2

Sequences: Convergence and Divergence

In Section 2.1, we consider (infinite) sequences, limits of sequences, and bounded and monotonic sequences of real numbers. In addition to certain basic properties of convergent sequences, we also study divergent sequences and in particular, sequences that tend to positive or negative infinity. We present a number of methods to discuss convergent sequences together with techniques for calculating their limits. Also, we prove the *bounded monotone convergence theorem* (BMCT), which asserts that every bounded monotone sequence is convergent. In Section 2.2, we define the limit superior and the limit inferior. We continue the discussion with Cauchy sequences and give examples of sequences of rational numbers converging to irrational numbers. As applications, a number of examples and exercises are presented.

2.1 Sequences and Their Limits

An infinite *(real) sequence* (more briefly, a sequence) is a nonterminating collection of (real) numbers consisting of a first number, a second number, a third number, and so on:

$$a_1, a_2, a_3, \ldots .$$

Specifically, if n is a positive integer, then a_n is called the nth term of the sequence, and the sequence is denoted by

$$\{a_1, a_2, \ldots, a_n, \ldots\} \quad \text{or, more simply,} \quad \{a_n\}\,.$$

For example, the expression $\{2n\}$ denotes the sequence $2, 4, 6, \ldots$. Thus, a sequence of real numbers is a special kind of function, one whose domain is the set of all positive integers or possibly a set of the form $\{n : n \geq k\}$ for some fixed $k \in \mathbb{Z}$, and the range is a subset of $\mathbb{R}$. Let us now make this point precise.

Definition 2.1. *A real* sequence $\{a_n\}$ *is a real-valued function* f *defined on a set* $\{k, k+1, k+2, \ldots\}$*. The functional values*

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$$f(k), f(k+1), f(k+2), \ldots$$

are called the terms *of the sequence. It is customary to write* $f(n) = a_n$ *for* $n \geq k$*, so that we can denote the sequence by listing its terms in order; thus we write a sequence as*

$$\{a_n\}_{n\geq k} \quad \text{or} \quad \{a_{n+k-1}\}_{n=1}^{\infty} \quad \text{or} \quad \{a_n\}_{n=k}^{\infty} \quad \text{or} \quad \{a_k, a_{k+1}, \ldots\}.$$

The number a_n is called the *general term* of the sequence $\{a_n\}$ (nth term, especially for $k = 1$). The set $\{a_n : n \geq k\}$ is called the *range of the sequence* $\{a_n\}_{n\geq k}$. Sequences most often begin with $n = 0$ or $n = 1$, in which case the sequence is a function whose domain is the set of nonnegative integers (respectively positive integers). Simple examples of sequences are the sequences of positive integers, i.e., the sequence $\{a_n\}$ for which $a_n = n$ for $n \geq 1$, $\{1/n\}$, $\{(-1)^n\}$, $\{(-1)^n + 1/n\}$, and the constant sequences for which $a_n = c$ for all n. The *Fibonacci sequence* is given by

$$a_0, \; a_1 = 1, \; a_2 = 2, \; a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 3.$$

The terms of this Fibonacci sequence are called *Fibonacci numbers*, and the first few terms are

$$1, \; 1, \; 2, \; 3, \; 5, \; 8, \; 13, \; 21.$$

2.1.1 Limits of Sequences of Real Numbers

A fundamental question about a sequence $\{a_n\}$ concerns the behavior of its nth term a_n as n gets larger and larger. For example, consider the sequence whose general term is

$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n}.$$

It appears that the terms of this sequence are getting closer and closer to the number 1. In general, if the terms of a sequence can be made as close as we please to a number a for n sufficiently large, then we say that the sequence *converges to* a. Here is a precise definition that describes the behavior of a sequence.

Definition 2.2 (Limit of a sequence). *Let* $\{a_n\}$ *be a sequence of real numbers. We say that the sequence* $\{a_n\}$ *converges to the real number* a*, or tends to* a*, and we write*

$$a = \lim_{n\to\infty} a_n \quad \text{or simply} \quad a = \lim a_n,$$

if for every $\epsilon > 0$*, there is an integer* N *such that*

$$|a_n - a| < \epsilon \quad \text{whenever } n \geq N.$$

In this case, we call the number a a limit of the sequence $\{a_n\}$. We say that the sequence $\{a_n\}$ converges (or is convergent or has limit) if it converges to some number a. A sequence diverges (or is divergent) if it does not converge to any number.

For instance, in our example above we would expect

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

The notions of convergence and limit of a sequence play a fundamental role in analysis.

If $a \in \mathbb{R}$, other notations for the convergence of $\{a_n\}$ to a are

$$\lim_{n\to\infty} (a_n - a) = 0 \quad \text{and} \quad a_n \to a \quad \text{as } n \to \infty.$$

The notation $a = \lim a_n$ means that *eventually* the terms of the sequence $\{a_n\}$ can be made as close to a as may be desired by taking n sufficiently large. Note also that

$$|a_n - a| < \epsilon \quad \text{for } n \geq N \iff a_n \in (a - \epsilon, a + \epsilon) \quad \text{for } n \geq N.$$

That is, a sequence $\{a_n\}$ converges to a if and only if every neighborhood of a contains all but a finite number of terms of the sequence. Since N depends on ϵ, sometimes it is important to emphasize this and write $N(\epsilon)$ instead of N. Note also that the definition requires some N, but not necessarily the smallest N that works. In fact, if convergence works for some N then any $N_1 > N$ also works.

To motivate the definition, we again consider $a_n = (n+1)/n$. Given $\epsilon > 0$, we notice that

$$\left| \frac{n+1}{n} - 1 \right| = \frac{1}{n} < \epsilon \quad \text{whenever } n > \frac{1}{\epsilon}.$$

Thus, N should be some natural number larger than $1/\epsilon$. For example, if $\epsilon = 1/99$, then we may choose N to be any positive integer bigger than 99, and we conclude that

$$\left| \frac{n+1}{n} - 1 \right| < \epsilon = \frac{1}{99} \quad \text{whenever } n \geq N = 100.$$

Similarly, if $\epsilon = 2/999$, then $1/\epsilon = 499.5$, so that

$$\left| \frac{n+1}{n} - 1 \right| < \epsilon = \frac{2}{999} \quad \text{whenever } n \geq N = 500.$$

Thus, N clearly depends on ϵ.

The definition of limit makes it clear that changing a finite number of terms of a given sequence affects neither the convergence nor the divergence of the sequence. Also, we remark that the number ϵ provides a quantitative measure of "closeness," and the number N a quantitative measure of "largeness."

We now continue our discussion with a fundamental question: *Is it possible for a sequence to converge to more than one limit?*

Theorem 2.3 (Uniqueness of limits). *The limit of a convergent sequence is unique.*

Proof. Suppose that $a = \lim a_n$ and $a' = \lim a_n$. Let $\epsilon > 0$. Then there exist two numbers N_1 and N_2 such that

$$|a_n - a| < \epsilon \quad \text{for } n \geq N_1 \quad \text{and} \quad |a_n - a'| < \epsilon \quad \text{for } n \geq N_2.$$

In particular, these two inequalities must hold for $n \geq N = \max\{N_1, N_2\}$. We conclude that

$$|a - a'| = |a - a_n - (a' - a_n)| \leq |a_n - a| + |a_n - a'| < 2\epsilon \quad \text{for } n \geq N.$$

Since this inequality holds for every $\epsilon > 0$, and $|a - a'|$ is independent of ϵ, we must have $|a - a'| = 0$, i.e., $a = a'$. ∎

Also, as a direct consequence of the definition we obtain the following: If $a_n \to a$, then $a_{n+k} \to a$ for any fixed integer k. Indeed, if $a_n \to a$ as $n \to \infty$, then for a given $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N$. That is,

$$|a_{n+k} - a| < \epsilon \quad \text{for all } n + k \geq N + k = N_1 \quad \text{or} \quad |a_m - a| < \epsilon \quad \text{for } m \geq N_1,$$

which is same as saying that $a_m \to a$ as $m \to \infty$.

Definition 2.4. *A sequence* $\{a_n\}$ *that converges to zero is called a* null sequence.

Examples 2.5. (i) The sequence $\{n\}$ diverges because no matter what a and ϵ we choose, the inequality

$$a - \epsilon < n < a + \epsilon, \quad \text{i.e., } |n - a| < \epsilon,$$

can hold only for finitely many n. Similarly, the sequence $\{2^n\}$ diverges.

(ii) The sequence defined by $\{(-1)^n\}$ is $\{-1, 1, -1, 1, \ldots\}$, and this sequence diverges by oscillation because the nth term is always either 1 or -1. Thus a_n cannot approach any one specific number a as n grows large. Also, we note that if a is any real number, we can always choose a positive number ϵ such that at least one of the inequalities

$$a - \epsilon < -1 < a + \epsilon \quad \text{or} \quad a - \epsilon < 1 < a + \epsilon$$

is false. For example, the choice $\epsilon = |1 - a|/2$ if $a \neq 1$, and $\epsilon = |1 + a|/2$ if $a \neq -1$, will do. If $a = 1$ or -1, choose ϵ to be any positive real number less than 1. Thus the inequality $|(-1)^n - a| < \epsilon$ will be false for infinitely many n. Hence $\{(-1)^n\}$ diverges.

(iii) The sequence $\{\sin(n\pi/2)\}_{n\geq 1}$ diverges because the sequence is

$$\{1, 0, -1, 0, 1, 0, \ldots\},$$

and hence it does not converge to any number, by the same reasoning as above.

(iv) The sequence $\{(-1)^n/n\}$ converges to zero, and so it is a null sequence. ●

Definition 2.6. *A sequence* $\{a_n\}$ *is* bounded *if there exists an* $R > 0$ *such that* $|a_n| \leq R$ *for all* n*. A sequence is* unbounded *if it is not bounded.*

Since a convergent sequence eventually clusters about its limit, it is fairly evident that a sequence that is not bounded cannot converge, and hence the next theorem is not too surprising; it will be used in the proof of Theorem 2.8.

Theorem 2.7. *Every convergent sequence is bounded. The converse is not true.*

Proof. Let $\{a_n\}_{n\geq 1}$ converge to a. Then there exists an $N \in \mathbb{N}$ such that $|a_n - a| < 1 = \epsilon$ for $n \geq N$. It follows that $|a_n| < 1 + |a|$ for $n \geq N$. Define $M = \max\{1 + |a|, |a_1|, |a_2|, \dots |a_{N-1}|\}$. Then $|a_n| < M$ for every $n \in \mathbb{N}$.

To see that the converse is not true, it suffices to consider the sequence $\{(-1)^n\}_{n\geq 1}$, which is bounded but not convergent, although the odd terms and even terms both form convergent sequences with different limits. ∎

2.1.2 Operations on Convergent Sequences

The sum of sequences $\{a_n\}$ and $\{b_n\}$ is defined to be the sequence $\{a_n + b_n\}$. We have the following useful consequences of the definition of convergence that show how limits team up with the basic algebraic operations.

Theorem 2.8 (Algebra of limits for convergent sequences). *Suppose that* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$*, where* $a, b \in \mathbb{R}$*. Then*

- $\lim_{n\to\infty}(ra_n + sb_n) = ra + sb$, $r, s \in \mathbb{R}$. **[Linearity rule for sequences]**
- $\lim_{n\to\infty}(a_n b_n) = ab$. **[Product rule for sequences]**
- $\lim_{n\to\infty} a_n/b_n = a/b$, *provided* $b \neq 0$. **[Quotient rule for sequences]**
- $\lim_{n\to\infty} \sqrt[m]{a_n} = \sqrt[m]{a}$, *provided* $\sqrt[m]{a}_n$ *is defined for all* n *and* $\sqrt[m]{a}$ *exists.*

Proof. The linearity rule for sequences is easy to prove. The quotient rule for sequences is easy if we prove the product rule for sequences (see also Questions 2.44(33) and 2.44(34)). We provide a direct proof.

We write

$$a_n b_n - ab = (a_n - a)b_n + (b_n - b)a.$$

Since every convergent sequence must be bounded, there exists an $M > 0$ such that $|b_n| \leq M$ (say), for all n. Let $\epsilon > 0$ be given. Again, since $b_n \to b$ as $n \to \infty$, there exists an N_2 such that

$$|b_n - b| < \frac{\epsilon}{2(|a| + 1)} \quad \text{for } n \geq N_2.$$

(We remark that we could not use $\epsilon/2|a|$ instead of $\epsilon/[2(|a| + 1)]$ because a could be zero.)

Also by the hypothesis that $a_n \to a$ as $n \to \infty$, there exists an N_3 such that

$$|a_n - a| < \frac{\epsilon}{2M} \quad \text{for } n \geq N_3.$$

Finally, for $n \geq \max\{N_2, N_3\} = N$, we have

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n - a|\,|b_n| + |b_n - b|\,|a| \\ &< \frac{\epsilon}{2M} M + \frac{\epsilon}{2(|a|+1)}|a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

The product rule clearly follows.

The proof of third part follows from Lemma 2.9. The proof of the final part is left as a simple exercise (see Questions 2.44(16)). ∎

Lemma 2.9 (Reciprocal rule). *If* $\lim_{n\to\infty} b_n = b$ *and* $b \neq 0$, *then the reciprocal rule holds:*

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

Proof. The proof is easy, and so we leave it as a simple exercise. ∎

Note that if $a_n = (-1)^n$ and $b_n = (-1)^{n-1}$, then $\{a_n^2\}$ and $\{a_n + b_n\}$ both converge, although individual sequences $\{a_n\}$ and $\{b_n\}$ diverge.

Example 2.10. Find the limit of each of these convergent sequences:

(a) $\left\{\dfrac{1}{n^p}\right\}$ $(p > 0)$. **(b)** $\left\{\dfrac{n^2 - 2n + 3}{5n^3}\right\}$. **(c)** $\left\{\dfrac{n^6 + 3n^4 - 2}{n^6 + 2n + 3}\right\}$.

Solution. **(a)** As n grows arbitrarily large, $1/n$ (and hence $1/n^p$) gets smaller and smaller for $p > 0$. Thus, $\lim_{n\to\infty} 1/n^p = 0$. Also, we note that if $\epsilon > 0$, then $|(1/n^p) - 0| < \epsilon$ or $n > 1/(\epsilon^{1/p})$. Thus, if N is any integer greater than $1/(\epsilon^{1/p})$, then

$$|(1/n^p) - 0| < \epsilon \quad \text{for all } n \geq N.$$

Thus, for each $p > 0$, $n^{-p} \to 0$ as $n \to \infty$. That is, $\{1/n^p\}$ is a null sequence for each $p > 0$.

(b) We cannot use the quotient rule of Theorem 2.8 because neither the limit for the numerator nor that for the denominator exists. On the other hand, we can divide the numerator and denominator by n^3 and then use the linearity rule and the product rule. We then have

$$\frac{n^2 - 2n + 3}{5n^3} = \frac{1}{5}\left(\frac{1}{n} - \frac{2}{n^2} + \frac{3}{n^3}\right) \to 0 \quad \text{as } n \to \infty.$$

(c) Divide the numerator and denominator by n^6, the highest power of n that occurs in the expression, to obtain

$$\lim_{n\to\infty} \frac{n^6 + 3n^4 - 2}{n^6 + 2n + 3} = \lim_{n\to\infty} \frac{1 + \frac{3}{n^2} - \frac{2}{n^6}}{1 + \frac{2}{n^5} + \frac{3}{n^6}} = 1.$$

In fact, if we set

$$a_n = 1 + \frac{3}{n^2} - \frac{2}{n^6} \quad \text{and} \quad b_n = 1 + \frac{2}{n^5} + \frac{3}{n^6},$$

then the linearity rule gives that $a_n \to 1$ and $b_n \to 1$ as $n \to \infty$. Finally, the quotient rule gives the desired limit, namely,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1. \qquad \bullet$$

Suppose that $\{a_n\}$ is a sequence of real numbers such that $a_n > 0$ for all but a finite number of n. Then there exists an N such that $a_n > 0$ for all $n \geq N$. If the new sequence $\{1/a_{n+N}\}_{n\geq 0}$ converges to zero, then we say that $\{a_n\}$ diverges to ∞ and write $\lim a_n = \infty$. Equivalently, if $\lim a_n$ does not exist because the numbers $a_n > 0$ become arbitrarily large as $n \to \infty$, we write $\lim_{n\to\infty} a_n = \infty$. We summarize the discussion as follows:

Definition 2.11 (Divergent sequence). *For given sequences $\{a_n\}$ and $\{b_n\}$, we have*

(a) $\lim_{n\to\infty} a_n = \infty$ *if and only if for each $R > 0$ there exists an $N \in \mathbb{N}$ such that $a_n > R$ for all $n \geq N$.*
(b) $\lim_{n\to\infty} b_n = -\infty$ *if and only if for each $R < 0$ there exists an $N \in \mathbb{N}$ such that $b_n < R$ for all $n \geq N$.*

We do not regard $\{a_n\}$ as a convergent sequence unless $\lim a_n$ exists as a finite number, as required by the definition. For instance,

$$\lim_{n\to\infty} n^3 = \infty, \quad \lim_{n\to\infty} (-n) = -\infty, \quad \lim_{n\to\infty} 3^n = \infty, \quad \lim_{n\to\infty} (\sqrt{n} + 5) = \infty.$$

We do not say that the sequence $\{n^2\}$ "converges to ∞" but rather that it "diverges to ∞" or "tends to ∞." To emphasize the distinction, we say that $\{a_n\}$ diverges to ∞ (respectively $-\infty$) if $\lim a_n = \infty$ (respectively $-\infty$). We note that $\lim(-1)^n n$ is unbounded but it diverges neither to ∞ nor to $-\infty$.

