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Source: *The American Mathematical Monthly*, Vol. 124, No. 1 (January 2017), pp. 24-36

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/10.4169/amer.math.monthly.124.1.24>

Accessed: 06-03-2017 22:56 UTC

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# The Falling Slinky

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Robert J. Vanderbei

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**Abstract.** It is an interesting and counterintuitive fact that a Slinky released from a hanging position does not begin to fall all at once but rather each part of the Slinky only starts to fall when the collapsed part above it reaches its level. The analyses published so far have given physical arguments to explain this property of Slinkies. In particular, they have relied on the fact that a perturbation to a Slinky travels through the Slinky as a wave and therefore has a certain propagation speed. Releasing a Slinky that was being held at the top is a perturbation at the top, and it takes time for that perturbation to propagate downward. This “high-level” analysis is, of course, correct. But, it is also interesting to analyze the dynamics from a purely mathematical perspective. We present such a careful mathematical analysis. It turns out that we can derive an explicit formula for the solution to the differential equation, and from that solution, we see that the effect of gravity exactly counteracts the tension in the Slinky. The mathematical analysis turns out to be as interesting as the physics.

**1. INTRODUCTION.** Consider a Slinky that is held at one end while the rest of it hangs freely. Assume that the Slinky is suspended in this manner until all transient oscillatory motions dissipate, that is, until the Slinky reaches a suspended static equilibrium. Suppose that the Slinky is then released and begins to fall. As it falls due to gravity, it also shrinks in extent due to the contracting force in the Slinky itself. Back in 1993, M. G. Calkin [1] published an analysis showing that the net effect of these two dynamics exactly counteract each other and that the full extent of the Slinky remains utterly at rest except for the top, which falls due to gravity and gathers more and more of the Slinky as it falls. The bottom of the Slinky therefore remains motionless until the full Slinky is completely compressed, at which point the whole thing falls downward together. This surprising effect fascinated Martin Gardner [3]. Here’s a video illustrating the phenomenon [7]:

<http://www.princeton.edu/~rvdb/WebGL/Slinky.html>

A few papers have already appeared (see, e.g., [2, 4, 5, 6]) describing this phenomenon and explaining in physical terms why it makes sense for the stretched part of the Slinky to remain motionless. In this paper, we describe a simple mathematical model for the dynamics of a real Slinky, and we give a careful mathematical explanation for this behavior. One’s first thought is that the gravitational effect should be quadratic in time, whereas the compressive effect should be sinusoidal, and therefore, any motionless property should be only approximate. But, it turns out that the dynamics are more complicated than that, and in fact, the two effects do indeed perfectly cancel each other out. It is also an interesting side note that the analysis requires the explicit formula for the roots of a cubic equation.

In Section 2, we start by modeling the Slinky as a chain of  $n + 1$  bodies each having mass  $m$ , and we assume that each adjacent pair of bodies is connected by a spring having spring constant  $k$ . We present a detailed analysis of this (overly simplified) discrete approximation to the problem. Then, in Section 3, we consider the continuum limit of the discrete approximation. Not surprisingly, this limit turns out to be an instance of

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<http://dx.doi.org/10.4169/amer.math.monthly.124.1.24>  
MSC: Primary 70F10



$$= \begin{bmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 1 & \end{bmatrix}.$$

As we analyze this differential equation, it will be convenient to have the vector  $y$  and its derivatives on the left of the equals sign and all other terms on the right:

$$m\ddot{y} + kAy = -mge. \quad (1)$$

**Static initial conditions.** Suppose that initially the masses are hanging in such a way that the downward force due to gravity exactly matches the upward force due to the springs so that the Slinky remains stationary—at least until we let go of body 0. Let's pick our coordinate system so that

$$y_0(0) = 0.$$

Setting the acceleration  $\ddot{y}$  equal to zero for each of the other bodies, we get

$$y_{j-1}(0) - 2y_j(0) + y_{j+1}(0) = \frac{mg}{k}, \quad j = 1, 2, \dots, n-1,$$

