Proposed problems for sequence limits.

AMM 11889

Let $a_n = \dfrac{(n+2)^{n+1}}{(n+1)^n}$ and

let $b_n(x) = n^{\sin^2 x}\left(a_{n+1}^{\cos^2 x} - a_n^{\cos^2 x}\right)$

Find $\lim_{n\to\infty} b_n(x)$ (Here n denotes a positive number and x a real number).

B-4, 2012 -

Suppose that $a_0 = 1$ and that

$a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$

Does $a_n - \log n$ have a finite limit as $n \to \infty$?

(Here $\log n = \ln n$).

B-2 - 2008

Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$

let $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. Evaluate

$$\lim_{n \to \infty} \frac{n!\, F_n(1)}{\ln n}$$

B-6 - 2006

Let k be an integer greater than 1.

Suppose $a_0 > 0$ and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}} \quad \text{for } n > 0.$$

Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}$$

MM - 1983

Let $Y_1 = 0$, $Y_n = \frac{1}{3 - Y_{n-1}}$ for all $n \geq 2$

Assume that $x_1 \notin \{Y_n, n \in \mathbb{N}\}$. and let

$$x_n = 3 - \frac{1}{x_{n-1}}, \quad \text{for } n \geq 2$$

Prove that x_n converges.

Example. Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Solution. The sequence $x_n = \sqrt[n]{n} - 1$ is clearly positive, so we only need to bound it from above by a sequence converging to 0. For that we employ the binomial expansion

$$n = (1 + x_n)^n = 1 + \binom{n}{1}x_n + \binom{n}{2}x_n^2 + \cdots + \binom{n}{n-1}x_n^{n-1} + x_n^n.$$

Forgetting all terms but one, we can write

$$n > \binom{n}{2}x_n^2,$$

which translates to $x_n < \sqrt{\frac{2}{n-1}}$, for $n \geq 2$. The sequence $\sqrt{\frac{2}{n-1}}$, $n \geq 2$, converges to 0, and hence by the squeezing principle, $(x_n)_n$ itself converges to 0, as desired. □

The third example was published by the Romanian mathematician T. Lalescu in 1901 in the *Mathematics Gazette, Bucharest*.

Example. Prove that the sequence $a_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$, $n \geq 1$, is convergent and find its limit.

Solution. The solution we present belongs to M. Ţena. It uses Stirling's formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot e^{\frac{\theta_n}{12n}}, \quad \text{with } 0 < \theta_n < 1,$$

which will be proved in Section 3.2.11. Taking the nth root and passing to the limit, we obtain

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e.$$

We also deduce that

$$\lim_{n\to\infty} \frac{n+1}{\sqrt[n]{n!}} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{n}{\sqrt[n]{n!}} = e.$$

Therefore,

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n = \lim_{n\to\infty} \left(\sqrt[n(n+1)]{\frac{((n+1)!)^n}{(n!)^{n+1}}}\right)^n = \lim_{n\to\infty} \left(\sqrt[n(n+1)]{\frac{(n+1)^n}{n!}}\right)^n$$
$$= \lim_{n\to\infty} \left(\sqrt[n+1]{\frac{n+1}{\sqrt[n]{n!}}}\right)^n = \lim_{n\to\infty} \left(\frac{n+1}{\sqrt[n]{n!}}\right)^{\frac{n}{n+1}}$$

$$= \left(\lim_{n\to\infty} \frac{n+1}{\sqrt[n]{n!}} \right)^{\lim_{n\to\infty} \frac{n}{n+1}} = e.$$

Taking the nth root and passing to the limit, we obtain

$$\lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} = 1,$$

and hence

$$\lim_{n\to\infty} \frac{a_n}{\sqrt[n]{n!}} = \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1 = 0.$$

Thus, if we set

$$b_n = \left(1 + \frac{a_n}{\sqrt[n]{n!}}\right)^{\frac{\sqrt[n]{n!}}{a_n}},$$

then $\lim_{n\to\infty} b_n = e$. From the equality

$$\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n = b_n^{a_n \frac{n}{\sqrt[n]{n!}}},$$

we obtain

$$a_n = \ln \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n (\ln b_n)^{-1} \left(\frac{n}{\sqrt[n]{n!}}\right)^{-1}.$$

The right-hand side is a product of three sequences that converge, respectively, to $1 = \ln e$, $1 = \ln e$, and $\frac{1}{e}$. Therefore, the sequence $(a_n)_n$ converges to the limit $\frac{1}{e}$. □

Apply these methods to the problems below.

313. Compute

$$\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right|.$$

314. Let k be a positive integer and μ a positive real number. Prove that

$$\lim_{n\to\infty} \binom{n}{k} \left(\frac{\mu}{n}\right)^k \left(1 - \frac{\mu}{n}\right)^{n-k} = \frac{\mu^k}{e^\mu \cdot k!}.$$

315. Let $(x_n)_n$ be a sequence of positive integers such that $x_{x_n} = n^4$ for all $n \geq 1$. Is it true that $\lim_{n\to\infty} x_n = \infty$?

316. Let $(a_n)_n$ be a sequence of real numbers with the property that for any $n \geq 2$ there exists an integer k, $\frac{n}{2} \leq k < n$, such that $a_n = \frac{a_k}{2}$. Prove that $\lim_{n\to\infty} a_n = 0$.

317. Given two natural numbers k and m let $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_m$ be positive numbers such that

$$\sqrt[n]{a_1} + \sqrt[n]{a_2} + \cdots + \sqrt[n]{a_k} = \sqrt[n]{b_1} + \sqrt[n]{b_2} + \cdots + \sqrt[n]{b_m},$$

for all positive integers n. Prove that $k = m$ and $a_1 a_2 \cdots a_k = b_1 b_2 \cdots b_m$.