Definition 2.12 (Oscillatory sequence). *A sequence that neither converges to a finite number nor diverges to either ∞ or $-\infty$ is said to* oscillate *or* diverge by oscillation. *An oscillating sequence with finite amplitude is called a* finitely oscillating sequence. *An oscillating sequence with infinite amplitude is called an* infinitely oscillating sequence.

For instance,

$$\{(-1)^n\}, \quad \{1 + (-1)^n\}, \quad \{(-1)^n(1 + 1/n)\}$$

oscillate finitely. We remark that an unbounded sequence that does not diverge to ∞ or $-\infty$ oscillates infinitely. For example, the sequences

$$\{(-1)^n n\}, \quad \{(-1)^n n^2\}, \quad \{(-n)^n\}$$

are all unbounded and oscillate infinitely.

Example 2.13. Consider $a_n = (n^2 + 2)/(n + 1)$. Then

$$a_n = n\left(\frac{1 + \frac{2}{n^2}}{1 + \frac{1}{n}}\right).$$

From the algebra of limits we observe that

$$\lim_{n\to\infty} \frac{1 + \frac{2}{n^2}}{1 + \frac{1}{n}} = 1.$$

On other hand, $\lim_{n\to\infty} a_n$ does not exist. Indeed, we can show that $a_n \to \infty$ as $n \to \infty$. According to the definition, we must show that for a given $R > 0$, there exists an N such that $a_n > R$ for all $n \geq N$. Now we observe that

$$a_n > R \Longleftrightarrow n + 1 + \frac{3}{n+1} > R + 2,$$

which helps to show that $a_n > R$ if $n \geq R+2$. So we can choose any positive integer N such that $N \geq R + 2$. We then conclude that $a_n \to \infty$ as $n \to \infty$. Similarly, we easily have the following:

(1) As in Example 2.10**(c)**, we write

$$\lim_{n\to\infty} \frac{n^7 + 2n^3 - 1}{n^6 + n^2 + 3n + 1} = \lim_{n\to\infty} \frac{1 + \frac{2}{n^4} - \frac{1}{n^7}}{\frac{1}{n} + \frac{1}{n^5} + \frac{3}{n^6} + \frac{1}{n^7}}.$$

The numerator tends to 1 as $n \to \infty$, whereas the denominator approaches 0. Hence the quotient increases without bound, and the sequence must diverge. We may rewrite in the present notation,

$$\lim_{n\to\infty} \frac{n^7 + 2n^3 - 1}{n^6 + n^2 + 3n + 1} = \infty.$$

(2) $\{n/3 + 1/n\}$, $\{n^3 - n\}$, $\{(n^2 + 1)/(n + 1)\}$, and $\{(n^3 + 1)/(n + 1)\}$ all diverge to ∞.
(3) $\left\{(-1)^n n^2\right\}$ diverges but neither to $-\infty$ nor to ∞.
(4) $a_n \to \infty \Longrightarrow a_n^2 \to \infty$.
(5) If $a_n > 0$ for all large values of n, then $a_n \to 0 \Longrightarrow 1/a_n \to \infty$. Is the converse true? ●

Finally, we let $a_n = \sqrt{n^2 + 5n} - n$ and consider the problem of finding $\lim a_n$. It would not be correct to apply the linearity property for sequences (because neither $\lim \sqrt{n^2 + 5n}$ nor $\lim n$ exists as a real number). At this place it important to remember that the linearity rule in Theorem 2.8 cannot be applied to $\{a_n\}$, since $\lim \sqrt{n^2 + 5n} = \infty$ and $\lim n = \infty$. It is also not correct to use this as a reason to say that the limit does not exist. The supporting argument is as follows. Rewriting a_n algebraically as

$$a_n = \left(\sqrt{n^2+5n}-n\right)\frac{\sqrt{n^2+5n}+n}{\sqrt{n^2+5n}+n} = \frac{5n}{\sqrt{n^2+5n}+n} = \frac{5}{\sqrt{1+\frac{5}{n}}+1},$$

we obtain $\lim_{n\to\infty}\left(\sqrt{n^2+5n}-n\right) = 5/2$.

Remark 2.14. We emphasize once again that Theorem 2.8 cannot be applied to sequences that diverge to ∞ or $-\infty$. For instance, if $a_n = n+1$, $b_n = n$, and $c_n = n^2$ for $n \geq 1$, then it is clear that the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ diverge to ∞, showing that the limits do not exist as real numbers. Also, it is tempting to say that

$$a_n - b_n \to \infty - \infty = 0 \quad \text{and} \quad c_n - b_n \to \infty - \infty = 0 \quad \text{as } n \to \infty.$$

Note that ∞ is not a real number, and so it cannot be treated like a usual real number. In our example, we actually have $a_n - b_n = 1$ for all $n \geq 1$, and

$$c_n - b_n = n(n-1) \to \infty \quad \text{as } n \to \infty. \qquad \bullet$$

2.1.3 The Squeeze/Sandwich Rule

In the following squeeze rule, the sequence $\{b_n\}$ is "sandwiched" between the two sequences $\{a_n\}$ and $\{c_n\}$.

Theorem 2.15 (Squeeze/Sandwich rule for sequences). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three sequences such that $a_n \leq b_n \leq c_n$ for all $n \geq N$ and for some $N \in \mathbb{N}$. If*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$. If $b_n \to \infty$, then $c_n \to \infty$. Also, if $c_n \to -\infty$, then $b_n \to -\infty$.

Proof. Let $\epsilon > 0$ be given. By the definition of convergence, there exist two numbers N_1 and N_2 such that

$$|a_n - L| < \epsilon \quad \text{for } n \geq N_1 \quad \text{and} \quad |c_n - L| < \epsilon \quad \text{for } n \geq N_2.$$

In particular, since $a_n \leq b_n \leq c_n$ for all $n \geq N$, we have

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon \quad \text{for } n \geq N_3 = \max\{N, N_1, N_2\},$$

showing that $|b_n - L| < \epsilon$ for $n \geq N_3$, as required.

We leave the rest as a simple exercise. ∎

Corollary 2.16. *If $\{c_n\}$ is a null sequence of nonnegative real numbers, and $|b_n| \leq c_n$ for all $n \geq N$, then $\{b_n\}$ is a null sequence.*

For instance, since $\{1/\sqrt{n}\}$ is null and $1/(1+\sqrt{n}) < 1/\sqrt{n}$ for all $n \geq 1$, $\{1/(1+\sqrt{n})\}$ is also a null sequence. Similarly, comparing $1/3^n$ with $1/n$, it follows easily that $\{1/3^n\}$ is a null sequence.

Corollary 2.17. *If* $\lim_{n\to\infty} a_n = 0$ *and* $|b_n - L| \leq a_n$ *for all* $n \geq N$*, then* $\lim_{n\to\infty} b_n = L$.

Proof. By the last corollary, it follows that $\{b_n - L\}$ is a null sequence, and so the desired conclusion follows. Alternatively, it suffices to observe that

$$|b_n - L| \leq a_n \Longleftrightarrow L - a_n \leq b_n \leq L + a_n$$

and apply the squeeze rule. ∎

For instance, using the squeeze rule, we easily have the following:

(a) $\lim_{n\to\infty} \cos n^2/n = 0$, because $-(1/n) \leq \cos n^2/n \leq 1/n$. With the same reasoning, one has

$$\lim_{n\to\infty} \frac{\sin(n\pi/2)}{n} = 0.$$

(b) $\lim_{n\to\infty} \{\sqrt{n+1} - \sqrt{n}\} = 0$ and $\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}) = 1/2$. Moreover,

$$0 < \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}.$$

Note that the above inequality is useful in estimating $\sqrt{n}$. For $n = 1$, this gives $\sqrt{2} < 1.5$, and for $n = 2, 4$, we have $\sqrt{3} < 1.875$ and $\sqrt{5} < 2.25$. Indeed, for $n = 2$, we have

$$\sqrt{3} < \sqrt{2} + \frac{\sqrt{2}}{4} = \frac{5\sqrt{2}}{4} < \frac{5 \times 1.5}{4} = \frac{7.5}{4} = 1.875.$$

(c) $\lim_{n\to\infty} n/2^n = 0$. Indeed, using induction we easily see that $2^n \geq n^2$ for $n \geq 4$, so that

$$0 < \frac{n}{2^n} \leq \frac{1}{n}.$$

(d) $\lim_{n\to\infty} b_n = 1$ if $b_n = 1/(\sqrt{n^2+1}) + 1/(\sqrt{n^2+2}) + \cdots + 1/(\sqrt{n^2+n})$. We note that

$$\frac{n}{\sqrt{n^2+n}} < b_n < \frac{n}{\sqrt{n^2+1}}, \quad \text{i.e.,} \quad \frac{1}{\sqrt{1+1/n}} < b_n < \frac{1}{\sqrt{1+1/n^2}}.$$

(e) $\lim_{n\to\infty} c_n = \infty$ if $c_n = 1/(\sqrt{n+1}) + 1/(\sqrt{n+2}) + \cdots + 1/(\sqrt{n+n})$. We note that

$$c_n > \frac{n}{\sqrt{n+n}} = \frac{\sqrt{n}}{\sqrt{2}} = b_n,$$

where $b_n \to \infty$ as $n \to \infty$.

Using the squeeze rule, Theorem 2.8, and a few standard examples allows one to calculate limits of important sequences.

Example 2.18. Show that

(a) $\lim_{n\to\infty} a^{1/n} = 1$ for $a > 0$. **(b)** $\lim_{n\to\infty} n^{1/n} = 1$. **(c)** $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Solution. **(a)** We consider the cases $a > 1$ and $a < 1$, since there is nothing to prove if $a = 1$. Suppose first that $a > 1$. Then $a^{1/n} \geq 1$, and so

$$a^{1/n} = 1 + x_n$$

for some sequence $\{x_n\}$ of positive real numbers. Then by the binomial theorem,

$$a = (1 + x_n)^n \geq 1 + nx_n \quad \text{for all } n \geq 1,$$

which is equivalent to

$$0 < a^{1/n} - 1 \leq \frac{a-1}{n} \qquad \text{for all } n \in \mathbb{N}.$$

Thus, $a^{1/n} \to 1$ as $n \to \infty$ if $a > 1$. For $0 < a < 1$, we have $(1/a)^{1/n} \to 1$ as $n \to \infty$, and therefore, by the reciprocal rule,

$$a^{1/n} = \frac{1}{(1/a)^{1/n}} \to \frac{1}{1} = 1 \quad \text{as } n \to \infty.$$

The sequence $\{a^{1/n}\}$ is referred to as the nth root sequence.

(b) Clearly $(1+1)^n \geq 1 + n > n$, so that $n^{1/n} - 1 < 1$ for $n \geq 1$. Also, for $n \geq 1$, we observe that $n^{1/n} \geq 1$, so that $n^{1/n} - 1 = x_n$ with $x_n \geq 0$. In particular, using the binomial theorem, we deduce that

$$n = (1 + x_n)^n \geq 1 + nx_n + \frac{n(n-1)}{2}x_n^2 \geq 1 + \frac{n(n-1)}{2}x_n^2,$$

which implies that

$$0 \leq x_n = n^{1/n} - 1 \leq \sqrt{\frac{2}{n}} \quad \text{for } n \geq 1.$$

By the squeeze rule, $x_n \to 0$ as $n \to 0$, since $1/\sqrt{n} \to 0$. We conclude that $n^{1/n} \to 1$ as $n \to \infty$, as desired.

(c) It follows that

$$0 < \frac{n!}{n^n} \leq \frac{1}{n}.$$

The second inequality is true because

$$n! = n(n-1)\cdots 2 \cdot 1 < n \cdot n \cdots n \cdot 1 = n^{n-1}.$$

The squeeze rule (with $a_n = 0, c_n = 1/n$) gives the desired conclusion. ●

Remark 2.19. We observe that case **(a)** of Example 2.18 may be obtained as a special case of case **(b)**. For instance, if $a \geq 1$, then for n large enough we have $1 \leq a < n$. Taking roots on both sides, we obtain

$$1 \leq a^{1/n} < n^{1/n} \quad \text{for large } n.$$

Again, by the squeeze rule, we see that $\lim_{n\to\infty} a^{1/n} = 1$.

As a consequence of **(a)** and **(b)** of Example 2.18 and the product rule for sequences, we can easily obtain that

$$\lim_{n\to\infty} (2n)^{1/n} = 1 \quad \text{and} \quad \lim_{n\to\infty} (3\sqrt{n})^{1/2n} = 1. \qquad \bullet$$

2.1.4 Bounded Monotone Sequences

Now we introduce some important terminology associated with sequences. A sequence $\{a_n\}$ is said to be

- *bounded above* if there exists an $M \in \mathbb{R}$ such that $a_n \leq M$ for all n,
- *bounded below* if there exists an $m \in \mathbb{R}$ such that $a_n \geq m$ for all n,
- *bounded* if it is bounded both below and above,
- *monotonically increasing* (or *simply increasing*) if $a_n \leq a_{n+1}$ for all n (see Figure 2.1),
- *monotonically decreasing* (or *simply decreasing*) if $a_n \geq a_{n+1}$ for all n (see Figure 2.2),

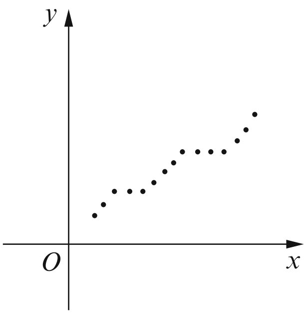

Fig. 2.1. An increasing sequence.

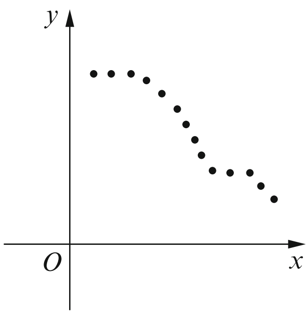

Fig. 2.2. A decreasing sequence.

- *strictly increasing* if $a_n < a_{n+1}$ for all n,
- *strictly decreasing* if $a_n > a_{n+1}$ for all n,
- *monotonic* if it is either increasing or decreasing,
- *strictly monotonic* if it is either strictly increasing or strictly decreasing,
- *alternating* if a_n changes sign alternately. In other words, a_n is of the form $a_n = (-1)^{n-1}b_n$ or $a_n = (-1)^n b_n (b_n \geq 0)$ for all n. That is, $a_n a_{n+1} < 0$ for all n.

Constant sequences are treated as both increasing and decreasing! We now demonstrate these definitions by giving several simple examples.

