# Quantum Walks - Fillman

## PROBLEMS

Some hints are on the last page. Try to solve problems without looking at hints first. Some notation: $\ell^2(\mathbb{Z})$ denotes the space of functions $f : \mathbb{Z} \to \mathbb{C}$ such that $\sum_n |f(n)|^2 < \infty$. The standard basis of $\ell^2(\mathbb{Z})$ is denoted by $\{\delta_n\}_{n\in\mathbb{Z}}$ and is defined by

$$\delta_n(m) = \begin{cases} 1 & n = m \\ 0 & n \neq m. \end{cases}$$

We will denote the standard basis of $\mathbb{C}^2$ by

$$e_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(1) Let

$$\mathcal{H}_1 = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$$

$$\mathcal{H}_2 = \ell^2(\mathbb{Z}, \mathbb{C}^2) := \left\{ f : f \text{ is a function from } \mathbb{Z} \text{ to } \mathbb{C}^2 \text{ such that } \sum_{n\in\mathbb{Z}} \|f(n)\|^2 < \infty \right\}$$

$$\mathcal{H}_3 = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$$

$$\mathcal{H}_4 = \ell^2(\mathbb{Z} \times \mathbb{Z}_2).$$

Show that $\mathcal{H}_1 \cong \mathcal{H}_2 \cong \mathcal{H}_3 \cong \mathcal{H}_4$ by constructing explicit unitary isomorphisms between them

(2) Given a sequence of $2 \times 2$ *unitary* matrices

$$Q_n = \begin{bmatrix} q_n^{11} & q_n^{12} \\ q_n^{21} & q_n^{22} \end{bmatrix} \in \mathbb{U}(2, \mathbb{C}),$$

define operators on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ by

$$S\delta_n \otimes e_\pm = \delta_{n\pm1} \otimes e_\pm$$
$$Q\delta_n \otimes e_+ = q_n^{11}\delta_n \otimes e_+ + q_n^{21}\delta_n \otimes e_-$$
$$Q\delta_n \otimes e_- = q_n^{12}\delta_n \otimes e_+ + q_n^{22}\delta_n \otimes e_-$$
$$W = SQ.$$

Show that $S$, $Q$, and $W$ are unitary.

(3) (Portions of the unitary RAGE theorem) Let $U$ be a unitary operator on $\ell^2(\mathbb{Z})$, $\psi \in \ell^2(\mathbb{Z})$, and $\mu = \mu_{U,\psi}$ the associated spectral measure.

(a) If $\mu = \mu_{\mathrm{pp}}$ show that: for every $\varepsilon > 0$, there exists $L \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} \sum_{|j|\geqslant L} |\langle \delta_j, U^n\psi\rangle|^2 \leqslant \varepsilon.$$

Explain why one could refer to this statement as "dynamical localization."

(b) If $\mu = \mu_{\mathrm{ac}}$, show that: for every $L \in \mathbb{N}$,

$$\lim_{n\to\infty} \sum_{|j|\leqslant L} |\langle \delta_j, U^n\psi\rangle|^2 = 0.$$

Why is this a statement of "dynamical delocalization"?

(4) With notation as in Exercise (2), suppose there is a bi-infinite sequence $\cdots n_{-1} < n_0 < n_1 < \cdots$ such that

$$q^{11}_{n_j} = q^{22}_{n_j} = 0 \text{ for every } j \in \mathbb{Z}.$$

First, what does this imply about $q^{12}_{n_j}$ and $q^{21}_{n_j}$? Discuss the structure of $W$. What can you say about the spectrum and spectral type of $W$? What can you say about the dynamics (behavior of $W^\ell$ as $\ell \to \infty$?)

