The Pigeonhole Principle

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Dirichlet's Pigeonhole Principle:

Assume we have more pigeons than pigeonholes. Then there will be a pigeonhole with at least two pigeons.

Warm-up problems:

1. Among any 13 people, there are two who were born in the same month.
2. If you pick five numbers from the integers 1 to 8, then two of them add up to 9.
3. Among any 5 integer numbers, there are two whose difference is divisible by 4. [At least how many numbers we need to have a difference divisible by n?]
4. 13 squares of side 1 are placed inside a circle of radius 2. Then there are two squares that have a point in common.
5. In any cocktail party with at least two people, there must be two people who have the same number of friends. [Assume that "friend" is symmetric: if A is a friend of B, then B is a friend of A.]

More problems:

6. Assume five points are selected in an equilateral triangle with side length 2. Then there are two points at most 1 unit apart.
7. We are given 11 infinite sequences of the digits $0, 1, \ldots, 9$. Then there are two sequences whose digits coincide in infinitely many places.
8. Each point of the plane is colored with one of 3 colors. Then there is a rectangle with all four vertices having the same color. [Note: can replace 3 with any other number. This is the same as another problem on this list!]
9. (Dirichlet's approximation theorem) Let α be a real number and $N > 0$ an integer. Then there is a rational number p/q with $1 \leq q \leq N$ such that
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN}$$
In particular, for α irrational this implies that there are infinitely many rational numbers p/q such that
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$
10. Among any 6 people, there are either 3 who are mutual friends or 3 who are mutual strangers. ["Friend" is a symmetric relation. Lookup Ramsey numbers for more about this.]
11. An athlete is training for a triathlon. Over a 30 day period, he pledges to train at least once per day, and 45 times in all. Then there will be a period of consecutive days where he trains exactly 14 times.
12. Given 9 points inside the unit square, there are three of them that form a triangle of area at most 1/8.