$$y_{n-1}(0) - y_n(0) = \frac{mg}{k}.$$

To find an explicit formula for the initial solution, we look for a solution of the form  $y_j(0) = aj + bj^2$ . Plugging this guess into the equations above, it is easy to deduce that

$$b = \frac{mg}{2k} \quad \text{and} \quad a = -\frac{mg}{k}(n + 1/2).$$

Hence,

$$y_j(0) = -\frac{mg}{k} \left( nj - \frac{1}{2}j(j-1) \right), \quad j = 0, 1, 2, \dots, n. \quad (2)$$

**Particular solution.** The general solution to a nonhomogeneous differential equation is the sum of a particular solution to the nonhomogeneous equation plus the general solution to the associated homogeneous equation. In this subsection, we will exhibit a particular solution to the nonhomogeneous equation.

If we let all  $n + 1$  masses start together superimposed on each other at the origin, then there are no spring forces, and all masses fall downward due to gravity. Hence, we expect that

$$y_P = -\frac{gt^2}{2}e$$

is a particular solution to the differential equation. Indeed, it is easy to see that

$$\ddot{y}_P = -ge$$

and that

$$Ae = 0.$$

Hence,

$$m\ddot{y}_P + kAy_P = -mge$$

as required.

**Homogeneous solution.** The associated homogeneous equation is given by

$$m\ddot{y} + kAy = 0.$$

We look for solutions of the form

$$y = e^{Bt}c$$

for some symmetric matrix  $B$  and some vector  $c$ . Differentiating twice, we get

$$\ddot{y} = B^2 e^{Bt}c = B^2 y,$$

and so

$$m\ddot{y} + kAy = mB^2 y + kAy = 0.$$

This equation is satisfied if we pick

$$B = \pm \sqrt{\frac{k}{m}} \sqrt{A}i,$$

where  $\sqrt{A}$  denotes the positive semidefinite square root of the positive semidefinite matrix  $A$  and  $i$  denotes, as usual, the square root of  $-1$ . Hence, the general solution to the homogeneous equation can be written as

$$y_H = e^{\sqrt{k/m}\sqrt{A}it}c_+ + e^{-\sqrt{k/m}\sqrt{A}it}c_-,$$

and the general solution to the original equation is the sum of a particular solution and the homogeneous solution:

$$y = e^{\sqrt{k/m}\sqrt{A}it}c_+ + e^{-\sqrt{k/m}\sqrt{A}it}c_- - \frac{gt^2}{2}e.$$

**Solution with given initial conditions.** To find the solution to our hanging Slinky that's been released at time  $t = 0$ , we just need to use the initial conditions to solve for  $c_+$  and  $c_-$ . Plugging  $t = 0$  into our solution, we get

$$y(0) = c_+ + c_-.$$

Next, using the fact that the velocity vector vanishes at  $t = 0$ , we get that

$$\dot{y}(0) = \sqrt{\frac{k}{m}} \sqrt{A} ic_+ - \sqrt{\frac{k}{m}} \sqrt{A} ic_- = 0,$$

from which we deduce that  $c_+ = c_-$ . Putting these facts together, we get that

$$c_+ = c_- = \frac{y(0)}{2},$$

and so

$$\begin{aligned} y(t) &= \frac{e^{\sqrt{k/m}\sqrt{A}it} + e^{-\sqrt{k/m}\sqrt{A}it}}{2} y(0) - \frac{gt^2}{2} e \\ &= \cos\left(\sqrt{k/m}\sqrt{A}t\right) y(0) - \frac{gt^2}{2} e \\ &= \left(I - \frac{k}{m} \frac{t^2}{2!} A + \frac{k^2}{m^2} \frac{t^4}{4!} A^2 - \dots\right) y(0) - \frac{gt^2}{2} e. \end{aligned}$$

Our main interest is in the position of the bottom-most mass,  $y_n$ , as a function of time:

$$y_n(t) = e_n^T y(t),$$

where  $e_n$  denotes the  $n + 1$  vector that is all zeros except for a one in the last position.