(1) $\{1/n\}_{n\geq 1}$ is strictly decreasing and bounded.
(2) $\{n\}_{n\geq 1}$ is strictly increasing and unbounded; however, it is bounded below by 1.
(3) $\{(-1)^{n-1}n\}_{n\geq 1}$ is neither increasing nor decreasing. Also, it is unbounded.
(4) $\{(-1)^n\}_{n\geq 1}$ is neither increasing nor decreasing nor convergent but is bounded.
(5) $\{(-1)^n/n\}_{n\geq 1}$ is convergent but is neither increasing nor decreasing.
(6) If $a_n = 2$ for $1 \leq n \leq 5$ and $a_n = n$ for $n \geq 6$, then $\{a_n\}_{n\geq 1}$ is increasing but not strictly.
(7) $\{n^{1/n}\}_{n\geq 1}$ is not monotone, as can be seen by examining the first four terms of the sequence.
(8) $\{n!/n^n\}$ is decreasing and bounded.
(9) $\{a_n\}$, $a_n = 8^n/n!$, is neither increasing nor decreasing, because

$$\frac{a_{n+1}}{a_n} = \frac{8}{n+1} \begin{cases} \geq 1 & \text{if } n \leq 7 \\ \leq 1 & \text{if } n \geq 7. \end{cases}$$

On the other hand, if we ignore the first six terms, it follows that $\{a_n\}_{n\geq 7}$ is decreasing. In such cases, we say that $\{a_n\}$ is eventually decreasing. Similarly, one can define eventually increasing sequences. Finally, we remark that (3)–(5) are examples of sequences that are alternating.

2.1.5 Subsequences

We now present two simple criteria that involve the notion of a subsequence for establishing that a sequence diverges. Let $\{a_n\}_{n\geq 1}$ be a sequence and $\{n_k\}_{k\geq 1}$ any strictly increasing sequence of positive integers; that is,

$$0 < n_1 < n_2 < n_3 < \cdots .$$

Then the sequence $\{a_{n_k}\}_{k\geq 1}$, i.e., $\{b_k\}_{k\geq 1}$, where $b_k = a_{n_k}$, is called a *subsequence* of $\{a_n\}_{n\geq 1}$. That is, a subsequence is obtained by choosing terms from the original sequence, without altering the order of the terms, through the map $k \mapsto n_k$, which determines the indices used to pick out the subsequence. For instance, $\{a_{7k+1}\}$ corresponds to the sequence of positive integers $n_k = 7k+1$, $k = 1, 2, \ldots$. Observe that every increasing sequence $\{n_k\}$ of positive integers must tend to infinity, because

$$n_k \geq k \quad \text{for } k = 1, 2, \ldots.$$

The sequences

$$\left\{\frac{1}{k^2}\right\}_{k\geq 1}, \quad \left\{\frac{1}{2k}\right\}_{k\geq 1}, \quad \left\{\frac{1}{2k+1}\right\}_{k\geq 1}, \quad \left\{\frac{1}{5k+3}\right\}_{k\geq 1}, \quad \left\{\frac{1}{2^k}\right\}_{k\geq 1}$$

are some subsequences of the sequence $\{1/k\}_{k\geq 1}$, formed by setting $n_k = k^2,\ 2k,\ 2k+1,\ 5k+3,\ 2^k$, respectively. Note that all the above subsequences converge to the same limit, 0, which is also the limit of the original sequence $\{1/k\}_{k\geq 1}$. *Can we conjecture that every subsequence of a convergent sequence must converge and converge to the same limit?* We have the following:

1. Every sequence is a subsequence of itself.
2. Let $a_k = 1+(-1)^k$, $k \geq 1$. Then $a_{2k} = 2$ and $a_{2k-1} = 0$, showing that the even sequence $\{a_{2k}\}$ and the odd sequence $\{a_{2k-1}\}$ are two convergent (constant) subsequences of $\{a_k\}$. Thus, a sequence may not converge yet have convergent subsequences with different limits.
3. Let $a_k = \sin(k\pi/2)$. Then $a_{2k-1} = (-1)^{k-1}$ and $a_{2k} = 0$ are two subsequences of a_k. Does the sequence $\{b_k^2\}$, where $b_k = (1+(-1)^{k-1})/2$, converge? Is $\{b_k\}$ a subsequence of $\{a_k\}$?

Definition 2.20 (Subsequential limits). *Let $\{a_k\}$ be a sequence. A subsequential limit is any real number or symbol ∞ or $-\infty$ that is the limit of some subsequence $\{a_{n_k}\}_{k\geq 1}$ of $\{a_k\}_{k\geq 1}$.*

For example, we have the following:

(1) 0 and 2 are subsequential limits of $\{1+(-1)^k\}$.
(2) $-\infty$ and ∞ are the only subsequential limits of $\{k(-1)^k\}$.
(3) $\{-\sqrt{3}/2, 0, \sqrt{3}/2\}$ is the set of subsequential limits of $\{a_k\}$, $a_k = \sin(k\pi/3)$. Here $\{a_{3k}\}$, $\{a_{3k+1}\}$, and $\{a_{3k+2}\}$ are convergent subsequences with limits 0, $-\sqrt{3}/2$, and $\sqrt{3}/2$, respectively.
(4) Every real number is a subsequential limit of some subsequence of the sequence of all rational numbers. Indeed, $\mathbb{R}\cup\{-\infty,\infty\}$ is the set of subsequential limits of the sequence of all rational numbers.

The following result, which shows that certain properties of sequences are inherited by their subsequences, is almost obvious.

Theorem 2.21 (Invariance property of subsequences). *If $\{a_n\}$ converges, then every subsequence $\{a_{n_k}\}$ of it converges to the same limit. Also, if $a_n \to \infty$, then $\{a_{n_k}\} \to \infty$ as well.*

Proof. Suppose that $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$. Note that $n_k \geq k$. Let $L = \lim a_n$ and $\epsilon > 0$ be given. Then there exists an N such that

$$|a_k - L| < \epsilon \quad \text{for } k \geq N.$$

Now $k \geq N$ implies $n_k \geq N$, which in turn implies that

$$|a_{n_k} - L| < \epsilon \quad \text{for } n_k \geq N.$$

Thus, a_{n_k} converges to L as $k \to \infty$. The proof of the second part follows similarly. ∎

Here is an immediate consequence of Theorem 2.21.

Corollary 2.22. *The sequence* $\{a_n\}$ *is divergent if it has two convergent subsequences with different limits. Also,* $\{a_n\}$ *is divergent if it has a subsequence that tends to* ∞ *or a subsequence that tends to* $-\infty$.

In order to apply this corollary, it is necessary to identify convergent subsequences with different limits or subsequences that tend to ∞ or $-\infty$. Now the question is whether the converse of Theorem 2.21 also holds.

We can prove the divergence of a sequence if we are able to somehow prove that it is unbounded. For instance (see also Questions 2.44(8)), consider $a_n = \sum_{k=1}^{n} 1/k$. There are several ways one can see that the sequence diverges. Clearly, $a_n > 0$ for all $n \in \mathbb{N}$, $\{a_n\}$ is increasing, and

$$\begin{aligned} a_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\ &> 1 + \frac{n}{2}, \end{aligned}$$

so that $\{a_n\}_{n\geq 1}$ is increasing and not bounded above. Therefore, it cannot be convergent, and so it must diverge (see also the bounded monotone convergence theorem (BMCT), which is discussed later in this section). We remark that we may group the terms in a number of ways and obtain that $\{a_n\}_{n\geq 1}$ is unbounded, for example,

$$\begin{aligned} a_{10^n-1} &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{9}\right) + \left(\frac{1}{10} + \cdots + \frac{1}{99}\right) \\ &\quad + \cdots + \left(\frac{1}{10^{n-1}} + \cdots + \frac{1}{10^n - 1}\right) \\ &> 9\left(\frac{1}{10}\right) + \frac{90}{100} + \cdots + \frac{9 \times 10^{n-1}}{10^n} = \left(\frac{9}{10}\right) n. \end{aligned}$$

We end this subsection with the following result, which is easy to prove.

Theorem 2.23. *A sequence is convergent if and only if there exists a real number* L *such that every subsequence of the sequence has a further subsequence that converges to* L.

Corollary 2.24. *If both odd and even subsequences of* $\{a_n\}$ *converge to the same limit* l*, then so does the original sequence.*

Note that $\{(-1)^n\}$ diverges, because it has two subsequences $\{(-1)^{2n}\}$ and $\{(-1)^{2n-1}\}$ converging to two different limits, namely 1 and -1.

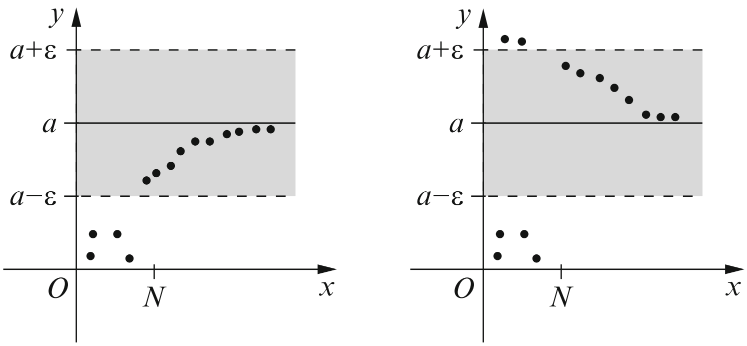

Fig. 2.3. Description for the bounded monotone convergence theorem.

2.1.6 Bounded Monotone Convergence Theorem

Until now, we have considered some basic techniques for finding the limit of a convergent sequence. In general, it is difficult to tell whether a given sequence converges. It is sometimes easy to show that a sequence is convergent even if we do not know its limit. For example, the following theorem is a starting point for our rigorous treatment of sequences and series, especially if we know that the given sequence is monotonic. However, we shall soon show that every bounded sequence has a convergent subsequence (see Theorem 2.42).

Theorem 2.25 (Monotone convergence theorem). *Every increasing sequence that is bounded above converges. Also, every decreasing sequence that is bounded below converges.*

Proof. Let $\{a_n\}_{n\geq 1}$ be an increasing sequence that is bounded above. According to the least upper bound property (Definition 1.18), since the range $A = \{a_n : n \in \mathbb{N}\}$ is bounded above, A has a least upper bound; call it a. We now prove that $a_n \to a$ as $n \to \infty$.

Clearly $a_n \leq a$ for all $n \in \mathbb{N}$, and by the definition of lub, given some $\epsilon > 0$ there exists an integer N such that $a_N > a - \epsilon$. Since $\{a_n\}$ is monotonically increasing,

$$a - \epsilon < a_N \leq a_n \leq a < a + \epsilon \quad \text{for } n \geq N.$$

That is, $|a_n - a| < \epsilon$ for $n \geq N$, and we conclude that $\{a_n\}$ converges to its least upper bound. That is, $\lim_{n\to\infty} a_n = a = \sup a_n$.

The proof for the case of decreasing sequences is identical, using the greatest lower bound instead of the least upper bound (see Figure 2.3).

Alternatively, it suffices to note that $\{b_n\}_{n\geq 1}$ is a decreasing sequence that is bounded below if and only if the sequence $\{-b_n\}_{n\geq 1}$ is increasing and bounded above. ∎

Remark 2.26. The monotonicity condition on the sequence $\{a_n\}$ in the above results need not be satisfied for all n. If this is true for all $n \geq N$, where N

is some suitably selected positive integer, then the conclusion of the above result is still true (see Figure 2.5). However, the tests in Theorem 2.25 tell us nothing about the limit, but they are often useful when we suspect that a sequence is convergent. ●

For instance, we easily obtain the following simple examples:

(1) If $a_n = 1 + 1/n$, then $\{a_n\}$ is clearly decreasing and bounded below (by 1, for example), and so it is convergent by Theorem 2.25. In this case, of course, we know already that it converges to 1.
(2) If $a_n = 1/\sqrt{n}$, then $\{a_n\}$ is clearly decreasing for $n \geq 1$ and bounded by 1. Consequently, the sequence $\{1/\sqrt{n}\}$ must converge.
(3) If $a_n = (2n-7)/(3n+2)$, then

$$a_n = \frac{1}{3n+2}\left(\frac{2}{3}(3n+2) - 7 - \frac{4}{3}\right) = \frac{2}{3} - \frac{25}{3(3n+2)},$$

so that $a_n \leq 2/3$ and $\{a_n\}$ is increasing. By Theorem 2.25, the sequence $\{a_n\}_{n\geq 1}$ must converge. Indeed, $a_n \to 2/3$ as $n \to \infty$.
(4) Consider

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Then $0 < a_n \leq n/(n+1)$ for all $n \geq 1$, since each term (except the first) in the sum is strictly less than $1/(n+1)$, and so $\{a_n\}$ is a bounded sequence. Also, for $n \geq 1$,

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{n+1} \\ &= \frac{1}{2n+1} - \frac{1}{2(n+1)} \\ &= \frac{1}{2(2n+1)(n+1)} > 0. \end{aligned}$$

Thus, $\{a_n\}$ is a bounded monotone sequence, and so it converges by Theorem 2.25. What is the limit of the sequence $\{a_n\}$?

The following equivalent form of Theorem 2.25 is the key to many important results in analysis. We shall soon see its usefulness in our subsequent discussion.

Theorem 2.27 (BMCT: Bounded monotone convergence theorem). *Every bounded monotonic sequence of real numbers converges. Equivalently, a monotonic sequence converges if and only if it is bounded.*

Consider the sequence $\{a_n\}_{n\geq 1}$, where $a_n = \sum_{k=1}^{n} 1/k$. This is clearly an increasing sequence. Does there exist an upper bound for this sequence? In fact, we have already proved that $\{a_n\}_{n\geq 1}$ is unbounded (see also Questions 2.44(8)). We also remark that a bounded sequence can converge without being monotone. For example, consider $\{(-1/3)^n\}_{n\geq 1}$.

Example 2.28. Show that $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$ (see also Theorem 2.34 and Example 2.43). Here $\{r^n\}$ is called a *power sequence.*

Solution. Observe that $-|r|^n \le r^n \le |r|^n$, and so it suffices to deal with $0 < r < 1$. In any case, define $a_n = |r|^n$ for $n \ge 1$. If $|r| < 1$, then we have

$$a_{n+1} = |r|a_n, \quad \text{i.e., } 0 \le a_{n+1} < a_n,$$

showing that $\{a_n\}$ is decreasing and bounded below by 0. Therefore, $\{a_n\}$ converges, say to a. Allowing $n \to \infty$ in the last equality, we see that

$$a = |r|a, \quad \text{i.e., } (1 - |r|)a = 0,$$

which gives $a = 0$, since $|r| < 1$.

Alternatively, we first notice that there is nothing to prove if $r = 0$. Thus for $0 < |r| = c < 1$, we can write $|r|$ in the form $c = 1/(1 + a)$ for some $a > 0$, so that by the binomial theorem,

$$0 < c^n = \frac{1}{(1+a)^n} \le \frac{1}{1+na} < \frac{1}{na},$$

and the result follows if we use the squeeze rule. ●

Because every monotone sequence converges, diverges to ∞, or diverges to $-\infty$, we have the following analogue of Theorem 2.25 for unbounded monotone sequences.

Theorem 2.29. *Every increasing sequence that is not bounded above must diverge to ∞. Also, every decreasing sequence that is not bounded below must diverge to $-\infty$.*

Proof. Let $\{a_n\}_{n\ge1}$ be an increasing sequence that is unbounded. Since the set $\{a_n : n \in \mathbb{N}\}$ is unbounded and it is bounded below by a_1, it must be unbounded above. Thus, given $R > 0$ there exists an integer N such that $a_N > R$. Since $\{a_n\}$ is monotonically increasing,

$$a_n \ge a_N > R \quad \text{for } n \ge N.$$

Since $R > 0$ is arbitrary, it follows that $\lim_{n\to\infty} a_n = \infty$.

The proof for decreasing sequences is identical and is left as an exercise. ∎

We may combine Theorems 2.27 and 2.29 in an equivalent form as follows.

Theorem 2.30. *Every monotone sequence converges, diverges to ∞, or diverges to $-\infty$. In other words, we say that $\lim_{n\to\infty} a_n$ is always meaningful for monotone sequences.*

Example 2.31. Set $a_n = (1 \cdot 3 \cdot 5 \cdots (2n-1))/(2 \cdot 4 \cdot 6 \cdots (2n))$. Then $\{a_n\}$ converges.

