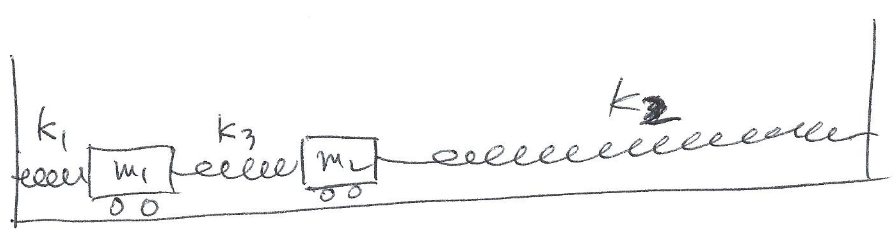
Analysis of spring - mass systems

We introduced the possible summer projects in this area by first considering the following two questions.

1. Having a mass and an unknown spring, is it possible inside an empty room to weight the mass assuming we can only take measurements by using a phone.

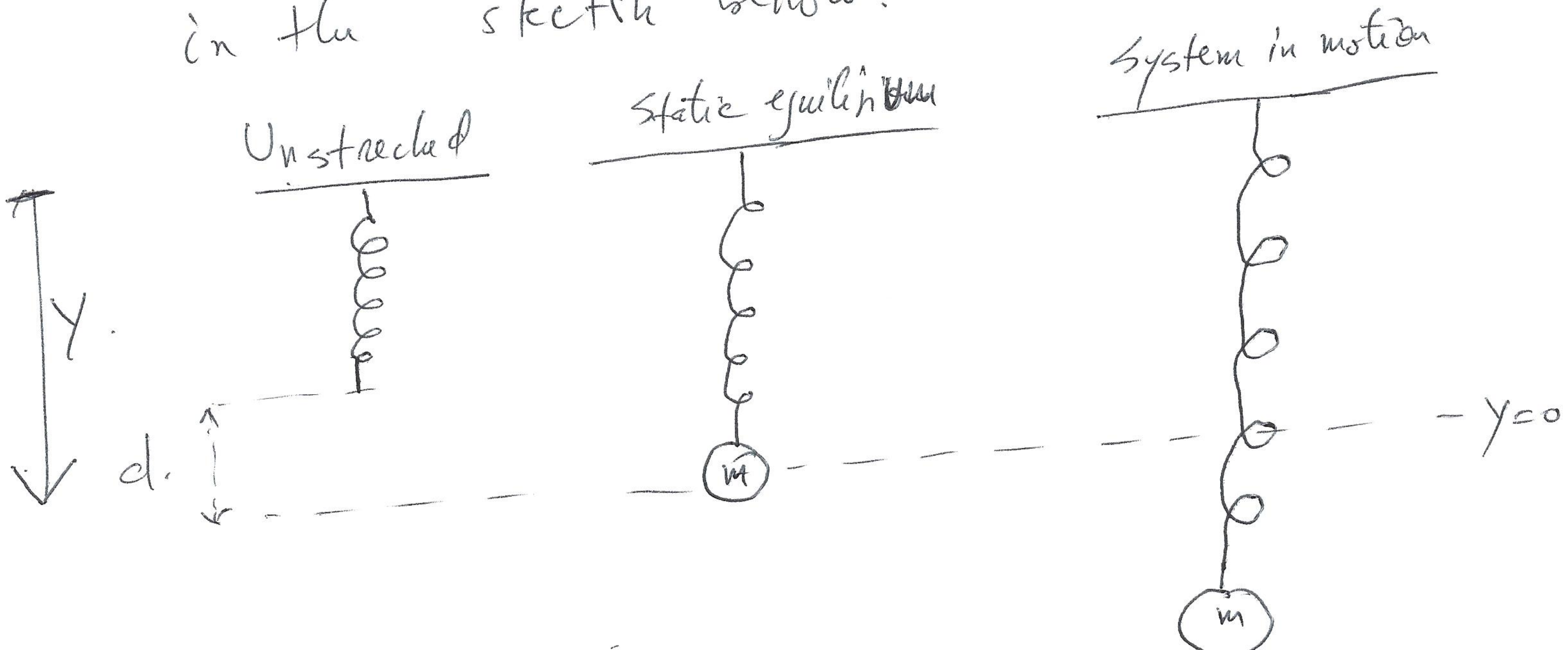
2. Asuming 2 masses and 3 springs as in figure bellow, what is the information we can obtain with one and with two measurements.



Assume m_1, m_2, k_1, k_2, k_3 are unknown and that there is no friction.

1. For the first problem we standed with a list of Questions so that we can Understand a possible mathematical formulation of the problem.

Then. we understood that there are two possible ways we can use the spring, assuming the mass can be connected to it. We choose the case when the spring and mass are hanging from the ceiling. as in the sketch bellow.



At equilibrium static balance of forces.

$$mg = kd.$$

Otherwise the force acting on the mass when the system is in motion (3-rd figure above) is. $F = mg - kd - kY$.

Using the fact that $mg = kd$ we have that the equation of motion is

$$F = -kY + f(t) - cY'$$

when f is any external force and cY' stands for air resistance with c still to be determined!

We observed that only by using the equilibrium equation. we can get

$$\frac{k}{m} = \frac{g}{d}.$$

Also we note that in general if we use only impulsive forces f ~~stable~~ the measurements can only give us $\frac{k}{m}$, $\frac{c}{m}$. but not individually k, m, c!!

