MENTOR Program: Mathematical Induction

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Inductive reasoning is prevalent throughout science. Two common inductive principles are used in mathematics. The ***principle of mathematical induction*** (PMI) asserts that if $P(n)$ is a statement satisfying

- $P(1)$ is true, and
- for every natural number k, if $P(k)$ is true, then $P(k+1)$ is true,

it follows that $P(n)$ is true for every natural number n. The ***principle of complete induction*** (PCI), a second and seemingly stronger inductive principle, asserts that if $P(n)$ is a statement satisfying

- $P(1)$ is true, and
- for every natural number k, if $P(j)$ is true for all natural numbers j with $1 \leqslant j \leqslant k$, then $P(k+1)$ is true,

it follows that $P(n)$ is true for every natural number n. It turns out that PMI and PCI are logically equivalent. (Why?)

Exercise 1 (warm up) Show that

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

for every natural number n.

Solution. Equation (1) holds for $n = 1$, since $1 = (1 \cdot 2)/2$. Now let k be a natural number and assume that (1) holds for $n = k$. For the induction step, we have

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= [1 + 2 + \cdots + k] + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= (k+1)\left(\frac{k}{2} + 1\right) \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

The induction is complete.

Exercise 2 Consider a puzzle consisting of three posts, with n concentric rings of decreasing diameter stacked on the first (ring size decreases from bottom to top). A ring at the top of a stack may be moved from one post to another, provided that it is not placed on top of a smaller ring. What is the minimal number of moves needed to transfer the entire initial stack onto the second post?

Solution. During the MENTOR session, we noticed that the minimal number of moves for 1, 2, 3, and 4 rings is 1, 3, 7, and 15, respectively. This suggests that the general answer is $2^n - 1$ moves for n rings. Let us prove this using induction. We have already checked the case $n = 1$. Now assume that the minimal number of moves for k rings is $2^k - 1$. To move $k+1$ rings from post one to post two, proceed as follows:

- move the top k rings on the stack from post 1 to post 3 ($2^k - 1$ moves using the induction hypothesis);
- move the remaining ring on post 1 to post 2 (one move);
- move the k rings on post 3 to post 2 ($2^k - 1$ moves using the induction hypothesis).

This algorithm requires

$$(2^k - 1) + 1 + (2^k - 1) = 2^{k+1} - 1$$

moves. Once you convince yourself that the algorithm produces the minimal number of moves for $k+1$ rings, the induction is complete.

Exercise 3 A group of people with assorted eye colors live on an island. They are all perfect logicians - if a conclusion can be logically deduced, they will do it instantly. No one knows the color of their eyes. Every night at midnight, a ferry stops at the island. Any islanders who have figured out the color of their own eyes then leave the island, and the rest stay. Everyone can see everyone else at all times and keeps a count of the number of people they see with each eye color (excluding themselves), but they cannot otherwise communicate. Everyone on the island knows all the rules in this paragraph.

On this island there are 100 blue-eyed people, 100 brown-eyed people, and the Guru (she happens to have green eyes). So any given blue-eyed person can see 100 people with brown eyes and 99 people with blue eyes (and one with green), but that does not tell him his own eye color; as far as he knows the totals could be 101 brown and 99 blue. Or 100 brown, 99 blue, and he could have red eyes.

The Guru is allowed to speak once (let's say at noon), on one day in all their endless years on the island. Standing before the islanders, she says the following: "I can see someone who has blue eyes."

Who leaves the island, and on what night?

Solution. First assume 1 blue-eyed person and 1 brown-eyed person live on the island. Once the Guru speaks, the blue-eyed person will immediately know that she has blue eyes and will therefore leave the first night. Now assume 2 blue-eyed people and 2 brown-eyed people live on the island. Nothing will happen the first night, since each blue-eyed person can see one other person with blue eyes. On the second night, each blue-eyed person will realize that her eyes are blue because the other blue-eyed person did not leave the first night! Thus both blue-eyed people will leave the island on the second night. Continuing inductively, if 100 blue-eyed people and 100 brown-eyed people live on the island, then all 100 blue-eyed people will leave on night 100.

Exercise 4 If n lines are drawn in a plane and no two lines are parallel, into how many regions do the lines separate the plane?

Solution. More information is needed in order for the problem to have a unique solution. Consider the case $n = 3$, for example. We obtain 7 regions if no more than two lines intersect at any given point, but 6 regions if all three lines intersect at a common point.

Let us make the additional assumption that no more than two lines intersect at any given point. With zero lines, we have 1 region. Adding the first line increases the number of regions by 1, for a total of 2. Adding the second line increases the number of regions by 2 - 1 for the new line and 1 for the intersection with the other line. This brings the total number of regions for two lines to 4. Now suppose k lines have already been drawn. Adding line $k + 1$ increases the number of regions by $k + 1$ - 1 for the new line and k due to the k new intersection points with existing lines. This reasoning leads to the conclusion that for n lines, the number of regions is

$$1 + (\text{number of lines}) + (\text{number of points of intersection}) = 1 + n + n(n-1)/2.$$

Exercise 5 $2N$ dots are placed around the outside of a circle. Then N of them are colored red and the remaining N are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the right point.

Solution. If $N = 1$, then starting at any point counterclockwise from the red dot and clockwise from the blue dot will produce a successful trip around the circle. Now assume that a good starting point exists whenever there are K red dots and K blue dots. If we start with $K + 1$ red dots and $K + 1$ blue dots, how do we show that a good starting point exists and thereby complete the inductive step? **Hint:** Remove a cleverly chosen red dot and a cleverly chosen blue dot and then use the inductive assumption.