Inequalities

Selected by prof. Andrew Török

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Facts to remember:

- A continuous function $f : M \to \mathbb{R}$ on a compact set M reaches its maximum and minimum values on M.
- A closed and bounded set of $\mathbb{R}^n$ is compact (the converse is also true).

Problems:

1. Find
$$\min_{a,b\in\mathbb{R}} \max\left(a^2+b, a+b^2\right)$$
2. Find all positive integers n for which the equation
$$nx^4 + 4x + 3 = 0$$
has a real root.
3. Let a, b, c be the sides of a right triagle, with c the hypothenuse. Prove that
$$a + b \le c\sqrt{2}$$
4. **Rearrangement Inequality.** Assume $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ are two families of real numbers (repetitions are allowed in both families). Say we arrange the a's in increasing order,
$$a_1 \le a_2 \le \cdots \le a_n$$
and permute the b's. Then the sum $a_1b_1 + a_2b_2 + \cdots + a_nb_n$ is maximized if the b's are also in increasing order, and minimized when the b's are in decreasing order.
[More precisely, should use "non-decreasing" instead of "increasing" and "non-increasing" instead of "decreasing".]
5. **Chebyshev's inequality.** If $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ then
$$n\left(\sum_{k=1}^n a_kb_k\right) \ge \left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right)$$
On the other hand, if $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, then the inequality above is reversed.
[Can be proven using the Rearrangement Inequality.]
6. If $a, b, c \ge 0$ and $(a+1)(b+1)(c+1) = 8$ then $abc \le 1$. Extend to more variables?
7. Prove that any polynomial (in one variable) with real coefficients that takes only non-negative values can be written as the sum of the square of two polynomials.
8. **Arithmetic Mean – Geometric Mean, a.k.a AM-GM** If $x_1, x_2, \dots, x_n \ge 0$ then
$$\frac{x_1 + x_2 + \cdots + x_n}{n} \ge \sqrt[n]{x_1x_2\dots x_n}$$
with equality only if all the variables are equal. [We proved this already using induction, now try to prove it directly.]

9. If $a, b, c \geq 0$ then
$$9a^2b^2c^2 \leq (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)$$

10. **Cauchy-Schwarz** For a_k, b_k complex numbers (but enough to prove it for $|a_k|, |b_k|$, so for real numbers)
$$\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \geq \left| \sum_{k=1}^{n} a_k b_k \right|^2$$
with equality if and only if
$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$$

11. Prove that the finite sequence $a_0, a_1, \ldots, a_n$ of positive numbers is a geometric progression if and only if
$$(a_0a_1 + a_1a_2 + \cdots + a_{n-1}a_n)^2 = (a_0^2 + a_1^2 + \cdots + a_{n-1}^2)(a_1^2 + a_2^2 + \cdots + a_n^2)$$

12. Let a_k be positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$, $n \geq 2$. Then
$$\frac{a_1}{1 + a_2 + a_3 + \cdots + a_n} + \frac{a_2}{1 + a_1 + a_3 + \cdots + a_n} + \cdots + \frac{a_n}{1 + a_1 + \cdots + a_{n-1}} \geq \frac{n}{2n-1}$$