Solution. Note that $a_n > 0$ for all $n \geq 1$ and

$$a_{n+1} = a_n \left(\frac{2n+1}{2n+2} \right) < a_n.$$

Thus, $\{a_n\}$ is decreasing and bounded below by 0. Applying Theorem 2.25, we see that $\{a_n\}$ converges. Note also that $a_n < 1$ for $n \geq 1$. ●

Often sequences are defined by formulas. There is still another way of specifying a sequence, by defining its terms "inductively" or "recursively." In such cases, we normally specify the first term (or first several terms) of the sequence and then give a formula that specifies how to obtain all successive terms. We begin with a simple example and later present a number of additional examples (see Examples 2.39 and 2.58 and Exercises 2.45).

Example 2.32. Starting with $a_1 = 1$, consider the sequence $\{a_n\}$ with $a_{n+1} = \sqrt{2a_n}$ for $n \geq 1$. We observe that

$$a_1 = 1, \quad a_2 = \sqrt{2}, \quad a_3 = \sqrt{2\sqrt{2}}, \quad a_4 = \sqrt{2\sqrt{2\sqrt{2}}}, \ldots,$$

which seems to suggest that the given sequence is positive and increasing. Hence, the sequence must converge if it is bounded and increasing. It is not clear how to find an upper bound. However, the following observation might be useful. "If an increasing sequence converges, then the limit must be the least upper bound of the sequence" (see the proof of Theorem 2.25). As a consequence, if the given sequence converges to a, then the limit a must satisfy

$$a = \sqrt{2a}, \quad \text{i.e., } a(a-2) = 0,$$

so that $a = 2$, for $a = 0$ is not possible. By the method of induction, it is easy to prove that $0 < a_n \leq 2$ for all $n \geq 1$. Consequently,

$$a_{n+1} = \sqrt{2a_n} = a_n(\sqrt{2/a_n}) \geq a_n \quad \text{for all } n \geq 1,$$

showing that the sequence $\{a_n\}$ is bounded and increasing. Thus, $\{a_n\}$ converges and in fact converges to 2. ●

The BMCT is an extremely valuable theoretical tool, as we shall see by a number of examples below.

Example 2.33 (The number e). Let $a_n = (1+1/n)^n$, $n \geq 1$. The sequence $\{a_n\}$ is called *Euler's sequence*. Note that $(1+x)^n \geq 1 + nx$ for $x \geq 0$ and $n \geq 1$, so that for $x = 1/n$, this gives

$$\left(1 + \frac{1}{n}\right)^n \geq 2 \quad \text{for } n \geq 1.$$

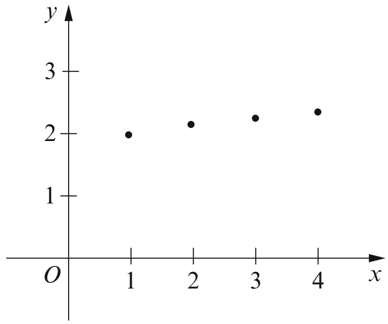

Fig. 2.4. Diagram for $a_n = (1 + 1/n)^n$.

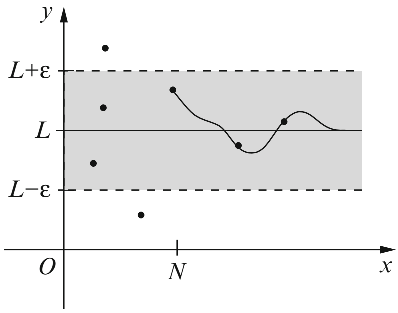

Fig. 2.5. a_n is eventually inside the strip.

If we plot the first few terms of this sequence on a sequence diagram, then it seems that the sequence $\{a_n\}$ increases and converges to a limit, which is less than 3 (see Figure 2.4).

First we show that the sequence is increasing (see Figure 2.4). This is an immediate consequence of the well-known arithmetic–geometric mean inequality

$$\left(\prod_{i=1}^{k} x_i\right)^{1/k} \leq \frac{1}{k}\sum_{i=1}^{k} x_i$$

if we choose $k = n + 1$, $x_1 = 1$, and $x_i = 1 + 1/n$ for $i = 2, \ldots, n + 1$. As an alternative proof, we may use the binomial theorem and obtain

$$\begin{aligned}
a_n &= 1 + \sum_{k=1}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \\
&= 1 + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{n^k}\frac{1}{k!} \\
&= 1 + 1 + \sum_{k=2}^{n} \left[1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-2}{n}\right)\left(1 - \frac{k-1}{n}\right)\right]\frac{1}{k!} \\
&< 2 + \sum_{k=2}^{n} \left[\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{k-2}{n+1}\right)\left(1 - \frac{k-1}{n+1}\right)\right]\frac{1}{k!} \\
&< 2 + \sum_{k=2}^{n+1} \left[\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{k-2}{n+1}\right)\left(1 - \frac{k-1}{n+1}\right)\right]\frac{1}{k!} \\
&= a_{n+1},
\end{aligned}$$

and so $\{a_n\}$ is increasing. Next, we show that the sequence is bounded. Since $k! = 1 \cdot 2 \cdot 3 \cdots k \geq 1 \cdot 2 \cdot 2 \cdots 2 = 2^{k-1}$ for $k \geq 2$, we have

$$2 < a_n < 1 + \sum_{k=1}^{n} \frac{1}{k!} < 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \frac{1-(1/2)^n}{1-(1/2)} < 1 + \frac{1}{1-1/2} = 3.$$

Thus, $\{a_n\}$ is an increasing bounded sequence. By BMCT, it follows that the sequence $\{a_n\}$ converges to a real number that is at most 3. It is customary to denote this limit by e, the base of the natural logarithm, a number that plays a significant role in mathematics. The above discussion shows that $2 < \mathrm{e} \leq 3$. The foregoing discussion allows us to make the following definition:

$$\mathrm{e} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Moreover, by considering the binomial expansion of $(1 + x/n)^n$, the above discussion may be continued to make the following definition of e^x for $x > 0$:

$$\mathrm{e}^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n, \quad x > 0.$$

Later, we shall show that this limit actually exists also for $x < 0$ (see Theorem 5.7). Thus, we easily have

$$\lim_{n\to\infty} \left(1 - \frac{1}{3n}\right)^{n+2} = \lim_{n\to\infty} \left[\left(1 - \frac{1}{3n}\right)^{3n}\right]^{1/3} \left(1 - \frac{1}{3n}\right)^2 = \mathrm{e}^{-1/3} \cdot 1$$

and

$$\lim_{n\to\infty} \left(1 + \frac{5}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{5}{5n}\right)^{5n} = \lim_{n\to\infty} \left[\left(1 + \frac{1}{n}\right)^n\right]^5 = \mathrm{e}^5.$$

Can we replace 5 in each step of the last of these equalities by a positive integer?

Moreover, by the product and the quotient rules for sequences, we have

$$\lim_{n\to\infty} \left(1 + \frac{1}{n+k}\right)^n = \lim_{n\to\infty} \frac{\left(1 + \frac{1}{n+k}\right)^{n+k}}{\left(1 + \frac{1}{n+k}\right)^k} = \frac{\lim_{n\to\infty} \left(1 + \frac{1}{n+k}\right)^{n+k}}{\lim_{n\to\infty} \left(1 + \frac{1}{n+k}\right)^k} = \mathrm{e},$$

where k is a fixed positive integer. Could k be any fixed integer? Could k be any positive real number? ●

Theorem 2.34 (Convergence of a geometric sequence). *If r is a fixed number such that $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. Further, $\{r^n\}$ diverges if $|r| > 1$. At $r = 1$, the sequence converges, whereas it diverges for $r = -1$.*

Proof. We have already proved the first part in Example 2.28 (see also Example 2.43). If $r = 1$, the sequence reduces to a constant sequence and so converges to 1. If $r > 1$, then $r^n \to \infty$ as $n \to \infty$, so the sequence diverges. Indeed, if $r > 1$, then $1/r < 1$, and so

$$\frac{1}{r^n} = \left(\frac{1}{r}\right)^n \to 0 \quad \text{as } n \to \infty,$$

which implies that $r^n \to \infty$ as $n \to \infty$.

For $r = -1$, the sequence $\{(-1)^n\}$ diverges, and if $r < -1$, then $\{r^n\}$ diverges, since $|r|^n \to \infty$ as $n \to \infty$. ∎

Example 2.35. For $p > 0$, we easily have

$$\lim_{n\to\infty} \frac{r^n}{n^p} = \begin{cases} 0 & \text{if } |r| \le 1, \\ \infty & \text{if } r > 1, \\ \text{does not exist} & \text{if } r < -1. \end{cases}$$

Indeed, for $|r| < 1$, let $a_n = r^n$ and $b_n = 1/n^p$. Then $\{a_n\}$ and $\{b_n\}$ are null sequences, and so is their product. For $r = 1, -1$, there is nothing to prove.

For $r > 1$, we write $r = 1 + x$ with $x > 0$. Let k be a positive integer such that $k > p$. Then for $n > 2k$,

$$(1+x)^n > \binom{n}{k} x^k = \frac{n(n-1)\cdots(n-k+1)}{k!} x^k > \left(\frac{n}{2}\right)^k \frac{x^k}{k!},$$

since $n - k + 1 > n/2$ for each k. Hence, since $k - p > 0$, it follows that

$$\frac{(1+x)^n}{n^p} > \frac{x^k}{2^k k!} n^{k-p} \to \infty \quad \text{as } n \to \infty.$$ •

Example 2.36. Find $\lim_{n\to\infty} r^n/(1 + r^{2n})$ for various values of r.

Solution. Set $a_n = r^n/(1+r^{2n})$. We need to find $\lim_{n\to\infty} a_n$ for various values of r. For $r = 1$, we have $a_n = 1/2$, showing that $\lim_{n\to\infty} a_n = 1/2$. For $r = -1$, we have $a_n = (-1)^n/2$, so that $\{a_n\}$ diverges. On the other hand, for $|r| < 1$, let $c_n = 1 + r^{2n}$. By Theorem 2.34, $\lim_{n\to\infty} c_n = 1$ and $\lim_{n\to\infty} r^n = 0$. Therefore, by the quotient rule,

$$\lim_{n\to\infty} \frac{r^n}{1 + r^{2n}} = \frac{\lim_{n\to\infty} r^n}{\lim_{n\to\infty}(1 + r^{2n})} = \frac{0}{1} = 0.$$

Similarly for $|r| > 1$, we have $1/|r| < 1$, and so using the above argument, we see that

$$\lim_{n\to\infty} \frac{r^n}{1 + r^{2n}} = \lim_{n\to\infty} \frac{1/r^n}{1 + 1/r^{2n}} = \frac{0}{1} = 0.$$

We conclude that $\{a_n\}_{n\ge1}$ converges for all $r \ne -1$. •

Theorem 2.37. *Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences such that $a_n \to L$ and $b_n \to M$ as $n \to \infty$. We have*

(a) $|a_n| \to |L|$ *as* $n \to \infty$*;*
(b) *if* $a_n \leq b_n$ *for all* $n \geq N_0$*, then* $L \leq M$*.*

Here **(b)** *is often referred to as the limit inequality rule.*

Proof. We prove case **(b)** by contradiction. Suppose that $a_n \to L$, $b_n \to M$, and $L > M$. Then with $\epsilon = (L - M)/2$, there exists an N such that

$$L - \epsilon < a_n < L + \epsilon \quad \text{and} \quad M - \epsilon < b_n < M + \epsilon \quad \text{for all } n \geq N.$$

In particular,

$$b_n < M + \epsilon = \frac{L + M}{2} = L - \epsilon < a_n \quad \text{for all } n \geq N,$$

which is a contradiction to the hypothesis that $a_n \leq b_n$ for all $n \geq N_0$. Therefore, our assumption is wrong, and hence we must have $L \leq M$.

The proof of case **(a)** follows from the fact that $\big||a_n| - |L|\big| \leq |a_n - L|$. ∎

Corollary 2.38. *Let $\{b_n\}$ be a convergent sequence such that $b_n \to M$ as $n \to \infty$, and $b_n \geq 0$ for all sufficiently large n. Then $M \geq 0$.*

Proof. Set $a_n = 0$ for all n in Theorem 2.37. ∎

Example 2.39. Consider the following sequences $\{a_n\}_{n \geq 1}$:

(a) $a_n = 1/n^2 + 1/(n+1)^2 + \cdots + 1/(2n)^2$;
(b) $a_1 = 1,\ a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$;
(c) $a_1 = 2,\ a_{n+1} = (1/2)(a_n + 2/a_n)$ for $n \geq 2$;
(d) $a_1 = \alpha$ and $a_{n+1} = (a_n + \beta/a_n)/2$ for $n \geq 1$, where $\alpha > 0$ is arbitrary and β is a fixed positive number.

In each case, determine whether the sequence converges.

Solution. **(a)** Clearly $0 < a_n < (n+1)/n^2$ for all $n \geq 1$, since each term (except the first) in the sum is strictly less than $1/n^2$, and so $\{a_n\}$ is a bounded sequence. Also, for $n \geq 1$,

$$a_{n+1} - a_n = \frac{1}{(2n+1)^2} + \frac{1}{(2n+2)^2} - \frac{1}{n^2} < \frac{1}{4n^2} + \frac{1}{4n^2} - \frac{1}{n^2} = -\frac{1}{2n^2} < 0,$$

that is, $a_{n+1} < a_n$ for all $n \geq 1$. Thus, $\{a_n\}$ is a bounded monotone sequence and so converges by Theorem 2.27.

Alternatively, we observe that for all $n \geq 1$,

$$\frac{n+1}{(2n)^2} \leq a_n \leq \frac{n+1}{n^2},$$

and so by the squeeze rule, we see that $\lim_{n \to \infty} a_n = 0$.

(b) Clearly $a_n > 0$ for all $n \geq 1$. Since $a_1 < 2$, by induction we obtain that $a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$ for all $n \geq 1$. Since

$$a_{n+1} - a_n = \sqrt{2+a_n} - a_n \geq 0 \Longleftrightarrow (2-a_n)(1+a_n) \geq 0,$$

and since $a_n \leq 2$, it follows that the sequence $\{a_n\}$ is monotonically increasing and bounded; hence it is convergent. We see that

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+a},$$

which gives $(a-2)(a+1) = 0$, or $a = 2$.

(c) First we observe that if the given sequence were convergent, then we would obtain its limit by allowing $n \to \infty$ in the given recurrence relation:

$$a = \frac{1}{2}\left(a + \frac{2}{a}\right), \quad \text{i.e., } a^2 = 2 \text{ or } a = \sqrt{2}.$$

Now we show that the given sequence indeed converges to $\sqrt{2}$. We have $a_1 = 2 > \sqrt{2}$, $a_n > 0$, and for $n \geq 1$,

$$a_{n+1} - \sqrt{2} = \frac{(a_n - \sqrt{2})^2}{2a_n} \geq 0.$$

(We remind the reader that it does not matter what positive value is assigned to a_1.) Thus, $a_n \geq \sqrt{2}$ for all $n \geq 2$, and therefore,

$$\frac{a_{n+1}}{a_n} = \frac{1}{2}\left(1 + \frac{2}{a_n^2}\right) \leq \frac{1}{2}(1+1) = 1, \quad \text{i.e., } a_{n+1} \leq a_n \text{ for } n \geq 2,$$

showing that $\{a_n\}$ is monotonically decreasing and bounded below by 0; hence it is convergent.

