SPECTRAL THEORY EXERCISES

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(1) Suppose that $V : \mathbb{Z} \to \mathbb{R}$ is bounded. Show that
$$[H_V \psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)$$
defines a bounded, self-adjoint operator in $\ell^2(\mathbb{Z})$.

(2) If $V \equiv 0$, show that H_V (the *free* Schrödinger operator) has no eigenvalues (as an operator in $\ell^2(\mathbb{Z})$).

(3) Suppose the potential V obeys $V(m+p) = V(m)$, $-p \le m \le p-1$. Show that every solution u of the difference equation
$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$
satisfies
$$\max \left\{ \left\| \begin{pmatrix} u(2p+1) \\ u(2p) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(p+1) \\ u(p) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(-p+1) \\ u(-p) \end{pmatrix} \right\| \right\} \ge \frac{1}{2} \left\| \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \right\|.$$
(Hint: use transfer matrices and apply the Cayley-Hamilton Theorem.)

(4) Show that if E is such that the transfer matrix norms $\|A(n;E)\|$ are uniformly bounded for $n \in \mathbb{Z}_+$ (resp., $n \in \mathbb{Z}_-$), then the difference equation
$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$
has no subordinate solution at $+\infty$ (resp., $-\infty$).

(5) Suppose that $\{H_\omega\}_{\omega \in \Omega}$ is an ergodic family of Schrödinger operators. Show that, for every E, $\mu\left(\{\omega : E \text{ is an eigenvalue of } H_\omega)\}\right) = 0$.

(6) Suppose that Ω is a compact metric space, T is a minimal homeomorphism, and $f : \Omega \to \mathbb{R}$ is continuous. Denote by $\{H_\omega\}_{\omega \in \Omega}$ the associated family of Schrödinger operators. Show that the spectrum is ω-independent, that is, there exists a set $\Sigma \subseteq \mathbb{R}$ such that for every $\omega \in \Omega$, $\sigma(H_\omega) = \Sigma$.
(Hint: use strong operator convergence.)

(7) Show that the Lyapunov exponents $L(E)$ associated with Schrödinger cocycles are always non-negative.

(8) Show that the integrated density of states is always continuous.

(9) Show that the almost sure spectrum of an ergodic family of Schrödinger operators is given by the points of increase of the integrated density of states, that is, $\Sigma = \operatorname{supp} \nu$.

(10) Using the fact that the transfer matrices of the Fibonacci Hamiltonian $M_k = M_k(E) = A(F_k; E)$ (with the k-th Fibonacci number F_k) obey the recursion $M_{k+1} = M_{k-1} M_k$, show that their half-traces $x_k = \frac{1}{2} \operatorname{Tr} M_k$ obey the recursion $x_{k+1} = 2 x_k x_{k-1} - x_{k-2}$.

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