Quantum Walks - Fillman

PROBLEMS

Some hints are on the last page. Try to solve problems without looking at hints first. Some notation: $\ell^2(\mathbb{Z})$ denotes the space of functions $f : \mathbb{Z} \to \mathbb{C}$ such that $\sum_n |f(n)|^2 < \infty$. The standard basis of $\ell^2(\mathbb{Z})$ is denoted by $\{\delta_n\}_{n \in \mathbb{Z}}$ and is defined by

$$\delta_n(m) = \begin{cases} 1 & n = m \\ 0 & n \neq m. \end{cases}$$

We will denote the standard basis of $\mathbb{C}^2$ by $e_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(1) Let

$$\mathcal{H}_1 = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$$

$$\mathcal{H}_2 = \ell^2(\mathbb{Z}, \mathbb{C}^2) := \left\{ f : f \text{ is a function from } \mathbb{Z} \to \mathbb{C}^2 \text{ such that } \sum_{n \in \mathbb{Z}} \|f(n)\|^2 < \infty \right\}$$

$$\mathcal{H}_3 = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$$

$$\mathcal{H}_4 = \ell^2(\mathbb{Z} \times \mathbb{Z}_2).$$

Show that $\mathcal{H}_1 \cong \mathcal{H}_2 \cong \mathcal{H}_3 \cong \mathcal{H}_4$ by constructing explicit unitary isomorphisms between them.

(2) Given a sequence of $2 \times 2$ unitary matrices

$$Q_n = \begin{bmatrix} q_{11}^{n} & q_{12}^{n} \\ q_{21}^{n} & q_{22}^{n} \end{bmatrix} \in \mathbb{U}(2, \mathbb{C}),$$

define operators on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ by

$$S \delta_n \otimes e_+ = \delta_{n+1} \otimes e_+$$

$$Q \delta_n \otimes e_+ = q_{11}^{n} \delta_n \otimes e_+ + q_{21}^{n} \delta_n \otimes e_-$$

$$Q \delta_n \otimes e_- = q_{12}^{n} \delta_n \otimes e_+ + q_{22}^{n} \delta_n \otimes e_-$$

$$W = SQ.$$

Show that $S$, $Q$, and $W$ are unitary.

(3) (Portions of the unitary RAGE theorem) Let $U$ be a unitary operator on $\ell^2(\mathbb{Z})$, $\psi \in \ell^2(\mathbb{Z})$, and $\mu = \mu_{U, \psi}$ the associated spectral measure.

(a) If $\mu = \mu_{pp}$ show that: for every $\varepsilon > 0$, there exists $L \in \mathbb{N}$ such that

$$\limsup_{n \to \infty} \sum_{|j| \geq L} |\langle \delta_j, U^n \psi \rangle|^2 \leq \varepsilon.$$

Explain why one could refer to this statement as “dynamical localization.”

(b) If $\mu = \mu_{ac}$, show that: for every $L \in \mathbb{N},$

$$\lim_{n \to \infty} \sum_{|j| \leq L} |\langle \delta_j, U^n \psi \rangle|^2 = 0.$$

Why is this a statement of “dynamical delocalization”?
(4) With notation as in Exercise (2), suppose there is a bi-infinite sequence \( \cdots \, n_{-1} < n_0 < n_1 < \cdots \) such that
\[
q^{11}_{n_j} = q^{22}_{n_j} = 0 \text{ for every } j \in \mathbb{Z}.
\]
First, what does this imply about \( q^{12}_{n_j} \) and \( q^{21}_{n_j} \)? Discuss the structure of \( W \). What can you say about the spectrum and spectral type of \( W \)? What can you say about the dynamics (behavior of \( W^\ell \) as \( \ell \to \infty \)?)

