Problem solving session, February 19, 2016.

We considered problems 11886 and 11877.

Some ideas were proposed and we will continue with this and possible extensions for research projects.

• 11886 — January 2016 — AMM — No. 1

Hypothesis 1: Suppose $n \geq 3$ and let $Y_1, \dots, Y_n$ be a list of real numbers such that $2Y_{k+1} \leq Y_k + Y_{k+2}$ for $1 \leq k \leq n-2$.

Hypothesis 2: Suppose that $\sum_{k=1}^{n} Y_k = 0$.

Conclusion: Prove that

$$\sum_{k=1}^{n} k^2 Y_k \geq (n+1) \sum_{k=1}^{n} k Y_k$$

and determine when equality holds.

--- Sol. by May 31, 2016 ----

<u>Idea</u>: Consider first the case $n=3$.

The problem states: Y_1, Y_2, Y_3 such that

$$Y_1 + Y_2 + Y_3 = 0 \quad \text{and} \quad Y_2 \leq \tfrac{1}{2}(Y_1 + Y_3).$$

Show $$Y_1 + 4Y_2 + 9Y_3 \geq 4(Y_1 + 2Y_2 + 3Y_3)$$

For $n=3$ the conclusion can be reewriten.

Show that

$$3Y_1 + 4Y_2 + 3Y_3 \le 0. \quad (1)$$

Using $Y_1+Y_2+Y_3=0 \Rightarrow (1)$ is gnivelut for

$$Y_2 \le 0. \quad (2)$$

Now from $Y_2 \le \frac{1}{2}(Y_1+Y_3)$ we obtain.

by using $Y_1+Y_3+Y_2=0$, we obtain.

$$\frac{3}{2}Y_2 \le 0 \Rightarrow Y_2 \le 0.$$

So (2) is true. !

Then we discussed the case $n=4$!

The problem in this case states

Y_1, Y_2, Y_3, Y_4 such that

$$\sum_{i=1}^{4} Y_i = 0 \quad \text{and} \quad \begin{cases} Y_2 \le \frac{1}{2}(Y_1+Y_3) & (*) \\ Y_3 \le \frac{1}{2}(Y_2+Y_4) & (**) \end{cases}$$

show $\sum_{k=1}^{4} k^2 Y_k \geq 5 \cdot \sum_{k=1}^{4} k Y_k. \quad (3)$

The conclusion (3) can be rewritten to obtain.

Show.

$$0 \geq 4Y_1 + 6Y_2 + 6Y_3 + 4Y_4 \quad (4)$$

Using $\sum_{i=1}^{4} Y_i = 0$ in (4) we obtain.

$$0 \geq Y_2 + Y_3. \quad (5)$$

From (*) and (* *) we obtain. (after adding them up).

$Y_2 + Y_3 \leq 0$. thus (5) is true.

The general solution could be developed along this lines, To be continued next time!)

• 11877 — No 10, AMM, December 2015

Let f be differentiable function $f: \mathbb{R} \to \mathbb{R}_+$ such that

Hypothesis 1 $\left[\quad \lim_{x\to\infty} \frac{x f'(x)}{f(x)} = 0 \quad (1) \right.$

Hypothesis 2. $\left[\right.$ Let g be a function on $\mathbb{R}$ such that

$$\lim_{x\to\infty} g(x) > -1 \qquad (2)$$

Prove that

$$\left. \lim_{x\to\infty} \frac{f(x+xg(x))}{f(x)} = 1. \right]$$ Conclusion.

— Sol by — April 30, 2016 ———

Ideas

Hypothesis 1 could be written as.

$$\lim_{x\to\infty} \frac{x f'(x)}{f(x)} = 0 \iff \lim_{x\to\infty} x (\ln f)' = 0 .$$

Idea: Observe that polynomials

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Cannot satisfy. Hypothesis 1. (1).

Inquiry

$$\begin{cases} \text{Can } f \text{ be linear?} & (3) \\ f(x+y) = f(x) + f(y) & \end{cases}$$

Answer: No. It can be shown that smooth functions (continuous) satisfying (3) are of the form.

$f(x) = ax.$ and we just showed that polynomials cannot satisfy Hypothesis 1 (1),

Idea $\lim_{x \to \infty} \frac{x f'(x)}{f(x)} = 0$ implies $\lim_{x \to \infty} \frac{f'(x)}{f(x)} = 0.$

Also $x + x g(x) = x(1 + g(x))$ and (2) implies that $x + x g(x) > 0$ for $x \to \infty.$

Idea Maybe we could employ Cauchy's theorem.

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \text{ for some } \xi \in (a,b)$$

Observe that

$$\frac{f(x+xg(x))}{f(x)}-1=\frac{f(x+xg(x))-f(x)}{f(x)}$$

$\exists$ c_x between $x+xg(x)$ and x such that.

$$\frac{f(x+xg(x))}{f(x)}-1=\frac{f'(c_x)}{f(x)}\cdot xg(x) \quad (3)$$

Observe that, after algebraic manipulations we could. have, (3) equivalent with.

$$\frac{f(x+xg(x))}{f(x)}-1=\frac{f'(c_x)\,c_x}{f(c_x)}\;\frac{xg(x)}{f(x)}\cdot\frac{f(c_x)}{c_x}$$

Note that by definition $c_x\to\infty$ as $x\to\infty$

So $$\frac{f'(c_x)\,c_x}{f(c_x)}\longrightarrow 0 \quad \text{as } x\to\infty \text{ by (1)}$$

To be continued!