Fictitious Domain Methods for the Simulation of Stokes Flow Past a Moving Disk

Roland Glowinski[1], Tsorng-Whay Pan[2], and Jacques Periaux[3]

Abstract. We address the numerical simulation of Stokes flow past moving rigid bodies. The method of choice is based on a combination of fictitious domain, time splitting and projection methods. The main advantage of this methodology is to allow the numerical treatment on a fixed structured mesh and the use of fast solvers. Preliminary numerical results for Stokes flow past a moving disk are presented.

1 INTRODUCTION

In this article which follows [1-5], we consider the numerical simulation of Stokes flow past a moving disk by using finite element methods combined with Lagrange multiplier-based fictitious domain method. The main advantage of fictitious domain methods (sometime called *domain imbedding methods*; cf. [6]) is that they allow the numerical treatment on a fixed structured mesh on a simple shape auxiliary domain containing the actual one, independent of the actual boundary of moving bodies, allowing therefore the use of fast solvers. Thus the time-consuming construction of a boundary-fitted mesh for each different position of moving rigid bodies can be skipped. Also we have developed a L^2 projection method combined with operator splitting methods to simplify the numerical algorithms used in [2-5] for the quasi-Stokes problem. In this approach, we only need to solve one elliptic problem associated to the preconditioner instead of three elliptic problems at each iteration in the conjugate gradient algorithm driven by pressure.

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This paper is organized as followed: In Section 2 we describe the fictitious domain formulation of the problem. In Section 3 we apply an operator splitting method to the fictitious domain formulation and obtain two types of problems, quasi-Stokes/L^2-projection problem and elliptic problem. In Section 4 we describe a conjugate gradient method for the quasi-Stokes/L^2-projection problem. Preliminary results for flow past a moving disk are presented in Section 5.

2 Fictitious domain formulation of the Stokes equations

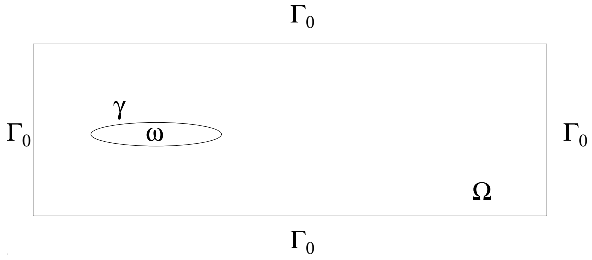

Figure 1.

In [2-5], we have used Lagrange multiplier/fictitious domain methods with finite element methods to simulate external incompressible viscous flow modeled by the Navier-Stokes equations. Here we would like to apply a similar methodology to incompressible viscous flows past moving rigid bodies. The Stokes equations are the following:

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega \setminus \overline{\omega(t)}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \setminus \overline{\omega(t)}, \tag{2}$$

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}), \mathbf{x}\in\Omega\setminus\overline{\omega(0)}, \tag{3}$$

$$\mathbf{u}=\mathbf{g}_0 \text{ on } \Gamma_0, \tag{4}$$

$$\mathbf{u}=\mathbf{g}_2 \text{ on } \gamma(t), \tag{5}$$

where $\nabla\cdot\mathbf{u}_0=0$. In (1)-(5), Ω is a bounded domain in $\mathbb{R}^d(d\geq 2)$, $\omega(t)$ is a *moving rigid body* in $\mathbb{R}^d(d\geq 2)$ (see Fig. 1), Γ_0 (resp., $\gamma(t)$) is the boundary of Ω (resp., $\omega(t)$), $\mathbf{u}=\{u_i\}_{i=1}^{i=d}$ is the flow velocity, p is the presure, $\mathbf{f}$ is a density of external forces.

For the fictitious domain formulation, we imbed $\Omega\setminus\overline{\omega(t)}$ in Ω and define

$$\mathbf{V}_{\mathbf{g}_0}=\{\mathbf{v}|\mathbf{v}\in(\mathbf{H}^1(\Omega))^d, \mathbf{v}=\mathbf{g}_0 \text{ on } \Gamma_0\},$$
$$\mathbf{V}_0=\{\mathbf{v}|\mathbf{v}\in(\mathbf{H}^1(\Omega))^d, \mathbf{v}=\mathbf{0} \text{ on } \Gamma_0\},$$
$$\Lambda(t)=(\mathbf{L}^2(\gamma(t)))^d,$$
$$\mathbf{L}_0^2(\Omega)=\{q|q\in L^2(\Omega); \int_\Omega q\,d\mathbf{x}=0\}.$$