(5) With notation as in Exercise (2), suppose \( \varphi \) is a vector with \( W \varphi = z \varphi \) for a scalar \( z \in \mathbb{C} \). Expanding \( \varphi \) in the standard basis of \( \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \), we may write
\[
\varphi = \sum_{n \in \mathbb{Z}} \varphi^+_n \delta_n \otimes e_+ + \varphi^-_n \delta_n \otimes e_-.
\]
Show that there are \( 2 \times 2 \) matrices \( T_n = T_n(W, z) \) such that
\[
\begin{bmatrix}
\varphi^+_{n+1} \\
\varphi^-_{n+1}
\end{bmatrix} = T_n \begin{bmatrix}
\varphi^+_n \\
\varphi^-_n
\end{bmatrix}.
\]

(6) Suppose \( U \) is a unitary operator on a Hilbert space \( \mathcal{H} \) and that \( \varphi, \psi \in \mathcal{H} \). Show that
\[
\sum_{\ell=0}^{\infty} e^{-2\ell/L} |\langle \varphi, U^\ell \psi \rangle|^2 = e^{2/L} \int_0^{2\pi} \left| \langle \varphi, (e^{i\theta+L^{-1}} - U)^{-1} \psi \rangle \right|^2 \frac{d\theta}{2\pi}.
\]
What is the significance of this equality from the dynamical point of view?

(7) Suppose \( W \) is a quantum walk with periodic coins, that is, there is \( p \in \mathbb{N} \) such that \( Q_{n+p} = Q_n \) for all \( n \). Let \( \alpha = (\alpha_n)_{n \in \mathbb{Z}} \) be the Verblunsky coefficients of the associated CMV matrix. Show that \( \alpha \) is periodic modulo a phase. That is, show that there is some \( \theta \in \mathbb{R} \) and \( q \in \mathbb{N} \) such that \( \alpha_{n+q} = e^{i\theta} \alpha_n \) for all \( n \).

(8) Let \( \mathcal{E} \) be an extended CMV matrix with Verblunsky coefficients \( \alpha = (\alpha_n)_{n \in \mathbb{Z}} \). The sieved matrix \( \tilde{\mathcal{E}} \) is the CMV matrix with coefficients \( \tilde{\alpha} \) defined by
\[
\tilde{\alpha}_{2n} = \alpha_n, \quad \tilde{\alpha}_{2n+1} = 0, \quad n \in \mathbb{Z}.
\]
(a) Show that \( \tilde{\mathcal{E}}^2 \cong \mathcal{E} \oplus \mathcal{E}^\top \). To do this, you should find two natural invariant subspaces \( \mathcal{E} \) and \( \mathcal{E}^\top \) and show that \( \tilde{\mathcal{E}}^2|_A \cong \mathcal{E} \) and \( \tilde{\mathcal{E}}^2|_B \cong \mathcal{E}^\top \).

(b) What does this tell you about the relationship between \( \sigma(\mathcal{E}) \) and \( \sigma(\tilde{\mathcal{E}}) \)?

(9) Let \( (A_n)_{n=1}^{\infty} \) denote a sequence of \( \text{SL}(2, \mathbb{C}) \) matrices obeying the recursion
\[
A_{n+1} = A_{n-1} A_n, \quad n \geq 2.
\]
Denoting \( x_n = \text{tr}(A_n) \), show that \( x_{n+1} = x_n x_{n-1} - x_{n-2} \) for all \( n \geq 3 \).
Hints. Quantum Walks - Fillman

(1) Try identifying suitable orthonormal bases for each space.

(2) Polarization

(3) (a) First, handle the case when $\psi$ is a linear combination of eigenfunctions of $U$.

(b) Cauchy–Schwarz, Riemann–Lebesgue

(4) As a warm-up, try the case when $\{n_j\} = \mathbb{Z}$, i.e., $q_{n1}^{11} = q_{n2}^{22} = 0$ for every $n \in \mathbb{Z}$.

(5) The equation $W\varphi = z\varphi$ is an equality of vectors. Write out the components.

(6) Parseval.

(7) How does periodicity of $Q$ affect the $\lambda$’s in the gauge transform?

(8) (a) Calculate $\tilde{E}^2\delta_{4n}$, $\tilde{E}^2\delta_{4n+1}$, $\tilde{E}^2\delta_{4n+2}$, and $\tilde{E}^2\delta_{4n+3}$ for general $n \in \mathbb{Z}$. This should be enough to identify some invariant subspaces.

(b) Spectral mapping theorem

(9) Cayley–Hamilton.