# 2018 Houston Summer School on Dynamical Systems

*Problem set: Statistical Properties*

**1.** Let $\mu$ be a probability measure on a measurable space $X$, and let $f\colon X \to X$ be a measurable map. Prove that the following definitions of "$f$-invariant" are equivalent.
(a) For every measurable $E \subset X$ we have $\mu(f^{-1}E) = \mu(E)$.
(b) For every measurable $\varphi\colon X \to \mathbb{R}$ we have $\int \varphi\, d\mu = \int \varphi \circ f\, d\mu$.

**2.** Prove that Lebesgue measure is invariant for each of the following.
(a) the doubling map $E_2\colon S^1 \to S^1$;
(b) a circle rotation $R_\alpha\colon S^1 \to S^1$;
(c) the toral automorphism given by $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$;
(d) the twist $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ on the torus;
(e) the map $(x, y) \mapsto (x + \alpha, y + x)$ on the torus.

**3.** Let $\mu$ be an invariant probability measure for $f$, and prove that the following definitions of "ergodic" are equivalent.
(a) If $E$ is a measurable set such that $f^{-1}(E) = E$, then $\mu(E) = 0$ or $\mu(E) = 1$.
(b) If $\varphi$ is a measurable function $\varphi$ such that $\varphi(x) = \varphi(f(x))$ for $\mu$-a.e. $x$, then $\varphi$ is constant $\mu$-a.e.
(c) If $\nu_1, \nu_2$ are invariant measures such that $\mu = a_1\nu_2 + a_2\nu_2$ for some $a_1, a_2 \geq 0$ with $a_1 + a_2 = 1$, then $\nu_1 = \nu_2 = \mu$.

**4.** Recall that given a measure $\mu$ on the circle $S^1 = \mathbb{R}/\mathbb{Z}$, the Fourier transform of $\mu$ is the function $\hat{\mu}\colon \mathbb{Z} \to \mathbb{C}$ given by $\hat{\mu}(\kappa) = \int e^{-2\pi i\kappa x}\, d\mu(x)$, and that the map $\mu \mapsto \hat{\mu}$ is 1-1. Use this fact to prove that an irrational rotation is uniquely ergodic.

**5.** Prove that $x \in \mathbb{R}/\mathbb{Z}$ is pre-periodic (that is, its trajectory terminates in a periodic orbit) for the doubling map if and only if it is rational. Characterize the periodic points.

**6.** Recall that the Fourier transform of $\varphi \in L^2(S^1)$ is $\hat{\varphi}(\kappa) = \int e^{-2\pi i\kappa x}\varphi(x)\, dx$, where the integral is taken with respect to Lebesgue measure, and that the map $\varphi \mapsto \hat{\varphi}$ is a bijection (in fact a unitary isomorphism) between $L^2(S^1)$ and $\ell^2(\mathbb{Z})$. Use this to prove that Lebesgue measure is ergodic for the doubling map.

**7.** Let $R_\theta(x) = x + \theta \pmod 1$ where $\theta$ is irrational. Let $dx$ be Lebesgue measure on $[0, 1]$ and suppose that $\phi$ is a continuous function on the circle with $\int \phi\, dx = 0$. Show that $\frac{1}{n}S_n\phi \to 0$ uniformly; that is, for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{1}{n}S_n\phi(x)| < \epsilon$ for *every* $x$.

[Note that this property, "uniform convergence of Birkhoff averages for all continuous functions for a continuous transformation on a compact space" is equivalent to unique ergodicity; see other problems on this list about irrational rotations on the circle.]

**8.** Use Fourier analysis to prove that Lebesgue measure is ergodic for the toral automorphism $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

**9.** Consider the map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ given by $f(x,y) = (x+\alpha, y+x)$, where $\alpha$ is irrational. Prove that $f$ is uniquely ergodic by following the steps below.

(a) Prove that Lebesgue measure $m$ is ergodic using Fourier analysis: use the fact that the characteristic function of any subset of $\mathbb{T}^2$ is in $L^2$, and hence has a Fourier transform in $\ell^2$, so that in particular the Fourier coefficients decay at infinity.

(b) Let $\nu$ be any invariant measure for $f$ and let $\nu_t$ be the image of $\nu$ under a rotation by $t$ in the second coordinate. Show that each $\nu_t$ is also $f$-invariant, and that the average over all values of $t$ is Lebesgue measure $m$ on $\mathbb{T}^2$. Use ergodicity to conclude that $\nu_t = m$ for a.e. $t$, and hence $\nu = m$.