Let $\mathbf{U}_0$ be an extension of $\mathbf{u}_0$ with $\nabla\cdot\mathbf{U}_0=0$ in Ω and $\tilde{\mathbf{f}}$ an extension of $\mathbf{f}$. Then we have equivalence between (1)-(5) and the following problem

For $t>0$, find $\mathbf{U}(t)\in\mathbf{V}_{\mathbf{g}_0}$, $P(t)\in\mathbf{L}_0^2(\Omega)$, $\lambda(t)\in\Lambda(t)$ such that

$$\int_\Omega \frac{\partial\mathbf{U}}{\partial t}\cdot\mathbf{v}\,d\mathbf{x}+\int_\Omega\nabla\mathbf{U}\cdot\nabla\mathbf{v}\,d\mathbf{x} -\int_\Omega P\nabla\cdot\mathbf{v}\,d\mathbf{x}=\int_\Omega\tilde{\mathbf{f}}\cdot\mathbf{v}\,d\mathbf{x} +\int_{\gamma(t)}\lambda\cdot\mathbf{v}\,d\gamma,\ \forall\mathbf{v}\in\mathbf{V}_0,\ a.e.\ t>0, \tag{6}$$

$$\nabla\cdot\mathbf{U}(t)=0 \text{ in } \Omega,\ \mathbf{U}(\mathbf{x},0)=\mathbf{U}_0(\mathbf{x}),\ \mathbf{x}\in\Omega, \tag{7}$$

$$\mathbf{U}(t)=\mathbf{g}_2(t) \text{ in } \gamma(t), \tag{8}$$

in the sense that $\mathbf{U}|_{\Omega\setminus\overline{\omega(t)}}=\mathbf{u}$, $P|_{\Omega\setminus\overline{\omega(t)}}=p$. The multiplier λ is the jump of $\dfrac{\partial\mathbf{U}}{\partial\mathbf{n}}-\mathbf{n}P$ at $\gamma(t)$ and the effect of the actual geometry is concentrated on $\int_{\gamma(t)}\lambda\cdot\mathbf{v}\,d\gamma$ in the right-hand-side of (6), and on (8).

3 Time discretization by Operator Splitting Method

To solve (6)-(8), we consider a time discretization by operator splitting methods, like the ones in, e.g., [7-10]. In the following, we consider a time discretization of (6)-(8) by the Peacaman-Rachford scheme. Let $\triangle t>0$ be a time discretization step and

$$\mathbf{V}_{\mathbf{g}_0^s}=\{\mathbf{v}|\mathbf{v}\in(\mathbf{H}^1(\Omega))^d, \mathbf{v}=\mathbf{g}_0(s\triangle t) \text{ on } \Gamma_0\},$$
$$\mathbf{V}_0=\{\mathbf{v}|\mathbf{v}\in(\mathbf{H}^1(\Omega))^d, \mathbf{v}=\mathbf{0} \text{ on } \Gamma_0\},$$
$$\Lambda^s=(\mathbf{L}^2(\gamma(s\triangle t)))^d,$$
$$\gamma^s=\gamma(s\triangle t).$$

We obtain the following scheme:

$$\mathbf{U}^0=\mathbf{U}_0 \textit{ is given;} \tag{9}$$

for $n\geq 0$, knowing $\mathbf{U}^n$ and λ^n, find $\mathbf{U}^{n+1/2}\in\mathbf{V}_{\mathbf{g}_0^{n+1/2}}$, $P^{n+1/2}\in\mathbf{L}_0^2(\Omega)$ such that

$$\int_\Omega\frac{\mathbf{U}^{n+1/2}-\mathbf{U}^n}{\triangle t/2}\cdot\mathbf{v}\,d\mathbf{x} -\int_\Omega P^{n+1/2}\nabla\cdot\mathbf{v}\,d\mathbf{x} =\int_\Omega\tilde{\mathbf{f}}^{n+1/2}\cdot\mathbf{v}\,d\mathbf{x}+\int_{\gamma^n}\lambda^n\cdot\mathbf{v}\,d\gamma -\int_\Omega\nabla\mathbf{U}^n\cdot\nabla\mathbf{v}\,d\mathbf{x},\ \forall\mathbf{v}\in\mathbf{V}_0, \tag{10}$$

$$\nabla\cdot\mathbf{U}^{n+1/2}=0 \textit{ in } \Omega; \tag{11}$$

next, find $\mathbf{U}^{n+1}\in\mathbf{V}_{\mathbf{g}_0^{n+1}}$ and $\lambda^{n+1}\in\Lambda^{n+1}$ such that

