1. Let $\Delta = \{(p_1, \ldots, p_N) : p_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^N p_i = 1\}$. Given $a_i \in \mathbb{R}$, define $F : \Delta \rightarrow \mathbb{R}$ by $F(p) = \sum_{i=1}^N -p_i \log p_i + \sum_{i=1}^N p_i a_i$. Prove that $\max_{p \in \Delta} F(p) = \log \sum_i e^{a_i}$, and that the maximum is achieved at $p_j = e^{a_j}/\sum_i e^{a_i}$.

2. Let $\Sigma \subset \{0,1\}^\mathbb{N}$ be the hard core lattice gas model, for which $x \in \Sigma$ iff $x$ does not contain two consecutive 1s. Put $\beta = 0$ and find the corresponding Gibbs measure; prove that it is a Markov measure by finding the relevant stochastic matrix and eigenvector.

3. Modify the Ising model by replacing the local energy function $U_k(x) = -x_k x_{k+1}$ with $U_k(x) = -x_k x_{k+1} - \frac{2}{N} x_k x_{k+2}$; that is, we add an interaction between sites that are a distance 2 apart. Prove that the corresponding invariant Gibbs measure $\mu$ is a ‘two-step Markov measure’ by finding $\{\pi_{ij} : i, j \in \{\pm 1\}\}$ and $\{P_{ijk} : i, j, k \in \{\pm 1\}\}$ such that for $w = w_1 \cdots w_n$ we have

$$\mu([w]) = \pi_{w_1 w_2} P_{w_1 w_2 w_3} P_{w_2 w_3 w_4} \cdots P_{w_{n-2} w_{n-1} w_n}.$$ 

Here $\pi_{ij}$ can be interpreted as the probability of beginning in the state $i$, then $j$, and $P_{ijk}$ can be interpreted as the probability of seeing the state $k$ next, given that the last two states were $i$ and $j$.

4. Let $X$ be a Markov shift on a finite alphabet given by a transition matrix $T$ of 0s and 1s. Suppose that $T$ is primitive (some power is positive) and let $\lambda$ be the Perron–Frobenius eigenvalue of $T$. Let $\mathcal{L}_n$ be the set of words $w$ of length $n$ such that $[w] \cap X \neq \emptyset$. Prove that there is a constant $C > 0$ such that $\#\mathcal{L}_n/\lambda^n \in [C^{-1}, C]$ for all $n \in \mathbb{N}$. Must the limit $\lim_{n \rightarrow \infty} \#\mathcal{L}_n/\lambda^n$ exist?

5. Fix $\beta \in \mathbb{R}$ and let $f : [0,1) \rightarrow [0,1)$ be the expanding interval map defined by

$$f(x) = \begin{cases} (1 + e^{-2\beta})x & 0 \leq x < 2/(1 + e^{-2\beta}), \\ 1 - (1 + e^{2\beta})(\frac{1}{2} - x) & 2/(1 + e^{-2\beta}) \leq x < \frac{1}{2}, \\ (1 + e^{2\beta})(x - \frac{1}{2}) & \frac{1}{2} \leq x < 1 - 2/(1 + e^{-2\beta}), \\ 1 - (1 + e^{-2\beta})(1 - x) & 1 - 2/(1 + e^{-2\beta}) \leq x < 1. \end{cases}$$

In other words, $f$ is the map uniquely defined by the following conditions:

- $f$ maps $I_0 := [0, \frac{1}{2})$ and $I_1 := [\frac{1}{2}, 1)$ monotonically onto $[0, 1]$;
- $f' = 1 + e^{-2\beta}$ on $I_{00} = I_0 \cap f^{-1}(I_0)$ and $I_{11} = I_1 \cap f^{-1}(I_1)$;
- $f' = 1 + e^{2\beta}$ on $I_{01} = I_0 \cap f^{-1}(I_1)$ and $I_{10} = I_1 \cap f^{-1}(I_0)$.

Write $P_{00} = P_{11} = e^{-\beta}/(e^{\beta} + e^{-\beta})$ and $P_{01} = P_{10} = e^{\beta}/(e^{\beta} + e^{-\beta})$. Prove that Lebesgue measure $m$ is $f$-invariant, and that writing $I_{w_1 \cdots w_n} = \bigcap_{k=1}^n f^{-(k-1)}(I_{w_k})$ for $w_1 \cdots w_n \in \{0,1\}^n$, we have $m(I_{w_1 \cdots w_n}) = \frac{1}{2} P_{w_1 w_2} P_{w_2 w_3} \cdots P_{w_{n-1} w_n} P_{w_n}$. In particular, $([0,1), f)$ is measure-theoretically isomorphic to the Markov chain defined by the stochastic matrix $P$, which was the Gibbs measure for the Ising model at inverse temperature $\beta$. 

---

2018 Houston Summer School on Dynamical Systems

Problem set: Thermodynamic formalism