**10.** Let $E_2$ be the doubling map. Prove that there are infinitely many ergodic measures $\mu$ such that (1) $\mu$ is not periodic, and (2) the support of $\mu$ is not the whole circle.

- ⋄ Recall that the support of $\mu$ is $\mathrm{supp}(\mu) = \{x \mid \mu(B(x,\epsilon)) > 0 \text{ for every } \epsilon > 0\}$.
- ⋄ Hint: code the system by the full 2-shift and consider Markov measures.

Prove that in fact, there are infinitely many closed subsets $A \subset S^1$ such that (1) $A$ is infinite, and (2) there is an ergodic measure $\mu$ such that $\mathrm{supp}(\mu) = A$.

**11.** Prove that every orbit for $R_\theta$ is dense if $\theta$ is irrational. Prove that not every orbit for $E_2$ is dense. Construct a dense orbit for $E_2$ using the symbolic representation on the full 2-shift.

**12.** Let $X$ be a separable metric space and $T\colon X \to X$ be continuous. Given $x \in X$ and $n \in \mathbb{N}$, let $\mathcal{E}_{x,n} = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k x}$ be the $n$th *empirical measure* for $x$. That is, the measure $\mathcal{E}_{x,n}$ is defined by

$$\int \phi(y)\, d\mathcal{E}_{x,n}(y) = \frac{1}{n}\sum_{k=0}^{n-1} \phi(T^k x)$$

for every continuous $\phi\colon X \to \mathbb{R}$. Suppose $n_j \to \infty$ and $\mu \in \mathcal{M}(X)$ are such that $\mathcal{E}_{x,n_j} \to \mu$ in the weak* topology. (Note that if $X$ is compact then existence of a convergent subsequence follows from compactness of $\mathcal{M}(X)$.)

(a) Show that $\mu$ is $T$-invariant.

(b) Give an example to show that this may fail if $T$ is not continuous.

(c) Say that $x$ is *generic* for $\mu$ if $\mathcal{E}_{x,n} \to \mu$ (without passing to a subsequence). Birkhoff's ergodic theorem says that if $\mu$ is ergodic and $G_\mu$ is the set of generic points for $\mu$, then $\mu(G_\mu) = 1$. Give an example showing that $G_\mu$ may be empty if $\mu$ is invariant but not ergodic.

(d) Let $\Sigma$ be the full shift on two symbols and let $\mu$ be any $\sigma$-invariant measure (not necessarily ergodic). Show that $\mu$ has a generic point.

**13.** Let $A$ be a finite alphabet and $\Sigma \subset A^{\mathbb{Z}}$ (or $A^{\mathbb{N}}$) be a closed $\sigma$-invariant subset. Given $n \in \mathbb{N}$, let

$$L_n = \{w = w_1 \cdots w_n \mid w_i \in A \text{ for each } 1 \le i \le n, \\ \text{and } w \text{ appears as a subword of some } x \in \Sigma\}.$$

(a) Let $a_n = \log \#L_n$ and show that

$$a_{n+m} \le a_n + a_m \text{ for every } n, m \in \mathbb{N}. \tag{$\star$}$$

(b) A sequence satisfying ($\star$) is called *subadditive*. Prove *Fekete's lemma*: if $a_n$ is a subadditive sequence, then $\lim \frac{1}{n} a_n$ exists and is equal to $\inf \frac{1}{n} a_n$ (which may be $-\infty$).

(c) Deduce that $h(\Sigma) = \lim_{n\to\infty} \frac{1}{n} \log \#L_n(\Sigma)$ exists for every shift $\Sigma$. This limit is called the *topological entropy* of the shift $\Sigma$.

(d) Let $\Sigma$ be the SFT on two symbols $\{0, 1\}$ where the only forbidden word is 11. Show that $\#L_n(\Sigma)$ is the Fibonacci sequence.

(e) Let $\Sigma$ be a topological Markov chain with transition matrix $M$ – that is, a sequence $x$ is in $\Sigma$ if and only if $M_{x_n,x_{n+1}} = 1$ for every $n$. Show that $\#L_n$ is the sum of all the entries of $M^{n-1}$.

(f) Let $\lambda$ be a positive real eigenvalue of $M$ with the property that $|\chi| \le \lambda$ for all eigenvalues $\chi$. (Existence of such an eigenvalue is part of the Perron–Frobenius theorem.) Prove that $h(\Sigma) = \log \lambda$.