Now, if we refer to the equations that defined our initial conditions, we see that all but the 0th element of  $Ay(0)$  is equal to  $-mg/k$  and the 0th element turns out to be  $mgn/k$ . So,

$$Ay(0) = -\frac{mg}{k} e + \frac{mg}{k} (n + 1) e_0,$$

where  $e_0$  denotes the  $n + 1$  vector that is one in the 0th position and zero elsewhere. From this, it follows that

$$e_n Ay(0) = -\frac{mg}{k}.$$

Hence, the quadratic term in the Taylor series expansion of the cosine exactly cancels the quadratic acceleration due to gravity term.

Finally, let us consider the other terms in the Taylor series expansion. Since the row sums of the  $A$  matrix all vanish, it follows, as mentioned before, that  $Ae = 0$ . Hence, for  $j \geq 1$ ,

$$A^{j+1}y(0) = \frac{mg}{k} (n + 1) A^j e_0.$$

It is easy to see that  $Ae_0$  is all zeros except for the first two elements. Similarly,  $A^2e_0$  is all zeros except for the first three elements. By induction, it is easy to check that  $A^j e_0$  is zero in all but the first  $j + 1$  elements. Hence, the last, i.e.,  $(n + 1)$ st, element remains zero until  $j = n$ . Therefore, the last component of the vector vanishes in the Taylor series until the  $t^{2n}$  term. Hence, the motion of the bottom mass in the chain appears to remain fixed until time is sufficiently great that the  $t^{2n}$  term becomes significant.

**3. CONTINUUM SOLUTION.** Now let's study the limit as  $n$  tends to infinity. Before taking limits, we need to scale things appropriately. First, rather than having each body have mass  $m$ , we will assume that each body has mass  $m/n$  so that the total

mass is roughly  $m$ . Also, we need to scale the spring constant appropriately. It turns out that the correct choice there is to increase the spring constant in direct proportion to the number of masses:  $kn$  instead of  $k$ . With these rescalings, the differential equation (1) becomes

$$\ddot{y} + \frac{k}{m} \frac{A}{1/n^2} y = -ge. \quad (3)$$

We also change our indexing from a simple index  $j$  that counts the bodies to a real number  $x$  that represents the fraction of the distance from the top of the Slinky to the bottom. So,  $x = 0$  represents the top,  $x = 1$  represents the bottom, and, in general, the index  $j$  is replaced by  $x = j/n$ . With this notation, all rows of the matrix  $A/(1/n^2)$  except the first and the last converge to the negative of the second derivative of height  $y$  with respect to the variable  $x$ :

$$\frac{A}{1/n^2} \longrightarrow -\frac{\partial^2}{\partial x^2}.$$

The first row (corresponding to  $x = 0$ ) and the last row (corresponding to  $x = 1$ ) require a different scaling. Rather than  $1/n^2$  in the denominator, we need  $1/n$ . So, for these two cases, we must divide (3) by  $n$  before taking limits. Doing this and taking limits, we see that the right-hand side and the first term on the left-hand side both vanish, and so we are left with

$$\frac{\partial y}{\partial x}(x, t) = 0, \quad \text{for } x \in \{0, 1\} \text{ and } t \geq 0.$$

In the continuum limit, the initial conditions given by (2) become

$$y(x, 0) = -\frac{mg}{k} \left( x - \frac{1}{2}x^2 \right).$$

To summarize, the differential equation that defines the continuum problem is a particular solution to the *wave equation*

$$\frac{\partial^2 y}{\partial t^2}(x, t) - \frac{k}{m} \frac{\partial^2 y}{\partial x^2}(x, t) = -g, \quad 0 < x < 1, \quad t > 0,$$

$$y(x, 0) = -\frac{mg}{k} \left( x - \frac{1}{2}x^2 \right), \quad 0 \leq x \leq 1,$$

$$\frac{\partial y}{\partial t}(x, 0) = 0, \quad 0 \leq x \leq 1,$$

$$\frac{\partial y}{\partial x}(x, t) = 0, \quad x \in \{0, 1\}, \quad t \geq 0.$$

To solve this wave equation, we must find a particular solution to the differential equation and then the most general solution to the associated homogeneous equation.

