MENTOR Program: Mathematical Induction

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Inductive reasoning is prevalent throughout science. Two common inductive principles are used in mathematics. The ***principle of mathematical induction*** (PMI) asserts that if $P(n)$ is a statement satisfying

- $P(1)$ is true, and
- for every natural number k, if $P(k)$ is true, then $P(k+1)$ is true,

it follows that $P(n)$ is true for every natural number n. The ***principle of complete induction*** (PCI), a second and seemingly stronger inductive principle, asserts that if $P(n)$ is a statement satisfying

- $P(1)$ is true, and
- for every natural number k, if $P(j)$ is true for all natural numbers j with $1 \leqslant j \leqslant k$, then $P(k+1)$ is true,

it follows that $P(n)$ is true for every natural number n. It turns out that PMI and PCI are logically equivalent. (Why?)

Exercise 1 (warm up) Show that

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

for every natural number n.

Exercise 2 Consider a puzzle consisting of three posts, with n concentric rings of decreasing diameter stacked on the first (ring size decreases from bottom to top). A ring at the top of a stack may be moved from one post to another, provided that it is not placed on top of a smaller ring. What is the minimal number of moves needed to transfer the entire initial stack onto the second post?

Exercise 3 A group of people with assorted eye colors live on an island. They are all perfect logicians - if a conclusion can be logically deduced, they will do it instantly. No one knows the color of their eyes. Every night at midnight, a ferry stops at the island. Any islanders who have figured out the color of their own eyes then leave the island, and the rest stay. Everyone can see everyone else at all times and keeps a count of the number of people they see with each eye color (excluding themselves), but they cannot otherwise communicate. Everyone on the island knows all the rules in this paragraph.

On this island there are 100 blue-eyed people, 100 brown-eyed people, and the Guru (she happens to have green eyes). So any given blue-eyed person can see 100 people with brown eyes and 99 people with blue eyes (and one with green), but that does not tell him his own eye color; as far as he knows the totals could be 101 brown and 99 blue. Or 100 brown, 99 blue, and he could have red eyes.

The Guru is allowed to speak once (let's say at noon), on one day in all their endless years on the island. Standing before the islanders, she says the following: "I can see someone who has blue eyes."

Who leaves the island, and on what night?

Exercise 4 If n lines are drawn in a plane and no two lines are parallel, into how many regions do the lines separate the plane?

Exercise 5 $2N$ dots are placed around the outside of a circle. Then N of them are colored red and the remaining N are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the right point.