(d) Since α and β are positive and $a_1 > 0$ (arbitrary), the principle of induction shows that $a_n > 0$ for all $n \geq 2$. Next for $n \geq 1$, we have

$$a_{n+1}^2 - \beta = \frac{1}{4}\left(a_n + \frac{\beta}{a_n}\right)^2 - \beta = \frac{(a_n^2 - \beta)^2}{4a_n^2} \geq 0,$$

so that $a_{n+1}^2 \geq \beta$ for all $n \geq 1$. Also, for $n \geq 2$,

$$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{\beta}{a_n}\right) = \frac{a_n^2 - \beta}{2a_n} \geq 0,$$

showing that $\{a_n\}_{n\geq 2}$ is decreasing and bounded below (since all terms are positive). By Theorem 2.25, we are assured that the sequence converges; call the limit L. Since $a_{n+1}^2 \geq \beta$ and $a_n > 0$, we must have $a_{n+1} \geq \sqrt{\beta}$ for $n \geq 1$ and hence $L \geq \sqrt{\beta}$ (see Theorem 2.37). Since $a_n \to L$ as $n \to \infty$, $a_{n+1} \to L$ as $n \to \infty$. Thus, by the linearity rule,

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{\beta}{a_n}\right) = \frac{1}{2}\left(L + \frac{\beta}{L}\right), \quad \text{i.e., } L = \sqrt{\beta}.$$

●

Remark 2.40. Example 2.39**(c)** (also 2.39**(d)** with $\beta = 2$ and Exercise 2.68(10)) provides a proof that there is a sequence of rational numbers that converges to the irrational number $\sqrt{2}$. Moreover, using the a_n from Example 2.39**(c)**, we note that

$$a_1 = 2, \quad a_2 = \frac{3}{2}, \quad a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{5}\right) = \frac{17}{12} \quad \text{and} \quad a_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{24}{17}\right) = \frac{577}{408},$$

so that a_4^2 is approximately 2.0006. Thus, the sequence $\{a_n\}$ defined in Example 2.39**(c)** provides a practical way of computing a rational approximation to $\sqrt{2}$. ●

2.1.7 The Bolzano–Weierstrass Theorem

It is useful to have necessary and sufficient conditions for the convergence of sequences. For monotone sequences, BMCT (see Theorem 2.27) shows that boundedness is such a condition. On the other hand, for general sequences, boundedness is necessary but not sufficient for convergence. Indeed, we have seen examples of bounded sequences that do not converge yet have convergent subsequences. To show that this is true in general, we need to prove a lemma. It is convenient first to introduce a definition. We say that $n \in \mathbb{N}$ is a *peak point* of $\{a_n\}$ if

$$a_n \geq a_k \quad \text{for all } k \geq n.$$

Lemma 2.41. *Every sequence of real numbers contains a monotonic subsequence.*

Proof. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. We need to construct a monotone subsequence. Then either the sequence $\{a_n\}$ has infinitely many peak points or it has only finitely many peak points.

Assume that there are infinitely many peak points n. Let n_1 be the first such n with this property (i.e., the smallest peak point) and n_2 the second (i.e., the smallest peak point with $n_2 > n_1$), etc. Thus,

(i) $a_{n_1} \geq a_k$ for all $k \in \mathbb{N}$ with $k \geq n_1$;
(ii) $a_{n_2} \geq a_k$ for all $k \in \mathbb{N}$ with $k \geq n_2 \, (> n_1)$.

From (i) and (ii), it follows that

$$a_{n_1} \geq a_{n_2}.$$

We now introduce n_{k+1} inductively as the smallest peak point such that $n_{k+1} > n_k$. Consequently,

$$a_{n_k} \geq a_{n_{k+1}},$$

and so $\{a_{n_k}\}_{k\geq 1}$ is a monotonically decreasing subsequence of $\{a_n\}$.

On the other hand, if there are only finitely many n such that

$$a_n \geq a_k \quad \text{for all } k \in \mathbb{N} \text{ with } k \geq n,$$

then we can choose an integer m_1 greater than all peak points, so that no terms of the sequence

$$\{a_{m_1}, a_{m_1+1}, a_{m_1+2}, \dots\}$$

have this property. Because m_1 itself is not a peak point, there exists an m_2 with $m_2 > m_1$ for which

$$a_{m_1} < a_{m_2}.$$

Again, m_2 is not a peak point bigger than all peak points, and so there exists an m_3 with $m_3 > m_2$ and

$$a_{m_3} > a_{m_2}.$$

Continuing the process, we obtain a sequence $\{a_{m_k}\}_{k\geq 1}$ that is a monotonically increasing subsequence of $\{a_n\}$. This completes the proof. ∎

We see that if a sequence is bounded, then even though it may diverge, it cannot behave "too badly." This fact follows from Lemma 2.41 together with BMCT.

Theorem 2.42 (Bolzano–Weierstrass). *Every bounded sequence of real numbers has a convergent subsequence (a subsequence with a limit in* $\mathbb{R}$*). That is, if* $\{a_n\}$ *is a sequence such that* $|a_n| \leq M$ *for all* $n \geq N$*, then there exist a number* l *in the interval* $[-M, M]$ *and a subsequence* $\{a_{n_k}\}$ *such that* $\{a_{n_k}\}$ *converges to* l*.*

Proof. Let $\{a_n\}$ be a bounded sequence of real numbers. By Lemma 2.41, it has a monotonic subsequence, say $\{a_{n_k}\}$. Because $\{a_n\}$ is bounded, so is every subsequence of $\{a_n\}$. Hence by BMCT, $\{a_{n_k}\}$ converges. ∎

Next we remark that $\{\sin n\}$ is a bounded sequence. What is the behavior of $\sin n$ as $n \to \infty$? According to Theorem 2.42, there must exist at least one number l in $[-1, 1]$ such that some subsequences $\{\sin n_k\}$ will converge to l. A discussion of this surprising fact is beyond the scope of this book. However, we can prove that every number l in $[-1, 1]$ has this property.

We note that the Bolzano–Weierstrass theorem says nothing about uniqueness, for if $a_n = (-1)^n$, then $a_{2n} \to 1$ and $a_{2n-1} \to -1$ as $n \to \infty$.

Example 2.43. Fix r such that $0 < r < 1$, and consider the sequence $\{a_n\}_{n\geq 1}$, where $a_n = r^n$. Then $a_n > 0$ for all $n \geq 1$, and the sequence is decreasing, because

$$a_n - a_{n+1} = (1 - r)r^n > 0.$$

Thus, $\{a_n\}$, being a decreasing sequence that is bounded below by zero, converges; call the limit a. Also, since

$$a_{2n} = (r^n)(r^n),$$

$\{a_{2n}\}$ converges to a^2. On the other hand, $\{a_{2n}\}$ is a subsequence of $\{a_n\}$, and hence by the uniqueness of the limit, we have $a^2 = a$, i.e., $a = 0$ or 1. Clearly $a \neq 1$, since $\{r^n\}$ is decreasing and $r < 1$. Hence $\{r^n\}$ converges to 0 whenever $0 < r < 1$ (see also Theorem 2.34).

By the squeeze rule, the inequalities

$$-|r|^n \leq r^n \leq |r|^n$$

show that $\lim_{n\to\infty} r^n = 0$ for $-1 < r < 0$ also.

The same idea may be used to show that $\lim_{n\to\infty} a^{1/n} = 1$ for $0 < a < 1$ (see also Example 2.18**(a)**). ●

2.1.8 Questions and Exercises

Questions 2.44.

1. If $a_n \to a$ as $n \to \infty$, must the set $\{n : a_n \notin (a - \epsilon, a + \epsilon)\}$, where $\epsilon > 0$, be finite?
2. Is it true that a sequence $\{a_n\}$ is null iff $\{|a_n|\}$ is null?
3. Is every convergent sequence null? How about the converse?
4. Is the sum of two null sequences always null?
5. Does an alternating sequence always converge? Does it always diverge?
6. Is every convergent sequence monotone? Is every monotone sequence convergent?
7. Can a bounded sequence be convergent without being monotone?
8. Does every divergent increasing sequence diverge to ∞? How about a divergent decreasing sequence?
9. Can we say that $\{a_5, a_4, a_1, a_2, a_3, a_6, a_7, \dots\}$ is a subsequence of $\{a_n\}_{n\geq 1}$?
10. Does every sequence have at most a countable number of subsequences? Does there exist a sequence with an uncountable number of subsequences?
11. Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences such that one converges to 0 while the other is bounded. Does $\{a_nb_n\}$ converge? If so, to what limit?
12. Suppose that $\{a_n\}$ is bounded and $\alpha \in (0, 1)$ is fixed. Does $\{\alpha^n a_n\}$ converge? If so, does it converge to 0?
13. Suppose that $\{a_n\}$ is a bounded convergent sequence such that $|a_n| \leq M$ and the sequence has limit a. Must $|a| \leq M$?
14. Suppose that $\{a_n\}$ is increasing and bounded above by M. Must we have $a_n \to L$ for some L? Must $L \leq M$?
15. Suppose that $\{a_n\}$ is decreasing and bounded below by m. Must we have $a_n \to l$ for some l? Must $l \geq m$?

16. Let $\{a_n\}$ be a sequence of nonnegative real numbers, $p \in \mathbb{N}$, and $a \in [0, \infty)$. Is it true that $\{a_n\}$ converges to a if and only if $\{a_n^{1/p}\}$ converges to $a^{1/p}$?
17. Let $\{a_n\}$ be a null sequence of nonnegative real numbers, and $p \in \mathbb{R}$. Must $\{a_n^p\}$ be a null sequence? Is $\{1/n^p\}$ a null sequence?
18. Let $\{a_n\}$ be a sequence of positive real numbers. Is it true that $\{a_n\}$ diverges to ∞ if and only if $\{1/a_n\}$ converges to 0?
19. If $\{a_n\}$ is a sequence of real numbers such that $\{a_n/n\}$ converges to l for some $l \neq 0$, must $\{a_n\}$ be unbounded?
20. If $\{a_n\}$ converges to 0, must $\{(-1)^n a_n\}$ converge to 0?
21. If $\{a_n\}$ converges to a nonzero real number a, must $\{(-1)^n a_n\}$ oscillate?
22. If $\{a_n\}$ diverges to ∞, must $\{(-1)^n a_n\}$ oscillate?
23. If $\{|a_n|\}$ converges to $|a|$, must $\{a_n\}$ be convergent either to a or to $-a$? How about when $a = 0$? Does the sequence $\{(-1)^n\}$ address your concern for this question?
24. If $\{a_n\}$ converges and $\{b_n\}$ diverges, must $\{a_n b_n\}$ be divergent? Must $\{a_n + b_n\}$ be divergent?
25. If $\{a_n\}$ and $\{b_n\}$ are divergent, must $\{a_n b_n\}$ be divergent? Must $\{a_n + b_n\}$ be divergent?
26. Suppose that $\{a_n\}$ is an unbounded sequence of nonzero real numbers. Does $\{a_n\}$ diverge to ∞ or $-\infty$? Must $\{|a_n|\}$ be divergent to ∞? Must $\{1/a_n\}$ be bounded?
27. Suppose that $\{a_n\}$ is bounded. Must $\{1/a_n\}$ be bounded? Must $\{a_n/n\}$ be convergent?
28. If $\{a_n\}$ and $\{a_n b_n\}$ are both bounded, must $\{b_n\}$ be bounded?
29. If $a_1 = 1$ and $a_{n+1} = a_n + (1/a_n)$ for $n \geq 1$, must $\{a_n\}$ be bounded?
30. If $\{a_n\}$ and $\{b_n\}$ are both increasing, must $\{a_n b_n\}$ be increasing?
31. Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers such that $|a_n - b_n| < 1/n$ for large n, and $a_n \to a$ as $n \to \infty$. Does $b_n \to a$ as $n \to \infty$?
32. If $\{a_n\}$ is a sequence such that $\{(a_n - 1)/(a_n + 1)\}$ converges to zero, does $\{a_n\}$ converge?
33. If $\{a_n\}$ converges to a, must $\{a_n^2\}$ converge to a^2? Does $\{a_n^p\}$ converge to a^p if $p \in \mathbb{N}$?
34. Suppose that $b_n \to b$ as $n \to \infty$ and $b \neq 0$. Must there exist an $R > 0$ and a positive integer N such that $|b_n| \geq R$ for all $n \geq N$?
35. If $\{a_n^2\}$ converges, must $\{a_n\}$ be convergent?
36. Suppose that $\{a_n^2\}$ converges and $a_n > 0$. Can $\{a_n\}$ be convergent? Can $\{a_n\}$ be convergent?
37. If $\{a_n^2\}$ converges to a, must $\{|a_n|\}$ converge to $\sqrt{a}$?
38. If $\{a_n^3\}$ converges to a^3, must $\{a_n\}$ converge to a?
39. Can there exist a divergent sequence that is monotone?
40. Can there exist a divergent sequence $\{s_n\}$ such that $s_{n+1} - s_n \to 0$ as $n \to \infty$?

41. If $\{a_n\}$ is an increasing sequence of real numbers that is bounded above and $L = \lim_{n\to\infty} a_n$, must we have $a_n \le L$ for all n?
42. If $\{a_n\}$ is a decreasing sequence of real numbers that is bounded below and $L = \lim_{n\to\infty} a_n$, must we have $a_n \ge L$ for all n?
43. If $0 < a < 1$, does it follow that $\lim_{n\to\infty} a^{1/2^n} = 1$? Does it follow that $\lim_{n\to\infty} a^{1/3^n} = 1$?
44. Let $a_n = (1 + 1/n)^n$ and $b_n = (1 + 1/n)^{n+k}$, where k is a fixed integer. Do we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = e$?

Exercises 2.45.

1. Show that
$$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}, \quad \lim_{n\to\infty} \frac{3n+1}{2n+1} = \frac{3}{2}, \quad \text{and} \quad \lim_{n\to\infty} \frac{n^3-3}{n^4} = 0.$$
If $\epsilon = 0.001$ is chosen, find N in each case such that for $n \ge N$ we have
$$\left|\frac{n}{2n+3} - \frac{1}{2}\right| < 0.001, \quad \left|\frac{3n+1}{2n+1} - \frac{3}{2}\right| < 0.001, \quad \text{and} \quad \left|\frac{n^3-3}{n^4}\right| < 0.001.$$
2. Construct three sequences such that $a_n \le b_n \le c_n$ for all $n \ge N$, $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = M$ for some real numbers L, M, but $\lim_{n\to\infty} b_n$ does not exist.
3. Suppose that $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ are two sequences of real numbers such that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = L$, where $0 < L \le \infty$. Show that $\lim_{n\to\infty} a_n b_n = \infty$. Using this, show that
$$\lim_{n\to\infty} \frac{n^3-3}{n+2} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{3^n}{n^2+(-1)^n} = \infty.$$
4. Which of the following sequences are monotone? bounded? convergent?
$$\left\{\frac{(-1)^n(n+2)}{n}\right\}, \ \left\{2^{(-1)^n}\right\}, \ \left\{\frac{n}{2^n}\right\}, \ \{\log(n+1) - \log n\}, \ \left\{\frac{3n-5}{2^n}\right\}.$$
5. For $p > 0$ and $|c| < 1$, prove that $\{c^n\}$, $\{n^p c^n\}$, and $\{n^p/n!\}$ are all null sequences.
6. Using BMCT, show that $a^{1/n} \to 0$ as $n \to \infty$, where $0 < a < 1$. Is it possible to use BMCT to show that $n^{1/n} \to 1$ as $n \to \infty$?
7. Which is larger in each of the following:
(i) 1000^{1000} or 1001^{999}? **(ii)** $\left(1 + \frac{1}{100000}\right)^{100000}$ or 2?
8. Define a_n recursively by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + \sqrt{a_n}}$ for all $n \ge 1$. Show that the sequence $\{a_n\}_{n\ge1}$ is convergent. Find its limit.
9. Define a_n recursively by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for all $n \ge 1$. Show that the sequence $\{a_n\}_{n\ge1}$ converges to 2.

10. For each of the following sequences, show that there is a number L such that $a_n \to L$. Find also the value of L.
    (a) $\{a_n\}$, where $a_1 = 1$ and $a_{n+1} = 1 + \sqrt{a_n}$ for $n \geq 1$.
    (b) $\{a_n\}$, where $a_1 = 3$ and $a_{n+1} = 3 + \sqrt{a_n}$ for $n \geq 1$.
    (c) $\{a_n\}$, where $a_1 = L$ $(L > 1)$ and $a_{n+1} = \sqrt{a_n}$ for $n \geq 1$.
    (d) $\{a_n\}$, where $a_1 > 0$, $a_2 > 0$, and $a_{n+2} = \sqrt{a_n} + \sqrt{a_{n+1}}$ for $n \geq 1$.
    (e) $\{a_n\}$, where $a_1 = 1$ and $a_{n+1} = \frac{1}{4}(2a_n + 3)$ for $n \geq 1$.
    (f) $\{a_n\}$, where $a_1 = 1$ and $a_{n+1} = a_n/(1 + a_n)$ for $n \geq 1$.
    (g) $\{a_n\}$, where $a_1 = \alpha > 0$ and $a_{n+1} = \sqrt{(\alpha\beta^2 + a_n^2)/(\alpha + 1)}$ $(\beta > \alpha)$.
11. Suppose that a sequence $\{a_n\}$ of real numbers satisfies $7a_{n+1} = a_n^3 + 6$ for $n \geq 1$. If $a_1 = \frac{1}{2}$, prove that the sequence increases and find its limit. What happens if $a_1 = \frac{3}{2}$ or $a_1 = \frac{5}{2}$?
12. Test each of the sequences given below for convergence. Find its limit if it converges.
    (a) $a_1 = 1$ and $a_{n+1} = \sqrt{5a_n}$.   **(b)** $a_1 = 1$ and $a_{n+1} = \sqrt{5}\, a_n$.
    (c) $a_1 = 1$ and $a_{n+1} = \sqrt{5 + a_n}$.
13. Show that if $a_1 > b_1 > 0$, $a_{n+1} = \sqrt{a_n b_n}$, and $b_{n+1} = (a_n + b_n)/2$, then $\{a_n\}$ and $\{b_n\}$ both converge to a common limit.
14. Let $\{a_n\}$ be a sequence of positive real numbers such that $a_{n+1} \leq r a_n$ for some $r \in (0, 1)$ and for all n. Prove that $\{a_n\}$ converges to 0.
15. In the following problems, state whether the given sequence $\{a_n\}$ is convergent or divergent. If it is convergent, then determine its limit. Here a_n equals

    (a) $2 + (-1)^n$.   **(b)** $n(2 + (-1)^n)$ **(c)** $n \cos\left(\dfrac{n\pi}{2}\right)$.