A particular solution is easy to produce:

$$y_p(x, t) = -\frac{1}{2}gt^2.$$

Checking that this is a solution is trivial. The general solution to the homogeneous equation is also easy to write down:

$$y_H(x, t) = f_{\pm} \left( x \pm \sqrt{k/m} t \right),$$

where  $f_{\pm}$  are arbitrary functions. Again, it is trivial to check that such a function satisfies the homogeneous wave equation

$$\frac{\partial^2 y_H}{\partial t^2}(x, t) - \frac{k}{m} \frac{\partial^2 y_H}{\partial x^2}(x, t) = 0.$$

So, the general solution to the nonhomogeneous equation is

$$y(x, t) = f_+ \left( x + \sqrt{k/m} t \right) + f_- \left( x - \sqrt{k/m} t \right) - \frac{1}{2} g t^2.$$

All that remains is to use the boundary conditions to discover the exact form of  $f_+$  and  $f_-$ . From the boundary conditions at  $t = 0$ , we see that

$$y(x, 0) = f_+(x) + f_-(x) = -\frac{mg}{k} x \left( 1 - \frac{x}{2} \right), \quad 0 \leq x \leq 1, \quad (4)$$

and

$$\frac{\partial y}{\partial t}(x, 0) = \sqrt{\frac{k}{m}} f'_+(x) - \sqrt{\frac{k}{m}} f'_-(x) = 0, \quad 0 \leq x \leq 1. \quad (5)$$

From (4), we see that  $f_+(0) + f_-(0) = 0$ . Without loss of generality, we may assume that  $f_+(0) = f_-(0) = 0$ . From (5), we deduce that  $f'_+(x) = f'_-(x)$  for  $0 \leq x \leq 1$ . Hence, it follows that

$$f_+(x) = f_-(x) = -\frac{mg}{2k} x \left( 1 - \frac{x}{2} \right), \quad 0 \leq x \leq 1.$$

Finally, from the boundary conditions at  $x = 0$  and  $x = 1$ , we get, after a trivial change of variables:

$$f'(t) = -f'(-t) \quad \text{and} \quad f'(1+t) = -f'(1-t), \quad \text{for } t > 0.$$

(Note: we have dropped the subscript  $\pm$  since the two functions are the same.) These last conditions tell us that we can extend the formula for  $f$  beyond the interval  $[0, 1]$  by successive reflection operations. It is now easy to check that  $f$  is a periodic function with period 2 defined by

$$f(x) = -\frac{mg}{4k} x(2-x), \quad 0 \leq x \leq 2.$$

In other words, the general formula for  $f$  is

$$f(x) = -\frac{mg}{4k} (x \bmod 2)(2 - x \bmod 2).$$

**The temporal evolution of the Slinky.** Finally, we can analyze the motion of the Slinky. Consider a fixed location  $x$  on the Slinky. From our analysis above, we have

$$y(x, t) = f(x + \sqrt{k/m} t) + f(x - \sqrt{k/m} t) - \frac{1}{2}gt^2.$$

It is easy to check that for  $0 \leq t \leq \sqrt{m/k} x$  and  $0 \leq x \leq 1$ , both arguments to the function  $f$  in the formula for  $y$  are in the main interval  $[0, 2]$ , and therefore, the formula is quite simple in this case:

$$\begin{aligned} y(x, t) &= -\frac{mg}{4k} \left( x + \sqrt{\frac{k}{m}} t \right) \left( 2 - x - \sqrt{\frac{k}{m}} t \right) \\ &\quad - \frac{mg}{4k} \left( x - \sqrt{\frac{k}{m}} t \right) \left( 2 - x + \sqrt{\frac{k}{m}} t \right) - \frac{1}{2}gt^2 \\ &= -\frac{mg}{2k} x(2 - x) \\ &= y(x, 0). \end{aligned}$$

To summarize: Not only does the bottom of the Slinky remain motionless for a certain period of time, but every point on the Slinky (except the very top) remains motionless for a period of time. For parts close to the top (small values of  $x$ ), the time of stationarity is brief, and as we move lower, this time period gets longer. The bottom of the Slinky remains motionless for  $0 \leq t \leq \sqrt{m/k}$ .