$$\int_\Omega\frac{\mathbf{U}^{n+1}-\mathbf{U}^{n+1/2}}{\triangle t/2}\cdot\mathbf{v}\,d\mathbf{x} +\int_\Omega\nabla\mathbf{U}^{n+1}\cdot\nabla\mathbf{v}\,d\mathbf{x} =\int_\Omega\tilde{\mathbf{f}}^{n+1}\cdot\mathbf{v}\,d\mathbf{x}+\int_{\gamma^{n+1}}\lambda^{n+1}\cdot\mathbf{v}\,d\gamma +\int_\Omega P^{n+1/2}\nabla\cdot\mathbf{v}\,d\mathbf{x},\ \forall\mathbf{v}\in\mathbf{V}_0, \tag{12}$$

$$\mathbf{U}^{n+1}=\mathbf{g}_2^{n+1} \textit{ on } \gamma^{n+1}. \tag{13}$$

In the above scheme, at each time step there are one *quasi-Stokes/L^2- projection problem* (10)-(11) and one elliptic problem (12)-(13) which is in fictitious domain formulation discussed in [1]. Concerning problem (10)-(11), it is worth noticing that we have been taking advantage of the time discretization by operator splitting to treat the incompressiblity in the larger domain Ω without being concerned, in some sense, about the constraint $\mathbf{u} = \mathbf{g}_2^s$ at γ^s. Similar methodology has been used in [2-5] to the nonlinear advection-diffusion problem arising from applying a θ-scheme to the Navier-Stokes equations. The fictitious domain method for the elliptic problem (12)-(13) has been discussed in [1].

4 The Stokes/L^2-projection problem

The quasi-Stokes/L^2-projection problem obtained from previous section is the following:

$$\begin{aligned}
&Find\, \mathbf{U} \in \mathbf{V}_{\mathbf{g}_0} and\, P \in \mathbf{L}_0^2(\Omega) such\, that\\
&\int_\Omega \alpha \mathbf{U}\cdot \mathbf{v}\, d\mathbf{x} - \int_\Omega P\nabla\cdot\mathbf{v}\, d\mathbf{x} \qquad (14)\\
&= \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x},\ \forall \mathbf{v}\in\mathbf{V}_0,\\
&\nabla\cdot\mathbf{U} = 0 \text{ in } \Omega, \qquad (15)
\end{aligned}$$

In problem (14)-(15), *the pressure* P is a *Lagrange multiplier* associated with (15). The above saddle-point problem (14)-(15) can be solved by a preconditioned conjugate gradient algorithm driven by the pressure P as follows (this algorithm makes fully sense after space descretization only):

Step 0: Initialization

$P^0 \in \mathbf{L}_0^2(\Omega)$ *given*; (16)

solve the projection problem:

$$\begin{aligned}
&\alpha\int_\Omega \mathbf{U}^0\cdot\mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} + \int_\Omega P^0\nabla\cdot\mathbf{v}\, d\mathbf{x}\\
&\forall\mathbf{v}\in\mathbf{V}_0;\ \mathbf{U}^0\in\mathbf{V}_{\mathbf{g}_0}, \qquad (17)
\end{aligned}$$

set $r^0 = \nabla\cdot\mathbf{U}^0$, *and define* $g^0 = \phi^0$ *where* ϕ^0 *is the solution of*

$$\left\{\begin{aligned}
&-\Delta\phi^0 = r^0 \ in\, \Omega,\\
&\frac{\partial\phi^0}{\partial\mathbf{n}} = 0\ on\, \Gamma_0;\ \int_\Omega \phi^0\, d\mathbf{x} = 0.
\end{aligned}\right. \qquad (18)$$

We take $w^0 = g^0$.

Then for $n \geq 0$, *assuming that* P^n, $\mathbf{U}^n$, r^n, w^n, g^n *are known, compute* P^{n+1}, $\mathbf{U}^{n+1}$, w^{n+1}, r^{n+1}, g^{n+1} *as follows:*

Step 1: Descent

solve the intermediate projection problem:

$$\begin{aligned}
&\alpha\int_\Omega \bar{\mathbf{U}}^n\cdot\mathbf{v}\, d\mathbf{x} = \int_\Omega w^n\nabla\cdot\mathbf{v}\, d\mathbf{x},\\
&\forall\mathbf{v}\in\mathbf{V}_0;\ \bar{\mathbf{U}}^n\in\mathbf{V}_0, \qquad (19)
\end{aligned}$$

set $\bar{r}^n = \nabla\cdot\bar{\mathbf{U}}^n$, *and define* $\bar{g}^n = \bar{\phi}^n$ *where* $\bar{\phi}^n$ *is the solution of*

$$\left\{\begin{aligned}
&-\Delta\bar{\phi}^n = \bar{r}^n \ in\, \Omega,\\
&\frac{\partial\bar{\phi}^n}{\partial\mathbf{n}} = 0\ on\, \Gamma_0;\ \int_\Omega \bar{\phi}^n\, d\mathbf{x} = 0.
\end{aligned}\right. \qquad (20)$$