    (d) $2^{2008/n}$.   **(e)** $\dfrac{3n^2 - \log n}{n^2 + 3n^{3/2}}$.   **(f)** $\sqrt{n + 3\sqrt{n}} - \sqrt{n}$.

    (g) $n^{2008/n}$.   **(h)** $n^{1/(n+2008)}$.   **(i)** $(n+1)^{1/(\log(1+n))}$.

    (j) $\dfrac{5^n + 6^n}{1 + 7^n}$.   **(k)** $(\log n)^{1/n}$.   **(l)** $\sqrt{n(n+1)} - n$.

    (m) $\dfrac{(n!)^{1/n}}{n}$.   **(n)** $\log n - \log(n+1)$.   **(o)** $\dfrac{1}{n} \sin\left(\dfrac{n\pi}{6}\right) + \dfrac{5n+1}{7n+6}$.

    (p) $(an + 7)^{1/n}$.   **(q)** $\dfrac{a^n - a^{-n}}{a^n + a^{-n}}$.   **(r)** $(n + 2008)^{1/n}$.

    (s) $\dfrac{a^n + n}{a^n - n}$.   **(t)** $\dfrac{a^n}{n!}$ $(a \in \mathbb{R})$.   **(u)** $n(a^{1/n} - 1)$.

2.2 Limit Inferior, Limit Superior, and Cauchy Sequences

Consider a sequence of real numbers $\{a_n\}_{n\geq 1}$. Then for each fixed $k \in \mathbb{N}$, let

$$M_k = \sup\{a_k, a_{k+1}, \dots\} := \sup\{a_n \colon n \geq k\}$$

if the sequence is bounded above, and $M_k = \infty$ if it is not bounded above. Clearly, $M_k \geq M_{k+1}$ for every k. Similarly, let

$$m_k = \inf\{a_k, a_{k+1}, \dots\} := \inf\{a_n \colon n \geq k\}$$

if the sequence is bounded below, and $m_k = -\infty$ if it is not bounded below. Clearly, $m_k \leq m_{k+1}$ for every k. Consequently,

$$m_1 \leq m_2 \leq \cdots \leq m_k \leq m_{k+1} \leq \cdots \leq M_{k+1} \leq M_k \leq \cdots \leq M_2 \leq M_1.$$

Since every monotone sequence *has a limit* (see Theorem 2.30 if we also allow $\pm\infty$), the limits

$$M = \lim_{k\to\infty} M_k \quad \text{and} \quad m = \lim_{k\to\infty} m_k$$

both exist. So $m \leq M$. We call M and m the limit superior and the limit inferior, respectively, of $\{a_n\}$. We denote these limits by

$$M = \limsup_{n\to\infty} a_n \text{ and } \overline{\lim_{n\to\infty}}\, a_n, \quad \text{and} \quad m = \liminf_{n\to\infty} a_n \text{ or } \underline{\lim}_{n\to\infty} a_n,$$

respectively. Thus,

$$\limsup_{n\to\infty} a_n = \lim_{k\to\infty} \sup_{n\geq k} a_n \quad \text{and} \quad \liminf_{n\to\infty} a_n = \lim_{k\to\infty} \inf_{n\geq k} a_n.$$

The right-hand sides of these are always meaningful, provided it is understood that the values of ∞ and $-\infty$ are allowed. Note that

$$\begin{cases} M = \infty & \text{if } \{a_n\} \text{ is not bounded above,} \\ m = -\infty & \text{if } \{a_n\} \text{ is not bounded below,} \\ M = -\infty & \text{if } \lim\limits_{n\to\infty} a_n = -\infty, \\ m = \infty & \text{if } \lim\limits_{n\to\infty} a_n = \infty. \end{cases}$$

For instance:

(a) For the sequence $\{a_n\}_{n\geq 1}$, where $a_n = 1/n$, we have

$$m_1 = \inf\{1, 1/2, 1/3, \dots\} = 0, \quad m_2 = \inf\{1/2, 1/3, 1/4, \dots\} = 0,$$

and $m_k = 0$ for each $k \geq 1$. Therefore, it is clear that

$$m = \lim m_k = 0, \quad \text{i.e.,} \ \liminf a_n = 0.$$

Similarly, we see that

$$M_1 = \sup\{1, 1/2, 1/3, \ldots\} = 1, \quad M_2 = \sup\{1/2, 1/3, 1/4, \ldots\} = \frac{1}{2},$$

and $M_k = 1/k$ for each $k \geq 1$. Therefore,

$$M = \lim M_k = 0, \quad \text{i.e., } \limsup a_n = 0.$$

(b) $\limsup_{n\to\infty}(-1)^n = 1$ and $\liminf_{n\to\infty}(-1)^n = -1$.
(c) $\lim_{n\to\infty} n^2 = \infty$, and so $\limsup_{n\to\infty} n^2 = \liminf_{n\to\infty} n^2 = \infty$.
(d) $\limsup_{n\to\infty}(-n) = -\infty$ and $\limsup_{n\to\infty} n = \infty$.
(e)

$$\limsup_{n\to\infty} r^n = \begin{cases} \infty & \text{if } |r| > 1, \\ 1 & \text{if } |r| = 1, \\ 0 & \text{if } |r| < 1, \end{cases} \quad \text{and} \quad \liminf_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1, \\ 1 & \text{if } r = 1, \\ 0 & \text{if } |r| < 1, \\ -1 & \text{if } r = -1, \\ -\infty & \text{if } r < -1. \end{cases}$$

(f) If $a_n = (-1)^n(1 + 1/n)$, then $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$. Also, we note that $a_{2n} \to 1$, $a_{2n-1} \to -1$ as $n \to \infty$, and the sequence $\{a_n\}$ has no subsequences that can converge to a limit other than 1 or -1. Note also that

$$\sup\{a_n : n \geq 1\} = \frac{3}{2} \quad \text{and} \quad \inf\{a_n : n \geq 1\} = -2.$$

The reader is warned not to confuse the supremum of a set with the limit superior of a sequence, and similarly the infimum of a set with the limit inferior of a sequence.
(g) $\limsup_{n\to\infty} (-1)^n/n = 0 = \liminf_{n\to\infty} (-1)^n/n$, because for $k \geq 1$,

$$M_k = \sup\left\{\frac{(-1)^k}{k}, \frac{-(-1)^k}{k+1}, \frac{(-1)^k}{k+2}, \ldots\right\} = \begin{cases} \dfrac{1}{k+1} & \text{if } k \text{ is odd}, \\ \dfrac{1}{k} & \text{if } k \text{ is even}, \end{cases}$$

and

$$m_k = \begin{cases} -\dfrac{1}{k} & \text{if } k \text{ is odd}, \\ -\dfrac{1}{k+1} & \text{if } k \text{ is even}, \end{cases}$$

so that $M_k \to 0$ and $m_k \to 0$ as $k \to \infty$.
(h) For the sequence $\{(-1)^n n\}_{n\geq 1} = \{\ldots, -5, -3, -1, 2, 4, 6, \ldots\}$, we have

$$\inf\{(-1)^n n : n \in \mathbb{N}\} = -\infty \quad \text{and} \quad \liminf(-1)^n n = -\infty$$

and

$$\sup\{(-1)^n n : n \in \mathbb{N}\} = \infty \quad \text{and} \quad \limsup(-1)^n n = \infty.$$

Lemma 2.46. *Suppose that* $\{a_n\}$ *is a sequence of real numbers with*

$$L = \limsup_{n\to\infty} a_n \quad \textit{and} \quad \ell = \liminf_{n\to\infty} a_n.$$

Then for every $\epsilon > 0$ *there exist integers* N_1 *and* N_2 *such that*

$$\begin{cases} a_n - L < \epsilon & \textit{for all } n \geq N_1, \\ a_n - L > -\epsilon & \textit{for infinitely many } n \geq N_1, \end{cases}$$

and

$$\begin{cases} a_n - \ell > -\epsilon & \textit{for all } n \geq N_2, \\ a_n - \ell < \epsilon & \textit{for infinitely many } n \geq N_2, \end{cases}$$

respectively.

Proof. By the definition of the limit superior, since $L = \lim_{k\to\infty} M_k$, there exists an integer N_1 such that

$$|\sup\{a_k, a_{k+1}, \ldots\} - L| = |M_k - L| < \epsilon \quad \text{for all } k \geq N_1,$$

so that

$$a_k \leq \sup\{a_k, a_{k+1}, \ldots\} < L + \epsilon \quad \text{for all } k \geq N_1.$$

That is,

$$a_k < L + \epsilon \quad \text{for all } k \geq N_1.$$

Again, since $M_k \geq M_{k+1}$ for every $k \geq 1$, we have

$$L \leq \sup_{k\geq 1} M_k. \tag{2.1}$$

In particular, this gives

$$L \leq M_1 = \sup\{a_1, a_2, a_3, \ldots\}.$$

Thus, by the definition of supremum, there exists an n_1 such that $a_{n_1} > M_1 - \epsilon$, so that

$$a_{n_1} > L - \epsilon.$$

Now taking $k = n_1$ in (2.1), we obtain that

$$L \leq M_{n_1} = \sup\{a_{n_1}, a_{n_1+1}, \ldots\},$$

and so there exists an n_2 such that

$$a_{n_2} > M_{n_1} - \epsilon > L - \epsilon.$$

Proceeding indefinitely, we obtain integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$a_{n_k} > L - \epsilon \quad \text{for all } k \in \mathbb{N},$$

which proves the second inequality for the case of limit superior.

Similarly, since $\ell = \lim_{k\to\infty} m_k$, there exists an integer N_2 such that

$$a_k \geq \inf\{a_k, a_{k+1}, \ldots\} > L - \epsilon \quad \text{for all } k \geq N_2.$$ ■

Theorem 2.47. *For any sequence of real numbers* $\{a_n\}$, *we have*

$$\lim_{n\to\infty} a_n = L \quad \textit{if any only if} \quad \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L.$$

Proof. If $L = \pm\infty$, then the equivalence is a consequence of the definitions of limit superior and limit inferior. Therefore, we assume that $\lim a_n = L$, where L is finite.

$\Rightarrow$: Given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon, \quad \text{i.e., } L - \epsilon < a_n < L + \epsilon \text{ for all } n \geq N,$$

and so

$$L - \epsilon < M_N = \sup\{a_N, a_{N+1}, \ldots\} \leq L + \epsilon.$$

Thus, $\{M_k\}_{k\geq N}$ is a bounded monotone sequence and hence converges. That is,

$$L - \epsilon \leq \lim_{N\to\infty} M_N = \limsup_{n\to\infty} a_n \leq L + \epsilon.$$

Since ϵ is arbitrary, $\limsup_{n\to\infty} a_n = L$. A similar argument gives $\liminf_{n\to\infty} a_n = L$.

$\Leftarrow$: Conversely, suppose that $L = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \ell$. Since $\ell = L$, by Lemma 2.46 we conclude that there exists $N = \max\{N_1, N_2\}$ such that

$$L - \epsilon < a_k < L + \epsilon \quad \text{for all } k \geq N.$$

This proves that $\lim_{k\to\infty} a_k = L$, as desired. ∎

For any bounded sequence $\{a_n\}$, we see that $\{M_k - m_k\}$ is increasing and converges to $M - m$. Thus, using Theorem 2.47, we may formulate the definition of convergence of a sequence as follows.

Theorem 2.48. *A sequence* $\{a_n\}$ *of real numbers is convergent if and only if it is bounded and* $\{M_k - m_k\}$ *converges to zero, where* $M_k = \sup\{a_n \colon n \geq k\}$ *and* $m_k = \inf\{a_n \colon n \geq k\}$.

Alternatively, Theorem 2.42 can be seen (without using Lemma 2.41) as an immediate consequence of the following result, which in particular, shows that there are subsequences converging to m and M. Moreover, m and M are, respectively, the smallest and the largest possible limits for convergent subsequences.

Theorem 2.49. *Let* $\{a_n\}$ *be a bounded sequence of real numbers and let*

$$S = \{x \in \mathbb{R} : a_{n_k} \to x \textit{ for some subsequence } a_{n_k}\}.$$

If $m = \liminf a_n$ *and* $M = \limsup a_n$, *then* $\{m, M\} \subset S \subset [m, M]$.

Proof. First we prove that $M \in S$. For this, we need to show that there exists a subsequence $\{a_{n_k}\}_{k\geq 1}$ such that for each given $\epsilon > 0$, there exists an integer N such that

$$|a_{n_k} - M| < \epsilon \quad \text{for all } k \geq N.$$

By Lemma 2.46, there exists an integer N_1 such that

$$a_k < M + \epsilon \;\; \text{for all } k \geq N_1 \tag{2.2}$$

and $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$a_{n_k} > M - \epsilon \quad \text{for all } k \in \mathbb{N}. \tag{2.3}$$

Combining (2.2) and (2.3), we infer that

$$M - \epsilon < a_{n_k} < M + \epsilon, \quad \text{i.e., } |a_{n_k} - M| < \epsilon \text{ for all } n_k \geq N,$$

and so M is the limit of a subsequence of $\{a_n\}$. The assertion about m has a similar proof. Thus, $\{m, M\} \subset S$.

Next we prove that $S \subset [m, M]$. We assume that $a_{n_k} \to x$ as $k \to \infty$. We shall show that $x \in [m, M]$. Equation (2.2) shows that

$$a_n < M + \epsilon \quad \text{for sufficiently large } n,$$

and so

$$a_{n_k} < M + \epsilon \quad \text{for sufficiently large } k.$$

The limit inequality rule gives that

$$x \leq M + \epsilon,$$

and since $\epsilon > 0$ is arbitrary, it follows that $x \leq M$. The proof for $m \leq x$ is similar. ∎

Corollary 2.50. *A sequence $\{a_n\}$ of real numbers converges if and only if S is a singleton set. That is, $\lim a_n$ exists.*

In view of Theorem 2.49, we have the following equivalent definition: *If $\{a_n\}$ is a bounded sequence of real numbers, then M and m, the limit superior and the limit inferior of $\{a_n\}$, are respectively the greatest and the least subsequential limits of $\{a_n\}$.*

Theorem 2.51. *Suppose that $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are two bounded sequences of real numbers. Then we have the following:*

(a) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
(b) $\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$.
(c) $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$ *and* $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$ *if* $a_n \leq b_n$ *for all* $n \geq 1$.

(d) $\limsup_{n\to\infty}(a_n b_n) \le \Big(\limsup_{n\to\infty} a_n\Big)\Big(\limsup_{n\to\infty} b_n\Big)$ *if* $a_n > 0$, $b_n > 0$.

(e) $\liminf_{n\to\infty}(a_n b_n) \ge \Big(\liminf_{n\to\infty} a_n\Big)\Big(\liminf_{n\to\infty} b_n\Big)$ *if* $a_n > 0$, $b_n > 0$.