A plot of  $y(x, t)$  versus  $t$ , for some choices of  $x$  between zero and one, is shown in Figure 2.

**4. THE REALISTIC CASE: INELASTIC COLLISIONS.** It is clear from Figure 2 that the bodies can freely pass through each other as the chain of bodies falls. A real Slinky is not “porous” like this. Instead, we should assume that the balls can’t pass through each other. It’s easy to code up a simulation of such a motion. A Matlab program that computes the path of the balls is shown in Figure 3. The output of this Matlab code is shown in Figure 4.

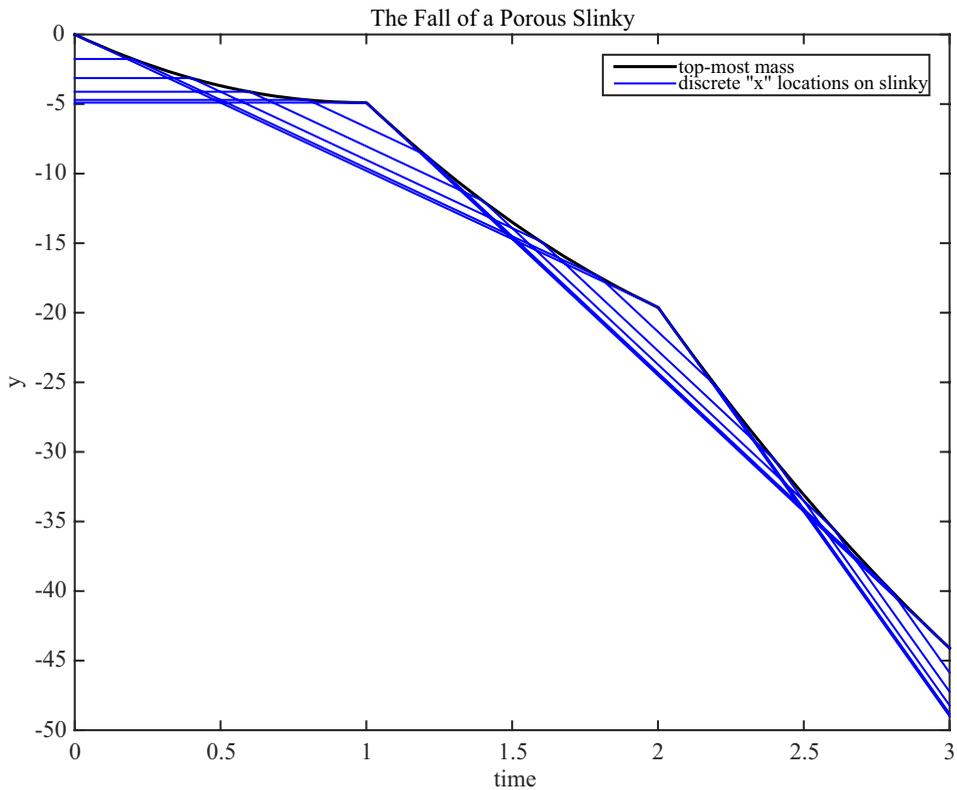
From these numerical computations, it is clear that each body in the nonporous chain remains stationary until the collapsed bodies from above collide with it. Based on this observation, we can derive a formula for the position of the top of the continuum Slinky as a function of time. Let  $\mu(t)$  denote the center of mass of the system at time  $t$ . For time  $t = 0$ , we compute the center of mass to be

$$\mu(0) = \int_0^1 y(x, 0) dx = -\frac{mg}{k} \int_0^1 \left( x - \frac{1}{2}x^2 \right) dx = -\frac{1}{3} \frac{mg}{k}.$$

