Quantum Information Theory: Exercises

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1. (Bloch sphere) Show that any pure one-qubit state can be written in the form (up to a global phase $e^{i\alpha}$)
$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle\,,$$
where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

2. (Pauli matrices) Verify the commutation relations
$$[\sigma^1, \sigma^2] = 2i\sigma^3;\ [\sigma^2, \sigma^3] = 2i\sigma^1;\ [\sigma^3, \sigma^1] = 2i\sigma^2.$$
Vefiry the anti-commutation relations for the Pauli matrices
$$\{\sigma^i, \sigma^j\} = 0,$$
where $i \neq j$ are both chosen from the set $\{1, 2, 3\}$. Also verify that for $j = 1, 2, 3$
$$(\sigma^j)^2 = I.$$

3. Show that Pauli matrices form a basis on the space of 2×2 matrices. In other words, show that any 2×2 self-adjoint matrix is of the form
$$A = \alpha_0 I + \alpha_1\sigma^1 + \alpha_2\sigma^2 + \alpha_3\sigma^3 = \alpha_0 I + \boldsymbol{\alpha}\cdot\boldsymbol{\sigma},$$
where I is the identity matrix, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, and $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$.

4. Determine the set of measurement operators corresponding to a measurement of the σ^1 observable.

5. (Hadamard gate) Verify that the Hadamard gate is unitary,
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
Also verify that $H^2 = I$. What are the eigenvalues and eigenvectors of H? Show that the Hadamard operator is its own inverse. Also show that $H\sigma^1 H = \sigma^3$ and $H\sigma^3 H = \sigma^1$.

6. Prove that a Bell state is indeed entangled, i.e. show that $|\Phi^+\rangle \neq |\psi\rangle\,|\phi\rangle$ for all single qubit states $|\psi\rangle$ and $|\phi\rangle$.

   Also show that Bell states form an orthonormal basis. Also, express two-qubit states $|00\rangle$, $|10\rangle$ in a superposition of Bell states.

7. Show that the average values of the observable $\sigma^1 \otimes \sigma^3$ for a two-qubit system measured in the state $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is zero.

8. Prove Schmidt decomposition theorem. Suppose that we have a two-qudit state (i.e. a vector in $\mathbb{C}^{2d}$)

$$|\psi\rangle^{AB},$$

where Hilbert spaces for systems A and B have the same dimension d, i.e $\mathbb{C}^d$. Then it is possible to express this state as follows:

$$|\psi\rangle^{AB} = \sum_{j=1}^{d} \lambda_j |\xi_j\rangle^A |\phi_j\rangle^B,$$

where the amplitudes λ_j are real, non-negative, and normalized as that $\sum_{j=1}^d |\lambda_j|^2 = 1$, the states $\{|\xi_j\rangle^A\}_j$ form an orthonormal basis for system A, and the states $\{|\phi_j\rangle^B\}_j$ form an orthonormal basis for system B.

The Schmidt rank of a bipartite state is equal to the number of non-zero coefficients λ_j in its Schmidt decomposition.

(*The proof of this theorem can be found as Theorem 3.6.1 in Mark Wilde, "Quantum information theory," Cambridge University Press, 2013 or Chapter 2.5 in Michael Nielsen and Isaac Chuang, "Quantum computation and quantum information", Cambridge university press, 2010*)

9. (Trace) If A is linear operator, show that

$$\mathrm{Tr}(A |\psi\rangle \langle\psi|) = \langle\psi| A |\psi\rangle .$$

Prove that $[A \otimes I, I \otimes B] = 0$ for any operators A and B.

Also show that $\mathrm{Tr}\{(M^A \otimes I^B)\rho^{AB}\} = \mathrm{Tr}\{M^A \rho^A\}$.

10. Prove the Characterization of density operators theorem. In other words, prove that an operator ρ is the density operator associated to some ensemble $\{p_j, |\psi_j\rangle\}$ if and only if it satisfies the conditions:

    (a) $\mathrm{Tr}\rho = 1$

    (b) ρ is a positive operator (i.e. $\langle\phi| \rho |\psi\rangle \geq 0$).

11. (Bloch sphere) Show that an arbitrary density matrix for a mixed state qubit on $\mathbb{C}^2$ may be written as

$$\rho = \frac{1}{2}(I + \boldsymbol{r} \cdot \boldsymbol{\sigma}),$$

where $\boldsymbol{r} \in \mathbb{R}^3$ is a real tree-dimensional vector such that $\|\boldsymbol{r}\| \leq 1$, This vector is known as the *Bloch vector* for the state ρ. The vector $\boldsymbol{\sigma}$ is a vector composing of Pauli matrices.

Show that a state ρ is pure if and only if $\|\boldsymbol{r}\| = 1$.

12. Let ρ be a density operator. Show that $\mathrm{Tr}(\rho^2) \leq 1$, with equality if and only if ρ is a pure state. Because of this, the expression $\mathrm{Tr}(\rho^2)$ is called *purity* of a state ρ.

13. Suppose $\Gamma = \mathbb{Z}^\nu$ for some integer $\nu \geq 1$ with metric $d(x,y) = |x-y| = \sum_{j=1}^{\nu} |x_j - y_j|$. Let the function F be
$$F(|x|) = (1+|x|)^{-\nu-1}.$$
Check the second condition for the F-function. In other words, show that
$$\|F\| = \sum_{y\in\Gamma} \frac{1}{(1+|y|))^{\nu+1}} < \infty.$$

14. Prove that for any $\alpha \geq 1$ and any $a, b \geq 0$,
$$\left(\frac{a+b}{2}\right)^\alpha \leq \frac{a^\alpha + b^\alpha}{2}.$$

15. For each $n \geq 1$, let $\{\alpha_t^{(n)} | t \in \mathbb{R}\}$ be a be a strongly continuous one-parameter group of automorphisms on $\mathcal{A}$. Assume there exists $T > 0$ such that for all $t \in [0,T]$, $\{\alpha_t^{(n)}\}$ converges strongly to an automorphism α_t on $\mathcal{A}$, uniformly for $t \in [0,T]$. This means that for all $A \in \mathcal{A}$, and $\epsilon > 0$ there exists N such that for all $n \geq N$ one has
$$\|\alpha_t^{(n)}(A) - \alpha_t(A)\| < \epsilon, \quad \text{for all } t \in [0,T].$$
Prove that $\{\alpha_t^{(n)}\}$ converge strongly to a strongly continuous one-parameter group of automorphisms on $\mathcal{A}$.