Problem set: statistical properties.

(1) Let μ be a probability measure on a measurable space X, and let $T\colon X \to X$ be a measurable map. Prove that the following definitions of μ being T-invariant are equivalent.

- For every measurable $A \subset X$ we have $\mu(T^{-1}A) = \mu(A)$.
- For every measurable $\phi\colon X \to \mathbb{R}$ we have $\int \phi \, d\mu = \int \phi \circ T \, d\mu$.

(2) Let μ be an invariant probability measure for T. Prove that the following definitions of ergodicity are equivalent:

- If A is a measurable set such that $T^{-1}(A) = A$, then $\mu(A) = 0$ or $\mu(A) = 1$.
- If ϕ is a measurable function ϕ such that $\phi(x) = \phi(T(x))$ for μ-a.e. x, then ϕ is constant μ-a.e.
- If ν_1, ν_2 are invariant measures such that $\mu = a\nu_2 + b\nu_2$ for some $a, b \geq 0$ with $a + b = 1$, then $\nu_1 = \nu_2 = \mu$.

(3) Prove that $x \in \mathbb{R}/\mathbb{Z}$ is pre-periodic (that is, its trajectory terminates in a periodic orbit) for the doubling map if and only if it is rational. What are the periodic points?

(4) Let $R_\theta(x) = x + \theta$, (mod 1), where θ is irrational. Let μ be Lebesgue measure on $[0,1]$ and suppose that ϕ is a continuous function on the circle with $\int \phi \, d\mu = 0$. Show that $\frac{1}{n}\sum_{j=0}^{n-1} \phi \circ T^j(x) \to 0$ uniformly; that is, for every $\epsilon > 0$ there is a positive integer N such that for all $n \geq N$, we have $|\frac{1}{n}\sum_{j=0}^{n-1} \phi \circ T^j(x)| < \epsilon$ for *every* x. This property, "uniform convergence of Birkhoff averages for all continuous functions for a continuous transformation on a compact space" is equivalent to unique ergodicity.

(5) Let $(X, \mu, \mathcal{B})$ be a probability space and $T\colon X \to X$ a measure-preserving map. Given measurable sets $A, B \subset X$, let

$$C_n(A,B) := |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)|$$

be the nth correlation function of A, B. Similarly, given L^2 test functions ϕ, ψ, let

$$C_n(\phi,\psi) := \left| \int \phi \cdot (\psi \circ T^n)\, d\mu - \int \phi \, d\mu \int \psi \, d\mu \right|.$$

Show that $C_n(A,B) \to 0$ for every A, B if and only if $C_n(\phi,\psi) \to 0$ for every $\phi, \psi \in L^2$. In this case the measure is called *mixing*. Our results on decay of correlations all involve $C_n(\phi,\psi)$ for sufficiently regular functions, instead of $C_n(A,B)$, or $C_n(\phi,\psi)$ for arbitrary L^2 functions. This is because even when $C_n(\phi,\psi)$ decays exponentially for Hölder continuous functions (or some other nice class), we may have very slow decay for $C_n(A,B)$, or for arbitrary measurable functions.

Demonstrate this phenomenon as follows: let $T\colon [0,1] \to [0,1]$ be the doubling map $T(x) = 2x$, (mod 1), and let μ be Lebesgue measure on $[0,1]$. Find measurable sets $A, B \subset [0,1]$ such that $C_n(A,B)$ only decays polynomially – that is, there are $\gamma, c > 0$ such that $C_n(A,B) \geq cn^{\gamma}$ for all n.

Note that this is equivalent to answering the same question where X is the full shift on two symbols and μ is $(\frac{1}{2}, \frac{1}{2})$-Bernoulli.

(6) Let ϕ be a bounded measurable function on a measure-preserving dynamical system (T, X, μ) with $\int \phi d\mu = 0$. Suppose for all $n > 0$

$$|\int \phi(x)\phi(T^n x)d\mu| \leq p(n)$$

where $\sum_n p(n) < \infty$. Let $S_n(x) = \sum_{j=0}^{n-1} \phi \circ T^j(x)$.

(a) Using the fact that if $j \geq i$, $\int \phi(T^j x)\phi(T^i x)d\mu = \int \phi(T^{j-i}x)\phi(x)d\mu$ show that

$$\int (S_n(x))^2 d\mu \leq Cn$$

for some constant C.

(b) Show that $\lim_{n\to\infty} \frac{1}{n} \int (S_n(x))^2 d\mu := \sigma^2$ exists.

(c) Use Chebychev's inequality

$$\mu(x : |S_n(x)/n| > \epsilon) \leq \frac{1}{n^2\epsilon^2} \int (S_n(x))^2 d\mu$$

to show that if

$$|\int \phi(x)\phi(T^n x)d\mu| \leq p(n)$$

where $\sum_n p(n) < \infty$ then

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = 0$$

for μ a.e. $x \in X$.

(7) Suppose that (T, X, μ) is a measure preserving of a metric and measure space X and let $\mathcal{B}$ be the Borel σ-algebra on X. Suppose that T is non-invertible and define $\mathcal{F}_j = T^{-j}\mathcal{B}$, which is a decreasing sequence of sub-σ-algebras of $\mathcal{B}$. Suppose that ϕ is a integrable function on (X, μ) and $E[\phi|\mathcal{F}_1] = 0$. Show that $E[\phi \circ T^j|\mathcal{F}_{j+1}] = E[\phi|\mathcal{F}] \circ T^j$ and hence $E[\phi \circ T^j|\mathcal{F}_{j+1}] = 0$.

(8) Let $(X_n)_{n\geq 1}$ be a sequence of square integrable random variables on a probability space (X, μ) and let $\mathcal{F}_n$ be a sequence of increasing σ-algebras with X_n being $\mathcal{F}_n$ measurable. Suppose that $E[X_n|\mathcal{F}_{n-1}] = 0$ for all $n \geq 1$. Show that $E[X_i X_j] = 0$ for

all $i \neq j$. *Hint: Use the fact that if X is $\mathcal{F}$-measurable and $E|X| < \infty$, $E|XY| < \infty$ then $E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$.*