Proof. **(a)** and **(b)**:

Method 1: As usual, for each fixed $k \in \mathbb{N}$, let

$$M_k = \sup\{a_k, a_{k+1}, \ldots\} \quad \text{and} \quad P_k = \sup\{b_k, b_{k+1}, \ldots\}.$$

Then

$$a_n \le M_k \quad \text{and} \quad b_n \le P_k \quad \text{for all } n \ge k,$$

and therefore

$$a_n + b_n \le M_k + P_k \quad \text{for all } n \ge k,$$

which shows that $M_k + P_k$ is an upper bound for

$$\{a_k + b_k, a_{k+1} + b_{k+1}, \ldots\}.$$

Consequently,

$$\sup\{a_k + b_k, a_{k+1} + b_{k+1}, \ldots\} \le M_k + P_k,$$

and thus

$$\limsup_{k\to\infty}\{a_k + b_k, a_{k+1} + b_{k+1}, \ldots\} \le \lim_{k\to\infty}(M_k + P_k) = \lim_{k\to\infty} M_k + \lim_{k\to\infty} P_k,$$

which, by the definition, is equivalent to **(a)**. The proof of **(b)** is similar and so will be omitted.

Method 2: Since $\{a_n+b_n\}_{n\ge1}$ is a bounded sequence (by hypothesis), Lemma 2.46 shows that there exist integers N_1, N_2, N_3, and N_4 such that

$$a_k < L_a + \epsilon/2 \quad \text{for all } k \ge N_1 \quad \text{and} \quad a_k > \ell_a - \epsilon/2 \quad \text{for all } k \ge N_2$$

and

$$b_k < L_b + \epsilon/2 \quad \text{for all } k \ge N_3 \quad \text{and} \quad b_k > \ell_b - \epsilon/2 \quad \text{for all } k \ge N_4,$$

respectively. Here

$$L_a = \limsup a_n, \quad \ell_a = \liminf a_n, \quad L_b = \limsup b_n, \quad \text{and} \quad \ell_b = \liminf b_n.$$

Thus,

$$a_k + b_k < L_a + L_b + \epsilon \quad \text{for all } k \ge \max\{N_1, N_3\}$$

and

$$a_k + b_k > \ell_a + \ell_b - \epsilon \quad \text{for all } k \ge \max\{N_2, N_4\}.$$

Since $\epsilon > 0$ is arbitrary, **(a)** and **(b)** follow.

(c) Since $a_n \le b_n$ for all $n \ge 1$, it follows that

$$M_k \le P_k \quad \text{and} \quad m_k \le p_k,$$

where $m_k = \inf\{a_k, a_{k+1}, \ldots\}$ and $p_k = \inf\{b_k, b_{k+1}, \ldots\}$. Taking the limit as $k \to \infty$ yields the desired conclusion. ■

Observe that if $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, then we have

$$a_n + b_n = 0 \quad \text{for all } n \ge 0, \quad \limsup a_n = 1 = \limsup b_n.$$

We may also consider

$$a_n = \begin{cases} 0 & \text{if } n = 2k, \\ (-1)^{k+1} & \text{if } n = 2k-1, \end{cases} \quad \text{and} \quad b_n = \begin{cases} (-1)^k & \text{if } n = 2k, \\ 0 & \text{if } n = 2k-1, \end{cases}$$

so that

$$a_n + b_n = \begin{cases} (-1)^k & \text{if } n = 2k, \\ (-1)^{k+1} & \text{if } n = 2k-1. \end{cases}$$

In either case, the equalities in **(a)** and **(b)** of Theorem 2.51 do not always hold.

If

$$a_n = \begin{cases} 1 \text{ if } n \text{ is odd}, \\ 2 \text{ if } n \text{ is even}, \end{cases} \quad \text{and} \quad b_n = \begin{cases} 2 \text{ if } n \text{ is odd}, \\ 1 \text{ if } n \text{ is even}, \end{cases}$$

we see that equality in each of **(d)** and **(e)** of Theorem 2.51 does not hold.

2.2.1 Cauchy Sequences

If a sequence $\{a_n\}$ of real numbers converges to a number a, then the terms a_n of the sequence are close to a for large n, and hence the terms of the sequence themselves are close to each other "near a." This intuition led to the concept of Cauchy[1] sequence, which helps us in deducing the convergence of a sequence without necessarily knowing its limit. Moreover, unlike theorems (such as BMCT) that deal only with monotone sequences, we have theorems on Cauchy sequences that deal with sequences that are not necessarily monotone.

Definition 2.52 (Cauchy sequence). *A sequence $\{a_n\} \subset \mathbb{R}$ is called a Cauchy sequence if for each $\epsilon > 0$ there is a positive integer N such that $m, n \ge N$ implies $|a_n - a_m| < \epsilon$. Equivalently, we say that a sequence $\{a_n\}$ is Cauchy if for each $\epsilon > 0$ there is a positive integer N such that*

$$|a_{n+p} - a_n| < \epsilon \quad \textit{for all } n \ge N \textit{ and for all } p \in \mathbb{N}.$$

[1] Augustin-Louis Cauchy (1789–1857) is one of the important mathematicians who placed analysis on a rigorous footing.

For example, if $a_n = (-1)^{n-1}/n$, then $\{a_n\}$ is Cauchy; for

$$|a_n - a_m| = \left|\frac{(-1)^{n-1}}{n} - \frac{(-1)^{m-1}}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{2}{n} \quad \text{if } m > n.$$

Our first result is algebraic.

Theorem 2.53. *Every convergent sequence is a Cauchy sequence.*

Proof. Suppose that $a_n \to a$ as $n \to \infty$, and let $\epsilon > 0$ be given. Then there exists an N such that

$$|a_n - a| < \frac{\epsilon}{2} \quad \text{for all } n \geq N.$$

Therefore, for $m, n \geq N$, we must have

$$|a_n - a_m| = |(a_n - a) - (a_m - a)| \leq |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and hence $\{a_n\}$ is a Cauchy sequence. ∎

Theorem 2.53 gives a necessary condition for convergence. Equivalently, if a sequence is not Cauchy, then it cannot be convergent. Thus, Theorem 2.53 can be used to show the divergence of several nontrivial sequences. For example, we have the following:

(a) Neither $\{n\}_{n\geq 1}$ nor $\{1 + (-1)^n\}_{n\geq 1}$ is Cauchy.

(b) If $s_n = \sum_{k=1}^{n} 1/k$, then $\{s_n\}_{n\geq 1}$ is not Cauchy, because for any $n \in \mathbb{N}$ (with $m = 2n$),

$$s_{2n} - s_n = \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > n\left(\frac{1}{2n}\right) = \frac{1}{2}.$$

Thus, the sequence $\{s_n\}$ is not convergent.

(c) Similarly, if $s_n = \sum_{k=1}^{n} 1/(2k-1)$, then $\{s_n\}_{n\geq 1}$ is not Cauchy (and hence is not convergent), because for any $n \in \mathbb{N}$,

$$\begin{aligned} s_{2n} - s_n &= \sum_{k=1}^{2n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k-1} \\ &= \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{2n+2n-1} \\ &> n\left(\frac{1}{4n-1}\right) > n\left(\frac{1}{4n}\right) = \frac{1}{4}. \end{aligned}$$

(d) Finally, consider the sequence $\{x_n\}$ given by

$$x_0 = 0 \quad \text{and} \quad x_{n+1} = \frac{10x_n + 6}{5} \quad \text{for } n \geq 0.$$

Then $\{x_n\}$ does not converge, because it is not Cauchy. Indeed,

$$x_n > 0 \quad \text{for all } n \geq 1 \quad \text{and} \quad x_{n+1} - x_n = x_n + \frac{6}{5} > \frac{6}{5},$$

showing that $\{x_n\}$ is not Cauchy.

We also remark that a sequence $\{s_n\}$ that satisfies the condition

$$s_{n+1} - s_n \to 0 \quad \text{as } n \to \infty$$

is not necessarily a Cauchy sequence (e.g., s_n as above or $s_n = \log n$).

Theorem 2.54. *Cauchy sequences are bounded.*

Proof. The proof is similar to that of the corresponding result for convergent sequences (see Theorem 2.7). For the sake of completeness we include a proof here. Consider a Cauchy sequence $\{a_n\}_{n\geq 1}$. Then by definition, there exists a positive integer $N \in \mathbb{N}$ such that

$$|a_m - a_n| < \epsilon = 1 \quad \text{for all } n > m \geq N.$$

That is, with $m = N$, we have $|a_n| < 1 + |a_N|$ for all $n > N$. We conclude that $\{a_n\}_{n\geq 1}$ is bounded. ∎

An interesting fact which that Cauchy sequences important is that the converse of Theorem 2.53 is also true. Our next task is to prove this result, which is also called the *general principle of convergence.*

Theorem 2.55 (Completeness criterion for sequences). *A sequence is convergent if and only if it is a Cauchy sequence.*

Proof. The first half of the theorem has already been proved. Thus, we have to show that every Cauchy sequence of real numbers converges. To do this, we begin with a Cauchy sequence $\{a_n\}$. Then $\{a_n\}$ is bounded by Theorem 2.54. Let $\epsilon > 0$. Then there exists an $N = N(\epsilon)$ such that

$$|a_n - a_m| < \frac{\epsilon}{2} \quad \text{whenever } n > m \geq N. \tag{2.4}$$

Method 1: In particular, taking $m = N$ in (2.4), it follows that

$$|a_n - a_N| < \frac{\epsilon}{2}, \quad \text{i.e., } -\frac{\epsilon}{2} + a_N < a_n < \frac{\epsilon}{2} + a_N \text{ for all } n > N.$$

This shows that $a_N - (\epsilon/2)$ and $a_N + (\epsilon/2)$ are, respectively, lower and upper bounds for the set

$$X_n = \{a_n, a_{n+1}, \ldots\} \quad \text{if } n > N.$$

Note that $X_n \supseteq X_{n+1} \supseteq \cdots$ and if $M_n = \sup X_n$, then $M_n \geq M_{n+1} \geq \cdots$. Thus, for $n > N$,

$$\underbrace{a_N - \frac{\epsilon}{2} \leq \inf\{a_n, a_{n+1}, \ldots\}} \leq \sup\{a_n, a_{n+1}, \ldots\} \leq a_N + \frac{\epsilon}{2},$$

which gives

$$\sup\{a_n, a_{n+1}, \ldots\} \leq \underbrace{a_N + \frac{\epsilon}{2} \leq \inf\{a_n, a_{n+1}, \ldots\} + \frac{\epsilon}{2} + \frac{\epsilon}{2}},$$

so that for $n > N$,

$$\sup\{a_n, a_{n+1}, \ldots\} \leq \inf\{a_n, a_{n+1}, \ldots\} + \epsilon.$$

Thus, by definition,

$$\limsup a_n \leq \sup\{a_n, a_{n+1}, \ldots\} \leq \inf\{a_n, a_{n+1}, \ldots\} + \epsilon \leq \liminf a_n + \epsilon.$$

Since this holds for every $\epsilon > 0$, we have

$$\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_n.$$

The reverse inequality always holds, so that

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

Hence $\{a_n\}$ converges by Theorem 2.47.

Method 2: Assume that $\{a_n\}$ is a Cauchy sequence. Then by the Bolzano–Weierstrass theorem (Theorem 2.42), $\{a_n\}$ has a convergent subsequence, say $\{a_{n_k}\}$. Let $a = \lim_{k\to\infty} a_{n_k}$. Then there exists an N_1 such that

$$|a_{n_k} - a| < \frac{\epsilon}{2} \quad \text{whenever } k > N_1.$$

We need to show that $a = \lim_{n\to\infty} a_n$. Choose k large enough that $n_k > N$ and $k > N_1$. Then because $\{a_n\}$ is Cauchy, (2.4) is also satisfied with $m = n_k$. Thus, $\{a_n\}$ converges, because

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{whenever } n > N. \qquad \blacksquare$$

Definition 2.56 (Contractive sequence). *A sequence $\{a_n\}_{n\geq 1}$ is said to be* contractive *if there exists a constant $\lambda \in (0,1)$ such that $|a_{n+1} - a_n| \leq \lambda|a_n - a_{n-1}|$ for all $n \geq 2$.*

Theorem 2.57. *Every contractive sequence is Cauchy (and hence convergent by Theorem 2.55). What happens if one allows $\lambda = 1$?*

Proof. Assume that $\{a_n\}_{n\geq 1}$ is a contractive sequence. We find that $a_1 \neq a_2$; otherwise, $\{a_n\}$ reduces to a zero sequence, which converges trivially. We see that

$$|a_{n+1} - a_n| \leq \lambda^{n-1}|a_2 - a_1|,$$

and so for $m > n \geq N$, we have

$$\begin{aligned} |a_m - a_n| &= |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+1} - a_n)| \\ &= [\lambda^{m-2} + \lambda^{m-3} + \cdots + \lambda^{n-1}]|a_2 - a_1| \\ &= \frac{\lambda^{n-1}(1 - \lambda^{m-n})}{1 - \lambda}|a_2 - a_1| \\ &< \frac{\lambda^{n-1}}{1-\lambda}|a_2 - a_1| \leq \frac{\lambda^{N-1}}{1-\lambda}|a_2 - a_1|. \end{aligned}$$

Since $\lambda \in (0, 1)$, given $\epsilon > 0$, we can choose $N = N(\epsilon)$ such that

$$\frac{\lambda^{N-1}}{1-\lambda}|a_2 - a_1| < \epsilon,$$

showing that $|a_m - a_n| < \epsilon$ for all $m > n \geq N$. Thus $\{a_n\}$ is a Cauchy sequence and hence converges.

Note that if $a_n = \sqrt{n}$, then

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n-1}} = a_n - a_{n-1},$$

but $\{\sqrt{n}\}$ is not a Cauchy sequence. ■

Example 2.58. Define a_n inductively by

$$a_{n+1} = \tfrac{1}{2}(a_n + a_{n-1}) \quad \text{for } n \geq 2,$$

where a_1 and a_2 are fixed real numbers. Does the sequence $\{a_n\}$ converge? If it converges, what is its limit?

Solution. For definiteness, we may assume that $a_1 < a_2$. For $n \geq 2$, we have

$$a_{n+1} - a_n = -\frac{1}{2}(a_n - a_{n-1}) = \cdots = \left(-\frac{1}{2}\right)^{n-1}(a_2 - a_1). \tag{2.5}$$

Method 1: If n is even, then the factor on the right, namely $(-1/2)^{n-1}(a_2 - a_1)$, is negative, and so $a_{n+1} - a_n < 0$, and if n is odd, this factor is positive, and so the reverse inequality holds. Thus $\{a_{2n}\}$ is decreasing, whereas $\{a_{2n+1}\}$ is increasing. Observe that $\{a_n\}$ is not a monotone sequence but is bounded. By BMCT, both $\{a_{2n+1}\}$ and $\{a_{2n}\}$ converge. In order to show that $\{a_n\}$

converges, it suffices to prove that these odd and even sequences converge to the same limit. We now begin by observing that (2.5) gives

$$a_{2n+1} = a_{2n} + \left(-\frac{1}{2}\right)^{2n-1}(a_2 - a_1),$$

showing that $\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} a_{2n}$. Therefore, $\{a_n\}$ converges to a limit l, say. To obtain the limit, it suffices to note from the definition that

$$a_{n+1} + \frac{a_n}{2} = a_n + \frac{a_{n-1}}{2} = \cdots = a_2 + \frac{a_1}{2}.$$

Now allow $n \to \infty$ and get that

$$l + \frac{l}{2} = a_2 + \frac{a_1}{2}, \quad \text{i.e.,} \quad l = \frac{2a_2 + a_1}{3}.$$

Method 2: One could directly prove the convergence of $\{a_n\}$ by showing that it is Cauchy. Indeed, using (2.5), it follows that for $m > n \geq 2$,

$$\begin{aligned} |a_m - a_n| &\leq |a_m - a_{m-1}| + \cdots + |a_{n+1} - a_n| \\ &= (a_2 - a_1)\left[\frac{1}{2^{m-2}} + \frac{1}{2^{m-1}} + \cdots + \frac{1}{2^{n-1}}\right] \\ &= \frac{a_2 - a_1}{2^{n-1}}\left[1 + \frac{1}{2} + \frac{1}{2^{m-n-1}}\right] \\ &= \frac{a_2 - a_1}{2^{n-1}}\left[\frac{1-(1/2)^{m-n}}{1-(1/2)}\right] < \frac{a_2 - a_1}{2^{n-2}}. \end{aligned}$$