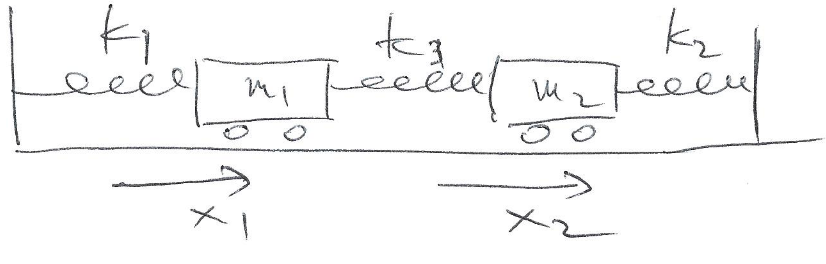
We then proceed towards solving the equation for $f(t)$ and concluded that with special f we can determine k, c and $\underline{m}$!

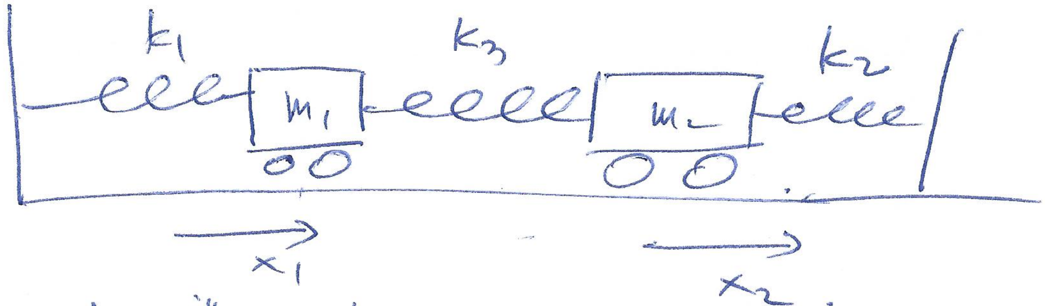
2. For the second model we reviewed the theory of solutions of ODE systems. Then we formulated the question necessary to understand the possible mathematical formulation for the model.

Then we formulated the model

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_3 (x_2 - x_1) \\ m_2 x_2'' = -k_2 x_2 - k_3 (x_2 - x_1) \end{cases}$$

when at the begining no forces or damping was assumed and only initial displacement when considered!





$$m_1 x_1'' = -k_1 x_1 + k_3 (x_2 - x_1)$$

$$m_2 x_2'' = -k_2 x_2 - k_3 (x_2 - x_1)$$

System

$$x_1'' = -\frac{k_1}{m_1} x_1 + \frac{k_3}{m_1}(x_2 - x_1) + \boxed{f_1} + \boxed{\frac{-c_1 x_1'}{m_1}}$$

$$x_2'' = -\frac{k_2}{m_2} x_2 - \frac{k_3}{m_2}(x_2 - x_1) + \boxed{f_2} + \boxed{\frac{-c_2 x_2'}{m_2}}$$

$$X = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} \qquad \begin{cases} x_1' = y_1 \\ y_1' = -\dfrac{(k_1+k_3)}{m_1} x_1 + \dfrac{k_3}{m_1} x_2 + \boxed{f_1} \\ \\ x_2' = y_2 \\ y_2' = \dfrac{-(k_2+k_3)}{m_2} x_2 + \dfrac{k_3}{m_2} x_1 + \boxed{f_2} \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{(k_1+k_3)}{m_1} & 0 & \dfrac{k_3}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_3}{m_2} & 0 & \dfrac{-(k_2+k_3)}{m_2} & 0 \end{bmatrix}$$

- $Y' = AY$, $A = n \times n$ matrix.

1. A has n distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$.

$$Y(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 + \cdots + c_n e^{\lambda_n t} V_n.$$

2. A has a complex eigenvalue with $U + iV$ corresponding eigenvector.

$$e^{\alpha t}[\cos(\beta t) U - \sin\beta t\, V]. \text{ and}$$

$$e^{\alpha t}[\sin(\beta t) U + \cos\beta t\, V]. \text{ are}$$

lin independent sol.

3. λ has multiplicity $k > 1$.

- If there are k lin independent eigenvectors associated with λ we produce k lin ind sol.
- If only $r < k$ lin ind eigenvectors then

$k - r = 1 \longrightarrow \Phi_2(t) = E_1 t e^{\lambda t} + E_2 e^{\lambda t}$

$k - \lambda = 2 \longrightarrow \Phi_2(t) = E_1 t e^{\lambda t} + E_2 e^{\lambda t}$, $\Phi_3(t) = \frac{1}{2} E_1 t^2 e^{\lambda t} + E_2 t e^{\lambda t} + E_3 e^{\lambda t}$.

$\vdots$

The system was then transformed as a first order system

$$X' = AX$$

$$X = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_3)}{m_1} & 0 & \frac{k_3}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_3}{m_2} & 0 & -\frac{(k_2+k_3)}{m_2} & 0 \end{bmatrix}$$

We analyzed a general solution and use matlab symbolic tool box to understand ~~the~~ the behavior for various k_1, k_2, k_3, m_1, m_2.

Possible summer research projects

- Assuming a system with N masses and N+1 springs, ~~deter~~ ~~if~~ if we know that all the masses are equal with m, given, except one which is equal to $M \neq m$ determine the necessary measurements on the system so that we find M and its location.

- Extend the previous analysis to the case of a continuous string with a defect or the case of a very long tube (with small cross-sectional area) with an inclusion inside.