Problem Solving – February 12, 2016

Further progress on 11876 and 11872.

Also, we propose several possible extensions for this problems, towards interesting research projects.

• 11876 — AMM, No 10 — December 2015

Let a and b be the roots of $x^2+x+\frac{1}{2}=0$, Find

$$\sum_{n=1}^{\infty} \frac{(-1)^n(a^n+b^n)}{n+2} \quad (1)$$

--- Sol by April 30 - 2016 ---

Hypothesis: a, b roots of $x^2+x+\frac{1}{2}=0$

Conclusion. Prove (1)

Ideas

Solve $x^2+x+\frac{1}{2}=0$ to get $a=\frac{-1+i}{2}$

$b=-\frac{1-i}{2}$

1. Calculate a^n+b^n

2. Observe that $a=-\frac{\sqrt{2}}{2}\cdot e^{-i\frac{\pi}{4}}$,

$b=-\frac{\sqrt{2}}{2}e^{+i\frac{\pi}{4}}$ (1)

To be continued

Plug in formulas (1) and obtain.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (a^n + b^n)}{n+2} = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n \cdot \frac{e^{-i\frac{n\pi}{4}} + e^{i\frac{n\pi}{4}}}{n+2}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n \frac{2\cos\frac{n\pi}{4}}{n+2}.$$

<u>Subproblems</u>:

Calculate

$$\sum_{n=1}^{\infty} \cos\frac{n\pi}{4}$$

$$\cancel{\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n} \quad \sum_{n=1}^{\infty} \frac{\cos n\frac{\pi}{4}}{n+2}$$

Then use these ideas to compute the final sum

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n \frac{2\cdot\cos\frac{n\pi}{4}}{n+2}.$$

~~Extensions~~

Extension of 11876 proposed by the students

I. Let a, b be the roots of $x^2+x+\frac{1}{2}=0$.
Find $\alpha>0$ such that.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (a^n+b^n)}{n+\alpha} \text{ is convergent}$$

and. find its sum

II. Let a, b be the roots of $Ax^2+Bx+C=0$.
Find A, B, C such that

$$\sum_{n=1}^{\infty} \frac{(-1)^n (a^n+b^n)}{n+2} \text{ is convergent}$$

and determine its sum

III. Let $a_1, a_2, --, a_n$ be the roots of.
$b_n x^n + b_{n-1} x_{n-1} + -- + b_1 x + b_0 = 0.$, $b_n \neq 0$.
Find $b_0, b_1, --, b_n$ such that.

$$\sum_{k=1}^{\infty} \frac{(-1)^k \left(\sum_{i=1}^{n} a_i^k\right)}{n+2} \text{ is convergent and}$$

determine its sum

• 11872 — AMM — No 10 — Dec 2015

Let f be a continuous function $f : [0,1] \to \mathbb{R}$ such that.

$$\int_0^1 f(x)\,dx = 0.$$

Prove that for all positive integers n there exists $c \in (0,1)$ such that.

$$n \int_0^c x^n f(x)\,dx = c^{n+1} f(c).$$

— Sol by March 31 2016 ————

Hypothesis: $f : [0,1] \longrightarrow \mathbb{R}$ continuous

$$\int_0^1 f(x)\,dx = 0$$

Conclusion: Prove that

$\exists\, c \in (0,1)$ such that for all n positive integers

$$n \int_0^c x^n f(x)\,dx = c^{n+1} f(c)$$

Ideas towards a solution

Try to show that the conclusion is true for $n=0$.

Conclusion for $n=0$:

$\exists\, c_0 \in (0,1)$ s.t $f(c_0)=0$.

This is true by applying the mean value theorem to f. and using

$$\int_0^1 f(x)\,dx = 0$$

Idea

Consider the particular case $n=1$.

Need to show that $\exists\, c_1 \in (0,1)$ such that.

$$\int_0^{c_1} x f(x)\,dx = c_1^2 f(c_1)$$

Integration by parts

$$\int_0^{c_1} x f(x)\,dx = c_1 F(c_1) - \int_0^{c_1} F(x)\,dx$$

where
$$F(x) = \int_0^x f(t)\,dt.$$

Thus, in other words we need to prove that:

$\exists c_1 \in (0,1)$ such that
$$c_1 F(c_1) - \int_0^{c_1} F(x)\,dx = c_1^2 F'(c_1).$$
where $F(x) = \int_0^x f(t)\,dt$

<u>Another Idea</u>

Consider $f(x) = \cos \pi x$ and prove the conclusion for this function

Then build an approximation of f with trigonometric functions

Another idea

Consider $f(x) = 2x - 1$.

Then prove the conclusion for this function.

Then prove the conclusion for any Polynomial P with $\int_0 P(x)\,dx = 0$.

Then use Taylor expansion for a smooth function f / or some polynomial approximation (Weierstrass Theorem)!

Another Hint

To be continued

Note that after a change of variables $y = \frac{x}{c}$ we reformulate the question as.

Does there exists $c \in (0,1)$ such that $(*)$

$$n \int_0^1 y^n f(yc)\,dy = f(c) \ ?$$

Extension of 11872 proposed by the students.

I. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous function such that.

$$\int_{-\infty}^{\infty} f(x)\,dx = 0.$$

Does then exists $c \in \mathbb{R}$ such that.

$$n \int_{-\infty}^{c} x^n f(x)\,dx = c^{n+1} f(c) \quad ?$$

II Let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ continuous such that.

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 0$$

Does then exists $(c_1, c_2) \in (0,1) \times (0,1)$ such that

$$n \int_0^{c_1} \int_0^{c_2} x^n y^n f(x,y)\,dx\,dy = \left(\sqrt{c_1^2 + c_2^2}\right)^{n+1} f(c_1, c_2) \quad ?$$

or does then exist, $(c_1, c_2) \in (0,1) \times (0,1)$ s.t.

$$n \int_0^{c_1} \int_0^{c_2} (x^n + y^n) f(x,y)\,dx\,dy = \left(\sqrt{c_1^2 + c_2^2}\right)^{n+1} f(c_1, c_2) \quad ?$$

III Observe that the initial question can be reformulated (after a change of variable $y = \frac{x}{c}$) as.

Does then exists $c \in (0,1)$ such that

$$n \int_0^1 y^n f(cy)\, dy = f(c) \quad ?$$

This can be extended to the following question in the complex domain.

Let $f : D \to \mathbb{C}$ continuous when

$D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$. Assume also that

$$\int_D f = 0 .$$

Does then exists $z_0 \in D$ such that.

$$n \int_D z^n f(z_0 \cdot z)\, dz = f(z) \quad ?$$

IV An interesting question is if the condition $\int_0^1 f(x)dx$ is necessary or it can be replaced with another!

$\underline{V}$ Does the result of 11872 hold for non-integer values of n?