subset of \( X \) then the closure \( \overline{S} \) is also connected.

(16) For any point \( x \in X \), the set \( \{x\} \) is trivially seen to be connected. By (14) from above, the union of all connected subsets of \( X \) which contain \( x \) is also a connected set, called the connected component of \( x \). It is clear from (15) that the connected component of any point is a closed subset of \( X \).

(17) For any \( x, y \in X \), the connected components of \( x \) and \( y \) are either the same, or they are disjoint. Therefore the collection of connected components in a topological space is a partition of the space into disjoint, closed, connected subsets. A space \( X \) in which the connected components are all singleton sets (i.e. sets consisting of only one element) is called totally disconnected.