Problem Solving. - Jan 29, 2016

- 11.884 - January 2016 AMM - No 1.

Let $f : [0,1] \longrightarrow \mathbb{R}$, $f \in C^2([0,1])$.
Prove that if

$f(\frac{1}{2}) = 0$ then

$$\int_0^1 (f''(x))^2 dx \geq 320 \left(\int_0^1 f(x) dx \right)^2$$

--- Sol. by May 31, 2016 ----

• 11886 — January 2016 — AMM — No. 1

Suppose $n \geqslant 3$ and let $y_1, \ldots, y_n$ be a list of real numbers such that $2y_{k+1} \leq y_k + y_{k+2}$ for $1 \leq k \leq n-2$.

Suppose that $\sum_{k=1}^{n} y_k = 0$.

Prove that

$$\sum_{k=1}^{n} k^2 y_k \geqslant (n+1) \sum_{k=1}^{n} k y_k \quad \text{and}$$

determine when equality holds.

--- Sol. by May 31, 2016 ----

• 11877 — No 10, AMM, December 2015

Let f be differentiable function $f: \mathbb{R} \to \mathbb{R}_+$ such that

$$\lim_{x \to \infty} \frac{x f'(x)}{f(x)} = 0$$

Let g be a function on $\mathbb{R}$ such that

$$\lim_{x \to \infty} g(x) > -1$$

Prove that

$$\lim_{x \to \infty} \frac{f(x + x g(x))}{f(x)} = 1.$$

— Sol by — April 30, 2016 ----

• 11876 — AMM, No 10 — December 2015

Let a and b be the roots of $x^2+x+\frac{1}{2}=0$.
Find
$$\sum_{n=1}^{\infty} \frac{(-1)^n(a^n+b^n)}{n+2} \qquad (1)$$

--- Sol by April 30 - 2016 -----

Hypothesis: a, b roots of $x^2+x+\frac{1}{2}=0$

Conclusion. Prove (1).

Ideas

Solve $x^2+x+\frac{1}{2}=0$ to get $a=\frac{-1+i}{2}$

$b=\frac{-1-i}{2}$

1. Calculate a^n+b^n

2. Observe that $a=\frac{-\sqrt{2}}{2}\cdot e^{-i\frac{\pi}{4}}$,

$b=\frac{-\sqrt{2}}{2}e^{+i\frac{\pi}{4}}$

To be continued

11872 — AMM — No 10 — Dec 2015

Let f be a continuous function $f: [0,1] \to \mathbb{R}$ such that:

$$\int_0^1 f(x)\,dx = 0.$$

Prove that for all positive integers n there exists $c \in (0,1)$ such that.

$$n \int_0^c x^n f(x)\,dx = c^{n+1} f(c).$$

— Sol by March 31 2016 ———.

Hypothesis: $f: [0,1] \longrightarrow \mathbb{R}$ continuous

$$\int_0^1 f(x)\,dx = 0$$

Conclusion: Prove that

$\exists\, c \in (0,1)$ such that for all n positive integers

$$n \int_0^c x^n f(x)\,dx = c^{n+1} f(c)$$

Ideas towards a solution

Try to show that the conclusion is true for $n=0$.

Conclusion for $n=0$:

$$\exists c_0 \in (0,1) \text{ s.t } f(c_0)=0.$$

This is true by applying the mean value theorem to f. and using

$$\int_0^1 f(x)dx = 0$$

Idea Consider the particular case $n=1$.

Need to show that $\exists c_1 \in (0,1)$ such that.

$$\int_0^{c_1} x f(x) dx = c_1^2 f(c_1)$$

Integration by parts

$$\int_0^{c_1} x f(x) dx = c_1 F(c_1) - \int_0^{c_1} F(x) dx$$

where

$$F(x) = \int_0^x f(t)\,dt.$$

Thus, in other words we need to prove that:

$\exists c_1 \in (0,1)$ such that

$$c_1 F(c_1) - \int_0^{c_1} F(x)\,dx = c_1^2 F'(c_1).$$

where $F(x) = \int_0^x f(t)\,dt$

Another Idea

Consider $f(x) = \cos \pi x$ and prove the conclusion for this function

Then build an approximation of f with trigonometric function

Another idea

Consider $f(x) = 2x - 1$.

Then prove the conclusion for this function.

Then prove the conclusion for any polynomial P with $\int_0 P(x)\,dx = 0$.

Then use Taylor expansion for a smooth function f / or some polynomial approximation (Weierstrass Theorem).

Another Hint

To be continued