**14.** Let $(X, \mathcal{B}, \mu)$ be a probability space and $T\colon X \to X$ a measure-preserving map. Given measurable sets $A, B \subset X$, let

$$C_n(A, B) := |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)|$$

be the $n$th correlation function of $A, B$. Similarly, given $L^2$ test functions $\phi, \psi$, let

$$C_n(\phi, \psi) := \left| \int \phi \cdot (\psi \circ T^n)\, d\mu - \int \phi\, d\mu \int \psi\, d\mu \right|.$$

(a) Show that $C_n(A, B) \to 0$ for every $A, B$ if and only if $C_n(\phi, \psi) \to 0$ for every $\phi, \psi \in L^2$. In this case the measure is called *mixing*.

(b) Our results on decay of correlations all involve $C_n(\phi, \psi)$ for sufficiently regular test functions, instead of $C_n(A, B)$, or $C_n(\phi, \psi)$ for arbitrary $L^2$ functions. This is because even when $C_n(\phi, \psi)$ decays exponentially for Hölder continuous functions (or some other nice class), we may have very slow decay for $C_n(A, B)$, or for arbitrary measurable functions.

Demonstrate this phenomenon as follows: let $T\colon [0,1] \to [0,1]$ be the doubling map $T(x) = 2x \pmod 1$, and let $\mu$ be Lebesgue measure on $[0,1]$. Find measurable sets $A, B \subset [0,1]$ such that $C_n(A,B)$ only decays polynomially – that is, there are $\gamma, c > 0$ such that $C_n(A,B) \geq cn^{\gamma}$ for all $n$.

*Note that this is equivalent to answering the same question where $X$ is the full shift on two symbols and $\mu$ is $(\frac{1}{2}, \frac{1}{2})$-Bernoulli.*

**15.** The *Dyck shift* $\Sigma$ is the two-sided shift on the four-symbol alphabet $A$ whose letters are the brackets (, [, ], and ), and whose allowable sequences are precisely those in which the brackets are opened and closed in the right order. That is, the word ( [ ] ) is allowable, but the word ( [ ) ] is not. Similarly, ( ( ( [ is allowable, but ( ] ( ( is not.

Let $B \subset A^* := \bigcup_n A^n$ be the set of *balanced* words; that is, the set defined by the following recursive procedure:

- ⋄ the empty word is in $B$;
- ⋄ if $w \in B$ then $(w) \in B$;
- ⋄ if $w \in B$ then $[w] \in B$.

Consider the subsets

$$\Sigma^R = \{x \in \Sigma \mid \text{ for every } n \in \mathbb{N} \text{ there is } m < n \text{ with } x_m x_{m+1} \cdots x_n \in B\},$$
$$\Sigma^L = \{x \in \Sigma \mid \text{ for every } m \in \mathbb{N} \text{ there is } n > m \text{ with } x_m x_{m+1} \cdots x_n \in B\}.$$

That is, $\Sigma^R$ is the set of sequences where every right bracket has a matching left bracket, and $\Sigma^L$ is the set of sequences where every left bracket has a matching right bracket.

(a) Let $X = \{0,1,2\}^{\mathbb{Z}}$ be the full shift on three symbols and define $h\colon \Sigma \to X$ by $h(x)_n = H(x_n)$, where $H\colon A \to \{0,1,2\}$ maps the symbol ( to 1, the symbol [ to 2, and both symbols ), ] to 0. Show that $h$ is 1-1 on $\Sigma^R$.

(b) Let $\mu$ be an $\sigma$-invariant probability measure on $\Sigma$ and show that $\mu(\Sigma^R \cup \Sigma^L) = 1$.

(c) Consider the directed graph $G$ whose vertices are non-negative integers and which has the following edges:

- ⋄ 2 edges from 0 to 0;
- ⋄ 2 edges from $n$ to $n+1$ for every $n \geq 0$;
- ⋄ 1 edge from $n$ to $n-1$ for every $n \geq 1$.

Let $a_n$ be the number of paths of length $n$ on this graph that start at 0. Show that $\#L_n(\Sigma) = a_n$.

*Hint:* it may help to label the two edges from 0 to 0 with the right brackets ) and ], the two edges from $n$ to $n+1$ with ( and [, and the edge from $n$ to $n-1$ with ") or ]".

(d) Show that $h(\Sigma) = \log 3$.

(e) If you know about measure-theoretic entropy and the variational principle, show that $\Sigma$ has two ergodic measures of maximal entropy, and that both are fully supported on $\Sigma$.