Once released, the center of mass of the Slinky accelerates downward at rate  $g$ , so at time  $t$ , we have

$$\mu(t) = \mu(0) - \frac{1}{2}gt^2 = -\frac{1}{3} \frac{mg}{k} - \frac{1}{2}gt^2. \quad (6)$$

We now assume that a certain proportion of the top of the Slinky has collapsed to the bottom position of that proportion. Let  $x_t$  denote this proportion at time  $t$ . So, we have



**Figure 2.** A plot of  $y(x, t)$  versus  $t$  for  $x = 0, 1/5, 2/5, \dots, 1$ . The mass of the Slinky is taken to be  $m = 1$  kg, distance is measured in meters, time in seconds, and the acceleration due to gravity is  $g = 9.8\text{m/s}^2$ . Also shown is the temporal evolution of the position of the uppermost mass, which changes as masses pass each other.

$$y(x, t) = \begin{cases} y(x_t, 0), & x \leq x_t, \\ y(x, 0), & x \geq x_t \end{cases}$$

and we can compute the center of mass of this partially collapsed Slinky:

$$\begin{aligned} \mu(t) &= \int_0^1 y(x, t) dx \\ &= \int_0^{x_t} y(x_t, 0) dx + \int_{x_t}^1 y(x, 0) dx \\ &= x_t y(x_t, 0) - \frac{mg}{k} \int_{x_t}^1 \left( x - \frac{x_t^2}{2} \right) dx \\ &= -x_t \frac{mg}{k} \left( x_t - \frac{x_t^2}{2} \right) - \frac{mg}{k} \left( \frac{1}{3} - \frac{x_t^2}{2} + \frac{x_t^3}{6} \right). \end{aligned} \quad (7)$$

Equating the right-hand sides of (6) and (7) and simplifying, we see that  $x_t$  must satisfy a cubic equation:

