2019 Houston Summer School on Dynamical Systems

Problem set: Uniform hyperbolicity

1. Let $I_0 = [0, 1/3]$ and $I_1 = [2/3, 1]$, and write $I = I_0 \cup I_1$. Define a map $f\colon I \to [0,1]$ by $f(x) = 3x \pmod 1$.

(a) Let $C = \{x \in [0,1] : f^n(x) \text{ is defined for every } n \geq 0\} = \bigcap_{n\geq 0} f^{-n}(I)$. Prove that C is the middle-third Cantor set.

(b) Let $\Sigma = \{0,1\}^{\mathbb{N}_0} = \{z_0 z_1 z_2 \cdots : z_n \in \{0,1\} \text{ for all } n \geq 0\}$ be the full shift on 2 symbols, with the metric $d(y,z) = 2^{-\min\{n : y_n \neq z_n\}}$. Define the shift map $\sigma\colon \Sigma \to \Sigma$ by $\sigma(z_0 z_1 z_2 \dots) = z_1 z_2 \dots$. Define $h\colon \Sigma \to C$ by $h(z) = \sum_{n=0}^{\infty} 2z_n/3^{n+1}$; prove that h is a homeomorphism and that $f \circ h = h \circ \sigma$.

2. (a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with eigenvalues $0 < \lambda_1 < \lambda_2$. Let P be the set of lines through the origin in $\mathbb{R}^2$; make this into a metric space by setting the distance between two lines to be the angle they make at the origin. Let $\ell_1, \ell_2 \in P$ be the eigenlines for λ_1, λ_2. Show that if $\ell \in P \setminus \{\ell_1, \ell_2\}$, then as $n \to \infty$ we have $T^n \ell \to \ell_2$ and $T^{-n}\ell \to \ell_1$.

(b) Formulate a more general version of this result that works in higher dimensions.

3. Let $c < \frac{\sqrt{17}}{2} - \sqrt{2} < \frac{3\sqrt{17}}{10\sqrt{2}}$. Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map that fixes the origin and satisfies $\|Dg_{(x,y)}\| \leq c$ for every $(x,y) \in \mathbb{R}^2$, where here $\|\cdot\|$ denotes the norm of a linear operator with respect to the Euclidean norm on $\mathbb{R}^2$. Define $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (2x, y/2) + g(x,y)$.

(a) Let $K = \{(v,w) \in \mathbb{R}^2 : |w| < |v|\}$, and prove that $Df_{(x,y)}(\overline{K}) \subset K$ for all $(x,y) \in \mathbb{R}^2$.

(b) Prove that there is $\lambda > 1$ such that $\|Df_{(x,y)}(v,w)\| \geq \lambda \|(v,w)\|$ for every $(x,y) \in \mathbb{R}^2$ and $(v,w) \in K$.

(c) Let $X = \{\varphi\colon \mathbb{R} \to \mathbb{R} : \varphi(0) = 0 \text{ and } \varphi \text{ is 1-Lipschitz}\}$. Given $\varphi \in X$, prove that $f(\operatorname{graph}\varphi)$ is the graph of some $\psi \in X$; the map $\varphi \mapsto \psi$ is called the *graph transform*.

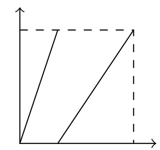
(d) Given $\varphi, \psi \in X$, let $d(\varphi,\psi) = \sup\{|\varphi(x) - \psi(x)|/|x| : x \neq 0\}$; prove that this makes X a complete metric space on which the graph transform is a contraction. Deduce that there is a unique fixed point $\varphi \in X$, and write $W^u = \operatorname{graph}\varphi$; then $f(W^u) = W^u$.

(e) Assuming f is invertible, prove that given any two $\mathbf{p}, \mathbf{q} \in W^u$, we have $d(f^{-1}\mathbf{p}, f^{-1}\mathbf{q}) \leq \lambda^{-1} d(\mathbf{p},\mathbf{q})$, and deduce that $W^u = \{\mathbf{p} \in \mathbb{R}^2 : f^{-n}(\mathbf{p}) \to (0,0) \text{ as } n \to \infty\}$.

(f) Correct any wrong or missing assumptions in the statement of the problem that resulted from the haste in which this was written the day before the summer school started.

4. Fix $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and define a map $f\colon [0,1) \to [0,1)$ by

$$f(x) = \begin{cases} px & 0 \le x < \frac{1}{p}, \\ q\left(x - \frac{1}{p}\right) & \frac{1}{p} \le x < 1. \end{cases}$$



Let $\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log(f^n)'(x)$, if the limit exists.

(a) Find the set of all possible values of $\lambda(x)$.

(b) Given $\alpha \in \mathbb{R}$, find the Lebesgue measure of $\{x : \lambda(x) = \alpha\}$.

(c) You can also think about trying to find the Hausdorff dimension of this set. (This is harder and requires some machinery we didn't discuss in the lecture.)

5. Complete the proof outlined in the second lecture that Lebesgue measure is ergodic for a hyperbolic toral automorphism, using the Hopf argument.

6. Let $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ be an Anosov diffeomorphism, meaning that for every $x \in \mathbb{T}^2$ there is a splitting $\mathbb{R}^2 = T_x\mathbb{T}^2 = E^u_x \oplus E^s_x$ such that $\|Df^n_x|_{E^s_x}\| \le C\lambda^n$ and $\|Df^{-n}_x|_{E^u_x}\| \le C\lambda^n$, where $C > 0$ and $\lambda < 1$ are independent of x. Then it can be shown, using an analogue of the argument in #3 above, that for every $x \in \mathbb{T}^2$ there is a curve W^s_x through x such that $T_x W^s_x = E^s_x$ and such that for every $y, z \in W^s_x$ we have $d(f^n y, f^n z) \le \lambda^n d(y,z)$ for all $n \ge 0$.

(a) Let $\varphi\colon \mathbb{T}^2 \to \mathbb{R}$ be Hölder continuous, and prove that there is $L > 0$ such that for every $y, z \in W^s_x$ and $n \ge 0$, we have $|S_n\varphi(y) - S_n\varphi(z)| \le L$, where $S_n\varphi = \varphi + \varphi\circ f + \cdots + \varphi \circ f^{n-1}$ is the nth Birkhoff sum.

(b) Mimic the argument in #2(a) to show that for every $\theta > 0$, there are $Q > 0$ and $\gamma \in (0,1)$ such that given any $y, z \in W^s_x$ and $v \in T_y\mathbb{T}^2 = \mathbb{R}^2$, $w \in T_z\mathbb{T}^2 = \mathbb{R}^2$ satisfying $\measuredangle(v, E^s_y) \ge \theta$ and $\measuredangle(w, E^s_z) \ge \theta$, we have $d(Df^n_y(v), Df^n_z(w)) \le Q\gamma^n$ for all $n \ge 0$.

(c) Suppose that f is $C^{1+\alpha}$, meaning that $x \mapsto Df_x$ is Hölder continuous, and combine the ideas from parts (a) and (b) to prove the following property of *absolute continuity of holonomy maps*: for every $\theta > 0$, there is $K > 0$ such that

- ◇ if V_1, V_2 are two curves such that the angle between V_i and W^s_x is at least θ at every point x on each V_i, and
- ◇ if for every $x \in V_1$ there exists a unique point $\pi(x) \in V_2 \cap W^s_x$, and
- ◇ if we write m_{V_1} and m_{V_2} for Lebesgue measure (length) along V_1 and V_2,

then $\pi_* m_{V_1} \ll m_{V_2}$, and the corresponding Radon–Nikodym derivative is bounded above by K. In fact you should find a formula for this derivative.