318. Prove that

$$\lim_{n\to\infty} n^2 \int_0^{\frac{1}{n}} x^{x+1} dx = \frac{1}{2}.$$

319. Let a be a positive real number and $(x_n)_{n\geq 1}$ a sequence of real numbers such that $x_1 = a$ and

$$x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} kx_k, \quad \text{for all } n \geq 1.$$

Find the limit of the sequence.

320. Let $(x_n)_{n\geq 1}$ be a sequence of real numbers satisfying

$$x_{n+m} \leq x_n + x_m, \quad n, m \geq 1.$$

Show that $\lim_{n\to\infty} \frac{x_n}{n}$ exists and is equal to $\inf_{n\geq 1} \frac{x_n}{n}$.

321. Compute

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n^2}\right)^{\frac{k}{n^2}+1}.$$

322. Let b be an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{11\ldots1}_{n-1}\underbrace{22\ldots2}_{n}5,$$

written in base b. Prove that the following condition holds if and only if $b = 10$:

> There exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.

We exhibit two criteria for proving that a sequence is convergent without actually computing the limit. The first is due to Karl Weierstrass.

Weierstrass' theorem. *A monotonic bounded sequence of real numbers is convergent.*

Below are some instances in which this theorem is used.

323. Prove that the sequence $(a_n)_{n\geq 1}$ defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1), \quad n \geq 1,$$

is convergent.

324. Prove that the sequence

$$a_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}}, \quad n \geq 1,$$

is convergent.

325. Let $(a_n)_n$ be a sequence of real numbers that satisfies the recurrence relation $a_{n+1} = \sqrt{a_n^2 + a_n - 1}$, for $n \geq 1$. Prove that $a_1 \notin (-2, 1)$.

326. Using the Weierstrass theorem, prove that any bounded sequence of real numbers has a convergent subsequence.

Widely used in higher mathematics is the following convergence test.

Cauchy's criterion for convergence. *A sequence $(x_n)_n$ of points in $\mathbb{R}^n$ (or, in general, in a complete metric space) is convergent if and only if for any $\epsilon > 0$ there is a positive integer n_ϵ such that whenever $n, m \geq n_\epsilon$, $\|x_n - x_m\| < \epsilon$.*

A sequence satisfying this property is called Cauchy, and it is the completeness of the space (the fact that it has no gaps) that forces a Cauchy sequence to be convergent. This property is what essentially distinguishes the set of real numbers from the rationals. In fact, the set of real numbers can be defined as the set of Cauchy sequences of rational numbers, with two such sequences identified if the sequence formed from alternating numbers of the two sequences is also Cauchy.

327. Let $(a_n)_{n\geq 1}$ be a decreasing sequence of positive numbers converging to 0. Prove that the series $S = a_1 - a_2 + a_3 - a_4 + \cdots$ is convergent.

328. Let a_0, b_0, c_0 be real numbers. Define the sequences $(a_n)_n$, $(b_n)_n$, $(c_n)_n$ recursively by

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{b_n + c_n}{2}, \quad c_{n+1} = \frac{c_n + a_n}{2}, \quad n \geq 0.$$

Prove that the sequences are convergent and find their limits.

329. Show that if the series $\sum a_n$ converges, where $(a_n)_n$ is a decreasing sequence, then $\lim_{n\to\infty} na_n = 0$.

The following fixed point theorem is a direct application of Cauchy's criterion for convergence.

Theorem. *Let X be a closed subset of $\mathbb{R}^n$ (or in general of a complete metric space) and $f : X \to X$ a function with the property that $\|f(x) - f(y)\| \le c\|x - y\|$ for any $x, y \in X$, where $0 < c < 1$ is a constant. Then f has a unique fixed point in X.*

Such a function is called contractive. Recall that a set is closed if it contains all its limit points.

Proof. Let $x_0 \in X$. Recursively define the sequence $x_n = f(x_{n-1})$, $n \ge 1$. Then

$$\|x_{n+1} - x_n\| \le c\|x_n - x_{n-1}\| \le \cdots \le c^n\|x_1 - x_0\|.$$

Applying the triangle inequality, we obtain

$$\begin{aligned}\|x_{n+p} - x_n\| &\le \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\le (c^{n+p-1} + c^{n+p-2} + \cdots + c^n)\|x_1 - x_0\| \\ &= c^n(1 + c + \cdots + c^{p-1})\|x_1 - x_0\| \le \frac{c^n}{1-c}\|x_1 - x_0\|.\end{aligned}$$

This shows that the sequence $(x_n)_n$ is Cauchy. Its limit x^* satisfies $f(x^*) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x^*$; it is a fixed point of f. A second fixed point y^* would give rise to the contradiction $\|x^* - y^*\| = \|f(x^*) - f(y^*)\| \le c\|x^* - y^*\|$. Therefore, the fixed point is unique. □

Use this theorem to solve the next three problems.

330. Two maps of the same region drawn to different scales are superimposed so that the smaller map lies entirely inside the larger. Prove that there is precisely one point on the small map that lies directly over a point on the large map that represents the same place of the region.

331. Let t and ϵ be real numbers with $|\epsilon| < 1$. Prove that the equation $x - \epsilon \sin x = t$ has a unique real solution.

332. Let c and x_0 be fixed positive numbers. Define the sequence

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{c}{x_{n-1}}\right), \quad \text{for } n \ge 1.$$

Prove that the sequence converges and that its limit is $\sqrt{c}$.