$$\frac{x_t^3}{3} - \frac{x_t^2}{2} + \frac{1}{2} \frac{k}{m} t^2 = 0.$$

```

n = 500;
dt = 0.0002;
endtime = 0.8;
iters = round(endtime/dt);

k = 1*n;
m = 1/n;
g = 9.8;

A = full(gallery('tridiag',n+1,-1,2,-1));
A(1,1) = 1;
A(end,end) = 1;
e = ones(n+1,1);

t = 0;

j = (0:n)';
y = -(m*g/k)*(n*j - 0.5*j.*(j-1));
ydot = zeros(n+1,1);
ydot2 = -(k/m)*A*y - g*e;

ylim([1.1*min(y) 1.1*max(y)]);
xlabel('time');
ylabel('y');

j0=1;
for iter = 1:iters
    ydot = ydot + ydot2*dt/2;
    y = y + ydot*dt/2;

    for i = (j0+1):n+1
        if y(j0) >= y(i)
            j0 = i-1;
            break;
        end
    end
    ybar = mean(y(1:j0));
    ydotbar = mean(ydot(1:j0));
    y(1:j0) = ybar;
    ydot(1:j0) = ydotbar;

    y = y + ydot*dt/2;
    ydot2 = -(k/m)*A*y - g*e;
    ydot = ydot + ydot2*dt/2;
    t = t + dt;
    TT(:,iter) = t(1:end);
    YY(:,iter) = y(1:end);
end
plot(TT',YY', 'b-'); hold on;
plot(mean(TT),mean(YY), 'r-'); hold off;

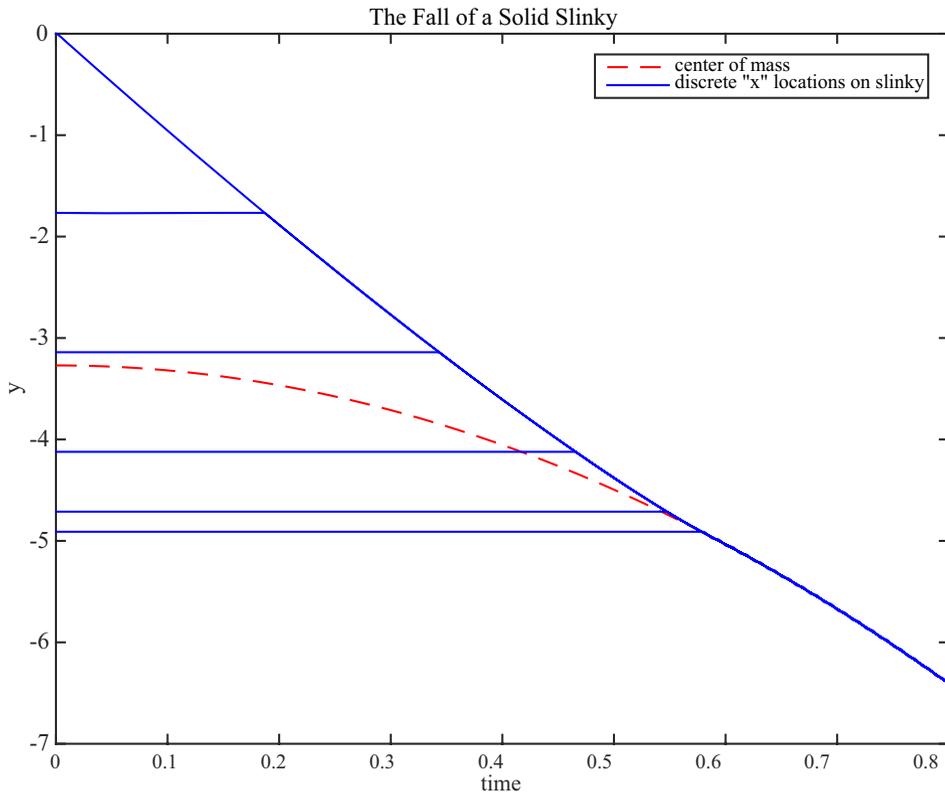
```

**Figure 3.** Matlab code for  $n$  nonporous balls.

The general solution to this cubic equation is

$$x_t = \frac{1}{2} - uC_t - \frac{1}{4uC_t},$$

where  $u$  is any of the three complex roots of unity and



**Figure 4.** A plot of  $y(x, t)$  versus  $t$  for five of 500 nonporous balls. The solid lines show the motion over time of every 100th ball. The dashed line shows evolution of the center of mass of the system. As before, the total mass is taken to be 1 kg, distance is measured in meters, time in seconds, and  $g = 9.8\text{m/s}^2$ .

$$C_t = \left( \frac{-\frac{1}{4} + \frac{3}{2} \frac{k}{m} t^2 + \sqrt{\frac{3}{4} \frac{k}{m} t^2 \left(-1 + 3 \frac{k}{m} t^2\right)}}{2} \right)^{1/3}.$$

We need to pick the correct root of unity. It must be picked so as to ensure that  $x_t$  lies between 0 and 1. Careful analysis of the case where  $t$  is small allows us to determine which of the three roots of unity is the correct one.

We start by considering the case  $t = 0$ . In this case, the cubic equation is easy:

$$\frac{x_0^3}{3} - \frac{x_0^2}{2} = 0 \quad \implies \quad x_0 = 0, 0, 3/2.$$

Let us try to determine which cube root of unity is associated with 3/2. For  $t = 0$ , we have

$$C_0 = -1/2,$$

and so we get

$$x_0 = \frac{1}{2} (1 + u + 1/u).$$

Clearly, the root  $u = 1$  yields the  $3/2$  solution, and the other two roots,  $u_{\pm} = e^{\pm 2\pi i/3}$ , both yield a zero solution. To determine which of these two roots is the correct one, we consider the case where  $\sqrt{k/m} t$  is tiny, say  $\varepsilon$ . In this case, we have

$$C_t \approx \left( -\frac{1}{8} + \frac{\sqrt{3}}{4} \varepsilon i \right)^{1/3} \approx -\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}} i.$$

