2018 Houston Summer School on Dynamical Systems

Problem set: Thermodynamic formalism

1. Let $\Delta_N = \{(p_1, \dots, p_N) : p_i \geq 0$ for all i and $\sum_{i=1}^N p_i = 1\}$. Given $a_i \in \mathbb{R}$, define $F\colon \Delta_N \to \mathbb{R}$ by $F(p) = \sum_{i=1}^N -p_i \log p_i + \sum_{i=1}^N p_i a_i$. Prove that $\max_{p \in \Delta_N} F(p) = \log \sum_i e^{a_i}$, and that the maximum is achieved at $p_j = e^{a_j} / \sum_i e^{a_i}$.

2. Let $\Sigma \subset \{0,1\}^{\mathbb{N}}$ be the hard core lattice gas model, for which $x \in \Sigma$ iff x does not contain two consecutive 1s. Put $\beta = 0$ and find the corresponding Gibbs measure; prove that it is a Markov measure by finding the relevant stochastic matrix and eigenvector.

3. Modify the Ising model by replacing the local energy function $U_k(x) = -x_k x_{k+1}$ with $U_k(x) = -x_k x_{k+1} - \frac{1}{2} x_k x_{k+2}$; that is, we add an interaction between sites that are a distance 2 apart. Prove that the corresponding invariant Gibbs measure μ is a 'two-step Markov measure' by finding $\{\pi_{ij} : i,j \in \{\pm 1\}\}$ and $\{P_{ijk} : i,j,k \in \{\pm 1\}\}$ such that for $w = w_1 \cdots w_n$ we have

$$\mu([w]) = \pi_{w_1 w_2} P_{w_1 w_2 w_3} P_{w_2 w_3 w_4} \cdots P_{w_{n-2} w_{n_1} w_n}.$$

Here π_{ij} can be interpreted as the probability of beginning in the state i, then j, and P_{ijk} can be interpreted as the probability of seeing the state k next, given that the last two states were i and j.

4. Let X be a Markov shift on a finite alphabet given by a transition matrix T of 0s and 1s. Suppose that T is primitive (some power is positive) and let λ be the Perron–Frobenius eigenvalue of T. Let $\mathcal{L}_n$ be the set of words w of length n such that $[w] \cap X \neq \emptyset$. Prove that there is a constant $C > 0$ such that $\#\mathcal{L}_n / \lambda^n \in [C^{-1}, C]$ for all $n \in \mathbb{N}$. Must the limit $\lim_{n\to\infty} \#\mathcal{L}_n / \lambda^n$ exist?

5. Fix $\beta \in \mathbb{R}$ and let $f\colon [0,1) \to [0,1)$ be the expanding interval map defined by

$$f(x) = \begin{cases} (1+e^{-2\beta})x & 0 \leq x < 2/(1+e^{-2\beta}), \\ 1 - (1+e^{2\beta})(\frac{1}{2} - x) & 2/(1+e^{-2\beta}) \leq x < \frac{1}{2}, \\ (1+e^{2\beta})(x - \frac{1}{2}) & \frac{1}{2} \leq x < 1 - 2/(1+e^{-2\beta}), \\ 1 - (1+e^{-2\beta})(1-x) & 1 - 2/(1+e^{-2\beta}) \leq x < 1. \end{cases}$$

In other words, f is the map uniquely defined by the following conditions:

- ◇ f maps $I_0 := [0, \frac{1}{2})$ and $I_1 := [\frac{1}{2}, 1)$ monotonically onto $[0,1)$;
- ◇ $f' = 1 + e^{-2\beta}$ on $I_{00} = I_0 \cap f^{-1}(I_0)$ and $I_{11} = I_1 \cap f^{-1}(I_1)$;
- ◇ $f' = 1 + e^{2\beta}$ on $I_{01} = I_0 \cap f^{-1}(I_1)$ and $I_{10} = I_1 \cap f^{-1}(I_0)$.

Write $P_{00} = P_{11} = e^{-\beta}/(e^{\beta} + e^{-\beta})$ and $P_{01} = P_{10} = e^{\beta}/(e^{\beta} + e^{-\beta})$. Prove that Lebesgue measure m is f-invariant, and that writing $I_{w_1 \cdots w_n} = \bigcap_{k=1}^n f^{-(k-1)}(I_{w_k})$ for $w_1 \cdots w_n \in \{0,1\}^n$, we have $m(I_{w_1 \cdots w_n}) = \frac{1}{2} P_{w_1 w_2} P_{w_2 w_3} \cdots P_{w_{n-1}} P_{w_n}$. In particular, $([0,1), f)$ is measure-theoretically isomorphic to the Markov chain defined by the stochastic matrix P, which was the Gibbs measure for the Ising model at inverse temperature β.