(5) With notation as in Exercise (2), suppose $\varphi$ is a vector with $W\varphi = z\varphi$ for a scalar $z \in \mathbb{C}$. Expanding $\varphi$ in the standard basis of $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$, we may write

$$\varphi = \sum_{n\in\mathbb{Z}} \varphi^+_n \delta_n \otimes e_+ + \varphi^-_n \delta_n \otimes e_-.$$

Show that there are $2 \times 2$ matrices $T_n = T_n(W, z)$ such that

$$\begin{bmatrix} \varphi^+_{n+1} \\ \varphi^-_n \end{bmatrix} = T_n \begin{bmatrix} \varphi^+_n \\ \varphi^-_{n-1} \end{bmatrix}.$$

(6) Suppose $U$ is a unitary operator on a Hilbert space $\mathcal{H}$ and that $\varphi, \psi \in \mathcal{H}$. Show that

$$\sum_{\ell=0}^{\infty} e^{-2\ell/L} |\langle \varphi, U^\ell \psi\rangle|^2 = e^{2/L} \int_0^{2\pi} \left| \langle \varphi, (e^{i\theta + L^{-1}} - U)^{-1}\psi\rangle \right|^2 \frac{d\theta}{2\pi}.$$

What is the significance of this equality from the dynamical point of view?

(7) Suppose $W$ is a quantum walk with periodic coins, that is, there is $p \in \mathbb{N}$ such that $Q_{n+p} = Q_n$ for all $n$. Let $\alpha = (\alpha_n)_{n\in\mathbb{Z}}$ be the Verblunsky coefficients of the associated CMV matrix. Show that $\alpha$ is periodic modulo a phase. That is, show that there is some $\theta \in \mathbb{R}$ and $q \in \mathbb{N}$ such that $\alpha_{n+q} = e^{i\theta}\alpha_n$ for all $n$.

(8) Let $\mathcal{E}$ be an extended CMV matrix with Verblunsky coefficients $\alpha = (\alpha_n)_{n\in\mathbb{Z}}$. The *sieved* matrix $\widetilde{\mathcal{E}}$ is the CMV matrix with coeffiencts $\widetilde{\alpha}$ defined by

$$\widetilde{\alpha}_{2n} = \alpha_n, \quad \widetilde{\alpha}_{2n+1} = 0, \ n \in \mathbb{Z}.$$

(a) Show that $\widetilde{\mathcal{E}}^2 \cong \mathcal{E} \oplus \mathcal{E}^\top$. To do this, you should find two natural invariant subspaces $A$ and $B$ of $\widetilde{\mathcal{E}}^2$ and show that $\widetilde{\mathcal{E}}^2|_A \cong \mathcal{E}$ and $\widetilde{\mathcal{E}}^2|_B \cong \mathcal{E}^\top$.

(b) What does this tell you about the relationship between $\sigma(\mathcal{E})$ and $\sigma(\widetilde{\mathcal{E}})$?

(9) Let $(A_n)_{n=1}^\infty$ denote a sequence of $\mathbb{SL}(2, \mathbb{C})$ matrices obeying the recursion

$$A_{n+1} = A_{n-1}A_n, \quad n \geqslant 2.$$

Denoting $x_n = \mathrm{tr}(A_n)$, show that $x_{n+1} = x_n x_{n-1} - x_{n-2}$ for all $n \geqslant 3$.

# Hints. Quantum Walks - Fillman

(1) Try identifying suitable orthonormal bases for each space.

(2) Polarization

(3) (a) First, handle the case when $\psi$ is a linear combination of eigenfunctions of $U$.
(b) Cauchy–Schwarz, Riemann–Lebesgue

(4) As a warm-up, try the case when $\{n_j\} = \mathbb{Z}$, i.e., $q_n^{11} = q_n^{22} = 0$ for *every* $n \in \mathbb{Z}$.

(5) The equation $W\varphi = z\varphi$ is an equality of vectors. Write out the components.

(6) Parseval.

(7) How does periodicity of $Q$ affect the $\lambda$'s in the gauge transform?

(8) (a) Calculate $\widetilde{\mathcal{E}}^2\delta_{4n}$, $\widetilde{\mathcal{E}}^2\delta_{4n+1}$, $\widetilde{\mathcal{E}}^2\delta_{4n+2}$, and $\widetilde{\mathcal{E}}^2\delta_{4n+3}$ for general $n \in \mathbb{Z}$. This should be enough to identify some invariant subspaces.
(b) Spectral mapping theorem

(9) Cayley–Hamilton.