Plugging into our equation for  $x_t$ , we get

$$\begin{aligned} x_t^{\pm} &\approx \frac{1}{2} - e^{\pm 2\pi i/3} \left( -\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}} i \right) - \frac{1}{4e^{\pm 2\pi i/3} \left( -\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}} i \right)} \\ &= \frac{1}{2} + e^{\pm 2\pi i/3} \left( \frac{1}{2} - \frac{\varepsilon}{\sqrt{3}} i \right) + e^{\mp 2\pi i/3} \frac{1}{2 - \frac{4\varepsilon}{\sqrt{3}} i} \\ &= \frac{1}{2} \left( 1 + e^{\pm 2\pi i/3} \left( 1 - \frac{2\varepsilon}{\sqrt{3}} i \right) + e^{\mp 2\pi i/3} \frac{1}{1 - \frac{2\varepsilon}{\sqrt{3}} i} \right) \\ &\approx \frac{1}{2} \left( 1 + e^{\pm 2\pi i/3} \left( 1 - \frac{2\varepsilon}{\sqrt{3}} i \right) + e^{\mp 2\pi i/3} \left( 1 + \frac{2\varepsilon}{\sqrt{3}} i \right) \right) \\ &= \frac{\varepsilon}{\sqrt{3}} i \left( -e^{\pm 2\pi i/3} + e^{\mp 2\pi i/3} \right) \\ &= \pm \varepsilon. \end{aligned}$$

From this computation, we see that  $u_+$  produces  $x_t$  values that are positive for small  $t$  whereas  $u_-$  produces negative values. Hence,  $u_+$  is the correct root.

Finally, we note that as a by-product of this analysis we can compute how long it takes for the Slinky to fully collapse. The moment of full collapse corresponds to  $x_t = 1$ . Hence, from our cubic equation, we get that

$$-\frac{1}{6} + \frac{1}{2} \frac{k}{m} t^2 = 0.$$

Solving for  $t$  we get

$$t = \sqrt{\frac{m}{3k}}.$$

Figure 4 shows the output produced by the Matlab code shown in Figure 3. The red curve plots the center of mass as a function of time.

Finally, the author has produced a WebGL online [7] integration of the differential equation. This dynamic animation can be seen here:

<http://www.princeton.edu/~rvdb/WebGL/Slinky.html>

**5. FINAL REMARKS.** Calkin's 1993 paper [1] provides a detailed analysis equivalent to both the porous and the nonporous continuum cases discussed here. He refers to these two cases as the "loosely wound spring" and the "tightly wound spring," respectively. The results are consistent with those presented here, but the derivations are

based on physical principles that have a mathematical basis but make the paper hard to follow for someone not familiar with these principles.

In 2001, Graham [4] modeled the Slinky as a pair of masses connected by a spring. His results are consistent with the results obtained below in case where  $n = 1$ .

In 2002, Sawicki [5] discussed the falling Slinky but only computed the stretch profile before releasing the Slinky.

In 2011, Unruh [6] provided an analysis of both the porous and the nonporous cases. For the porous case, he started directly with the wave equation and derived the same results as given here. For the nonporous case, his analysis differs from that presented here.

In 2012, Cross and Wheatland [2] provided a detailed analysis of the nonporous case. Their results are equivalent to those obtained here, but again, the derivation depends on understanding certain physical principles that might not be well-understood by a general mathematical audience.

It is clear that the results in this paper are not new to the physics community. But, the unique behavior of the falling Slinky is sufficiently surprising that a rigorous analysis should also be inspiring to mathematicians.

**ACKNOWLEDGMENTS.** The author would like to thank his colleague, Jeremy Kasdin, for introducing him to the problem and for buying him a Slinky to play with. The author would also like to thank the editor and the referees for their excellent, helpful, and timely comments.

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