Now let $\epsilon > 0$ be given. Choose N large enough that

$$\frac{a_2 - a_1}{2^{N-2}} < \epsilon.$$

Thus for all $m > n \geq N$, we have

$$|a_m - a_n| < \epsilon,$$

showing that $\{a_n\}$ is a Cauchy sequence and therefore converges. To get the limit value, by (2.5), we may write a_{n+1} as

$$\begin{aligned} a_{n+1} &= a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n+1} - a_n) \\ &= a_1 + (a_2 - a_1)\left[1 - \frac{1}{2} + \cdots + \left(-\frac{1}{2}\right)^{n-1}\right] \\ &\to a_1 + (a_2 - a_1)\left(\frac{1}{1 + 1/2}\right) = \frac{2a_2 + a_1}{3} \quad \text{as } n \to \infty, \end{aligned}$$

so that $\{a_n\}$ converges to $(2a_2 + a_1)/3$, as desired. ●

Lemma 2.59. *Let* $\{a_n\}$ *be a sequence of positive numbers. Then we have*

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} a_n^{1/n} \leq \alpha := \limsup_{n\to\infty} a_n^{1/n} \leq L := \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Proof. We need to prove that $\alpha \leq L$. This is obvious if $L = \infty$, and so we assume that $0 \leq L < \infty$. To prove $\alpha \leq L$, it suffices to show that

$$\alpha \leq \lambda \quad \text{for any } \lambda \text{ with } L < \lambda. \tag{2.6}$$

So we let $L < \lambda$. Then since

$$L = \limsup \frac{a_{n+1}}{a_n} = \lim_{k\to\infty} \left[\sup \left\{ \frac{a_{n+1}}{a_n} : n \geq k \right\} \right] < \lambda,$$

there exists a natural number N such that

$$\sup \left\{ \frac{a_{n+1}}{a_n} : n \geq N \right\} < \lambda,$$

which gives

$$\frac{a_{n+1}}{a_n} < \lambda \quad \text{for all } n \geq N,$$

so that for $n \geq N$,

$$a_n = a_N \left(\frac{a_{N+1}}{a_N} \right) \left(\frac{a_{N+2}}{a_{N+1}} \right) \cdots \left(\frac{a_n}{a_{n-1}} \right) < \lambda^{n-N} a_N.$$

Therefore,

$$a_n^{1/n} < \lambda^{1-N/n} a_N^{1/n} \quad \text{for } n \geq N,$$

where λ and a_N are fixed. Since $\lim_{n\to\infty} a^{1/n} = 1$ for $a > 0$ (see Example 2.18**(a)**), it follows that

$$\alpha = \limsup a_n^{1/n} \leq \lambda.$$

Consequently, (2.6) holds. The proof for the first inequality in the statement is similar, whereas the middle inequality in Lemma 2.59 is trivial. ■

Corollary 2.60. *Let* $\{a_n\}$ *be a sequence of positive numbers. If* $L = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$, *then* $\lim\limits_{n\to\infty} a_n^{1/n} = L$.

Example 2.61. Consider a_n defined by

$$a_n = \frac{n^n}{(n+1)(n+2)\cdots(n+n)}.$$

Suppose we wish to compute $\lim a_n^{1/n}$ (see also Example 7.16**(a)**). It is easier to apply Corollary 2.60. Now we have (by Example 2.33)

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^n(n+1)^2}{n^n(2n+1)(2n+2)} = \frac{(1+1/n)^n(1+1/n)^2}{(2+1/n)(2+2/n)} \to \frac{\mathrm{e}}{4} \quad \text{as } n \to \infty,$$

and so $\lim a_n^{1/n} = \mathrm{e}/4$. Similarly, it is easy to see that

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{\mathrm{e}}. \qquad \bullet$$

We shall provide a direct proof of Corollary 2.60 later, in Section 8.1. However, it is natural to ask the following: if $a_n > 0$ for all n and $\lim_{n\to\infty} a_n^{1/n}$ exists, does $\lim_{n\to\infty} a_{n+1}/a_n$ exist? Clearly not. For example, set

$$a_n = 3^{-n+(-1)^n}.$$

Then $a_n > 0$ and $a_n^{1/n} = 3^{c_n/n} = \mathrm{e}^{(c_n/n)\log 3}$, where

$$\frac{c_n}{n} = \frac{-n+(-1)^n}{n} = -1 + \frac{(-1)^n}{n} \to -1 \quad \text{as } n \to \infty,$$

which shows that $a_n^{1/n} \to e^{-\log 3} = 1/3$. On the other hand,

$$\frac{a_{n+1}}{a_n} = \frac{3^{c_{n+1}}}{3^{c_n}} = 3^{c_{n+1}-c_n} = 3^{-1-2(-1)^n} = \begin{cases} 3 & \text{if } n \text{ is odd,} \\ 3^{-3} & \text{if } n \text{ is even.} \end{cases}$$

This shows that

$$\frac{1}{27} = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 < \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 3,$$

and $\lim_{n\to\infty} a_{n+1}/a_n$ does not exist. The above construction helps to generate many more examples. For instance, consider $a_n = 2^{-n+(-1)^n}$.

2.2.2 Summability of Sequences

Our aim here is to attach "in some sense" a limit to divergent sequences, while realizing at the same time that any "new limit" we define must agree with the limit in the ordinary sense when it is applied to a convergent sequence. More precisely, if $\{s_n\}$ possibly diverges, we introduce "another method of summation" by replacing $\lim_{n\to\infty} s_n$ by

$$\lim_{n\to\infty} \sigma_n, \quad \text{where } \sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k.$$

Here the $\{\sigma_n\}$ are called *Cesàro means*[2] (of order 1). Note that $\{\sigma_n\}$ is precisely the average of the first n terms of the sequence $\{s_n\}$, and hence $\{\sigma_n\}$ is also called a *sequence of averages*.

[2] Ernesto Cesàro (1859–1906) was an Italian mathematician who worked on this problem in early stage of his career.

Definition 2.62. *If $\{s_n\}_{n\geq 1}$ is a sequence of real numbers, then we say that $\{s_n\}_{n\geq 1}$ is $(C,1)$ summable to L if the new sequence $\{\sigma_n\}_{n\geq 1}$ converges to L, where*

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k.$$

In this case, we write

$$s_n \to L \quad (C,1) \quad \text{or} \quad s_n \to L \quad (\text{Cesàro}) \quad \text{or} \quad \lim_{n\to\infty} s_n = L \quad (C,1).$$

Next, consider a sequence $\{s_n\}$ of real numbers such that $\sigma_n \to 0$ as $n \to \infty$ but $\{s_n\}$ is not convergent.

Example 2.63. Suppose that $s_n = (-1)^{n-1}$ for $n \geq 1$. Then

$$\sigma_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{1}{n} & \text{if } n \text{ is odd} \end{cases}, \quad n \in \mathbb{N},$$

and so $\sigma_n \to 0$ as $n \to \infty$. Thus, $\{(-1)^{n-1}\}_{n\geq 1}$ is $(C,1)$ summable to 0, and we write

$$\lim_{n\to\infty} (-1)^{n-1} = 0 \quad (C,1).$$ ●

All convergent sequences are $(C,1)$ summable to their limits. More precisely, we have the following result.

Theorem 2.64. *If $s_n \to x$, then $s_n \to x$ $(C,1)$.*

Proof. Suppose that $s_n \to x$ as $n \to \infty$. We need to prove that

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k \to x \quad \text{as } n \to \infty.$$

Clearly, it suffices to prove the theorem for the case $x = 0$. So we assume that $s_n \to 0$. Then given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|s_n| < \epsilon/2$ for all $n > N$. Now for $n > N$,

$$\begin{aligned} |\sigma_n| &= \left|\frac{1}{n}\sum_{k=1}^{n} s_k\right| \leq \frac{1}{n}\left[\sum_{k=1}^{N}|s_k| + \sum_{k=N+1}^{n}|s_k|\right] \\ &= \frac{1}{n}\left(\sum_{k=1}^{N}|s_k|\right) + \frac{1}{n}(n-N)\frac{\epsilon}{2} < \frac{M}{n} + \frac{\epsilon}{2}, \quad M = \sum_{k=1}^{N}|s_k|. \end{aligned}$$

Note that M is independent of n and $1/n \to 0$ as $n \to \infty$. Consequently, given $\epsilon > 0$, there exists an N_1 such that

$$\left|\frac{1}{n}\sum_{k=1}^{n} s_k\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n \geq N_1,$$

and so $\sigma_n \to 0$ whenever $s_n \to 0$. ∎

As a consequence of Theorem 2.64, we easily have

(a) $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} k^{1/k} = 1$;
(b) $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} n/(\sqrt{n^2+k}) = 1$;
(c) $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} 1/(2k-1) = 0$.

Theorem 2.64 can also be obtained as a consequence of the following result.

Theorem 2.65. *Let $\{s_n\}$ be a sequence of real numbers and $\{\sigma_n\}$ its Cesàro means of order 1. Then we have*

$$\liminf_{n\to\infty} s_n \le \liminf_{n\to\infty} \sigma_n \le \alpha := \limsup_{n\to\infty} \sigma_n \le L := \limsup_{n\to\infty} s_n. \tag{2.7}$$

In particular, Theorem 2.64 holds.

Proof. We need to prove that $\alpha \le L$. This is obvious if $L = \infty$, and so we assume that $L < \infty$. In order to prove $\alpha \le L$, it suffices to show that

$$\alpha \le \lambda \quad \text{for any } \lambda \text{ with } L < \lambda.$$

So we let $L < \lambda$. By the definition of L, it follows that there exists an N such that $s_n < \lambda$ for all $n > N$. Now for $n \ge N$,

$$\sigma_n = \frac{1}{n}\left[\sum_{k=1}^{N} s_k + \sum_{k=N+1}^{n} s_k\right] < \frac{M}{n} + \frac{1}{n}(n-N)\lambda, \quad M = \sum_{k=1}^{N} s_k.$$

Fix N, and allow $n \to \infty$, and take limit superior on each side to obtain

$$\alpha \le \lambda \quad \text{for any } \lambda \text{ with } L < \lambda.$$

It follows that $\alpha \le L$. The proof for the first inequality in (2.7) is similar, whereas the middle inequality in (2.7) is trivial.

In particular, if $\lim_{n\to\infty} s_n$ exists, then so does $\lim_{n\to\infty} \sigma_n$, and they are equal, proving the second assertion. ■

Now we ask whether a sequence $\{s_n\}$ that diverges to ∞ can be $(C,1)$ summable.

Example 2.66 (Not all divergent sequences are $(C,1)$ summable). For instance, consider $a_n = 1$ for all $n \ge 1$. Then

$$s_n = \sum_{k=1}^{n} a_k = n \quad \text{and} \quad \sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k = \frac{1}{n}\sum_{k=1}^{n} k = \frac{n+1}{2}.$$

Note that $\{s_n\}$ is a divergent sequence. Since $\{\sigma_n\}$ is not convergent, it follows that $\{s_n\}$ is not $(C,1)$ summable. ●

We have seen examples of divergent series that are not $(C,1)$ summable, but repeating the process of following arithmetic means may lead to a convergent sequence. This idea leads to $(C,2)$ summable sequences, and further extension leads to (C,k) summable sequences. We shall discuss this briefly in Chapter 9.

2.2.3 Questions and Exercises

Questions 2.67.

1. Is every convergent sequence bounded? Is every bounded sequence convergent?
2. Do sequences always have a convergent subsequence?
3. Must a scalar multiple of a Cauchy sequence be Cauchy? Must a sum of two Cauchy sequences always be Cauchy?
4. If $\{a_{3n-2}\}$, $\{a_{3n-1}\}$, and $\{a_{3n}\}$ converge to the same limit a, must $\{a_n\}$ converge to a?
5. Can an unbounded sequence have a convergent subsequence? Can it have many convergent subsequences?
6. Let $\{a_n\}$ be a Cauchy sequence that has a subsequence $\{a_{n_k}\}$ converging to a. Must we have $a_n \to a$?
7. Suppose that we are given a sequence of rational numbers that converges to an irrational number r. Is it possible to obtain many such sequences each converging to the same limit r?
8. Suppose that $\beta > 0$ is given. Is it possible to construct a sequence of rational numbers converging to $\sqrt{\beta}$?
9. Does there exist an example of a bounded sequence having four subsequences converging to different limits?
10. Let $a_n = (-1)^n$. For each fixed N, do we have $|a_n - a_N| = 0$ for infinitely many values of n? Does $\{a_n\}$ satisfy the Cauchy criterion for convergence?
11. Let $a_n = \sqrt{n}$ and $p \in \mathbb{N}$ be fixed. Then
$$a_{n+p} - a_n = \sqrt{n+p} - \sqrt{n} = \frac{p}{\sqrt{n+p} + \sqrt{n}} \to 0 \quad \text{as } n \to \infty.$$
Does $\{a_n\}$ satisfy the Cauchy criterion for convergence?
12. Is every bounded monotone sequence Cauchy? Is every Cauchy sequence monotone?
13. Is the sequence $\{a_n\}$, $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$, Cauchy?
14. If $a_{n+1} - a_n \to 0$ as $n \to \infty$, must $\{a_n\}$ be convergent?
15. Does $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n}(1/k)$ exist? If so, what is this limit? If not, must it be ∞?
16. Does $\lim_{n\to\infty}(1/\sqrt{n})\sum_{k=1}^{n}(1/\sqrt{k})$ exist? If so, what is this limit?
17. Must a constant sequence be $(C, 1)$ summable?

Exercises 2.68.

1. Suppose that p is an integer. Show that if $|r| < 1$, then the sequence $\{n^p r^n\}_{n\geq 1}$ converges to zero. In particular, $r^n \to 0$ as $n \to \infty$ if $|r| < 1$.
2. Construct three divergent sequences each having a convergent subsequence.
3. If the subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ converge to a, prove that $\{a_n\}$ also converges to a.

4. Suppose that $\{a_n\}$ is a sequence of real numbers and $\lim_{n\to\infty} a_n = a$, $a \neq 0$. For any sequence $\{b_n\}$, show that
   (a) $\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
   (b) $\liminf_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$.
   (c) $\limsup_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \limsup_{n\to\infty} b_n$.
   (d) $\liminf_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \liminf_{n\to\infty} b_n$.
5. If $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are both Cauchy, then show that $\{a_n\}$ need not be Cauchy. How about if $\{a_{2n}\}$ and $\{a_{2n+1}\}$ both converge to the same limit?
6. Show that the following sequences are Cauchy:
   (a) $a_n = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}$. **(b)** $a_n = \sum_{k=0}^{n} \frac{1}{k!}$. **(c)** $a_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1}$.
7. Define $a_n = \sin(n\pi/2)$. Extract subsequences of $\{a_n\}$ each having the stated property below:
   (a) converging to 1. **(b)** converging to -1.
   (c) converging to 0. **(d)** divergent.
8. Suppose that $\{a_n\}$ is a sequence such that
$$|a_{n+2} - a_{n+1}| \leq \frac{3}{n}|a_{n+1} - a_n| \quad \text{for } n \geq 1.$$
   Show that $\{a_n\}$ is Cauchy.
9. If $|a_n| < 1/2$ and $|a_{n+1} - a_{n+2}| \leq (1/8)|a_{n+1}^2 - a_n^2|$ for all $n \in \mathbb{N}$, prove that the sequence $\{a_n\}$ converges.
10. Let $a_1 = 1$ and $a_{n+1} = 1 + 1/(1 + a_n)$ for all $n \geq 1$. Is $\{a_n\}$ a Cauchy sequence? If so, find its limit.
11. Define $a_1 = 1$ and $a_{n+1} = 1/(3 + a_n)$ for $n \geq 1$. Show that $\{a_n\}$ converges. Also, find the limit of the sequence.
12. If $\{x_n\}$ is a sequence of real numbers such that $x_{n+1} - x_n \to x$, show that $x_n/n \to x$.
13. Show that
   (a) $\lim_{n\to\infty} \frac{1}{n} \prod_{k=1}^{n} (2n + k)^{1/n} = \frac{27}{4\mathrm{e}}$. **(b)** $\lim_{n\to\infty} \frac{1}{n} \prod_{k=1}^{n} (a + k)^{1/n} = \frac{1}{\mathrm{e}}$.
14. Show that if $\{s_n\}$ and $\{t_n\}$ are $(C, 1)$ summable to S and T, respectively, then $\{s_n \pm t_n\}$ is $(C, 1)$ summable to $S \pm T$.

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