MENTOR Program: More Mathematical Induction

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Inductive reasoning is prevalent throughout science. Two common inductive principles are used in mathematics. The ***principle of mathematical induction*** (PMI) asserts that if $P(n)$ is a statement satisfying

- $P(1)$ is true, and
- for every natural number k, if $P(k)$ is true, then $P(k+1)$ is true,

it follows that $P(n)$ is true for every natural number n. The ***principle of complete induction*** (PCI), a second and seemingly stronger inductive principle, asserts that if $P(n)$ is a statement satisfying

- $P(1)$ is true, and
- for every natural number k, if $P(j)$ is true for all natural numbers j with $1 \leqslant j \leqslant k$, then $P(k+1)$ is true,

it follows that $P(n)$ is true for every natural number n. It turns out that PMI and PCI are logically equivalent. (Why?)

Exercise 1 $2N$ dots are placed around the outside of a circle. Then N of them are colored red and the remaining N are colored blue. Going around the circle clockwise, you keep a count of how many red and blue dots you have passed. If at all times the number of red dots you have passed is at least the number of blue dots, you consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if you start at the right point.
Hint: For the inductive step, remove a pair of cleverly chosen points.

Exercise 2 Prove that the number of binary sequences of length n with an even number of 1s is equal to the number of binary sequences of length n with an odd number of 1s.

Exercise 3 (Fibonacci sequence) Let $(F_n)_{n=0}^{\infty}$ be the sequence defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$. Prove that $F_n^2 = F_{n-1}F_{n+1} - (-1)^n$ for every integer $n \geqslant 1$.

Exercise 4 (Geometric-arithmetic mean inequality) Show that if $a_1, a_2, \ldots, a_n$ are nonnegative numbers, then

$$(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \leqslant \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Hint: First prove this for powers of two and then use 'backward' induction to fill in the gaps.

Exercise 5 Let S be the set of positive integers. Let $h : S \to S$ be a bijective (one-to-one and onto) function. Prove that there do not exist functions $f : S \to S$ and $g : S \to S$ with

- f injective (one-to-one),
- g surjective (onto), and
- $f(n)g(n) = h(n)$ for all $n \in S$.

Hint: Argue by contradiction.