We compute then

$$\rho_n = \int_\Omega r^n g^n\, d\mathbf{x} \Big/ \int_\Omega \bar{r}^n w^n\, d\mathbf{x}, \qquad (21)$$

and set

$$\begin{aligned}
P^{n+1} &= P^n - \rho_n w^n, &(22)\\
\mathbf{U}^{n+1} &= \mathbf{U}^n - \rho_n\bar{\mathbf{U}}^n, &(23)\\
g^{n+1} &= g^n - \rho_n\bar{g}^n, &(24)\\
r^{n+1} &= r^n - \rho_n\bar{r}^n. &(25)
\end{aligned}$$

If

$$\int_\Omega r^{n+1}g^{n+1}\, d\mathbf{x} \Big/ \int_\Omega r^0 g^0\, d\mathbf{x} \leq \epsilon, \qquad (26)$$

take $P = P^{n+1}$, $\mathbf{U} = \mathbf{U}^{n+1}$. *If not, compute*

Step 2: The new descent direction

$$\gamma_n = \int_\Omega r^{n+1}g^{n+1}\, d\mathbf{x} \Big/ \int_\Omega r^n g^n\, d\mathbf{x}, \qquad (27)$$

and set

$$w^{n+1} = g^n + \gamma_n w^n. \qquad (28)$$

Do $n = n+1$ *and go back to* (19).

The main advantage of this methodology is to allow the numerical treatment of the pressure on a fixed structured mesh, independent of the actual boundary of the moving bodies. It is very easy to implement the above preconditioned L^2-projection algorithm. For the elliptic problem (12)-(13), we still use the same structured mesh. Thus the time-consuming construction of a boundary-fitted mesh for each different position of moving rigid bodies at different time step can be skipped.

5 Numerical experiments

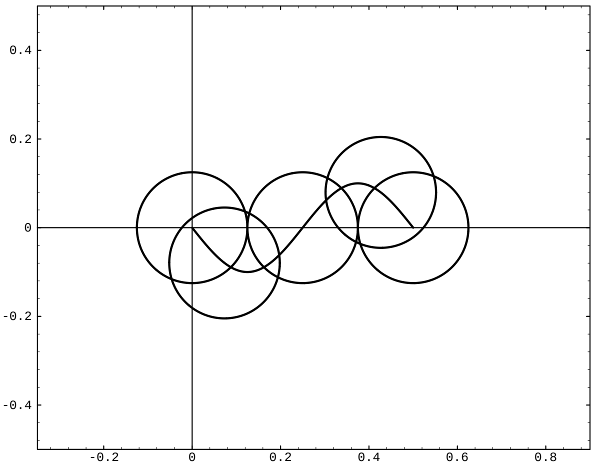

Figure 2.

In the test problem, let $\Omega = (-0.35, 0.9) \times (-0.5, 0.5)$ (see Figure 2) and ω a disk of radius 0.125. The center of disk is moving between $(0, 0)$ and $(0.5, 0)$ with zero angular velocity along a prescribed trajectory $(x(t), y(t))$ (also see Figure 2) given as follows

$$x(t) = 0.25(1 - \cos(\frac{\pi t}{2})),$$
$$y(t) = -0.1 \sin(\pi(1 - \cos(\frac{\pi t}{2}))).$$

Several different positions of disk have been show in Figure 2. The boundary conditions are

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; on \, \Gamma_0.$$

As a finite dimensional subspace of $\mathbf{V}$, we choose $\mathbf{V}_h = \{\mathbf{v}_h | \mathbf{v}_h \in H^1_{0h} \times H^1_{0h}\}$ where

$$H^1_{0h} = \{\phi_h | \phi_h \in C^0(\bar{\Omega}), \phi_h|_T \in P_1, \ \forall T \in \mathcal{T}_h, \\ \phi_h = 0 \text{ on } \Gamma_0\},$$

where $\mathcal{T}_h$ is a triangulation of Ω (see, e.g., Figure 3), P_1 being the space of the polynomials in x_1, x_2 of degree ≤ 1. A traditional way of approximating the pressure is to take it in the space

$$H^1_{2h} = \{\phi_h | \phi_h \in C^0(\bar{\Omega}), \phi_h|_T \in P_1, \ \forall T \in \mathcal{T}_{2h}\},$$

where $\mathcal{T}_{2h}$ is a triangulation twice coarser than $\mathcal{T}_h$.

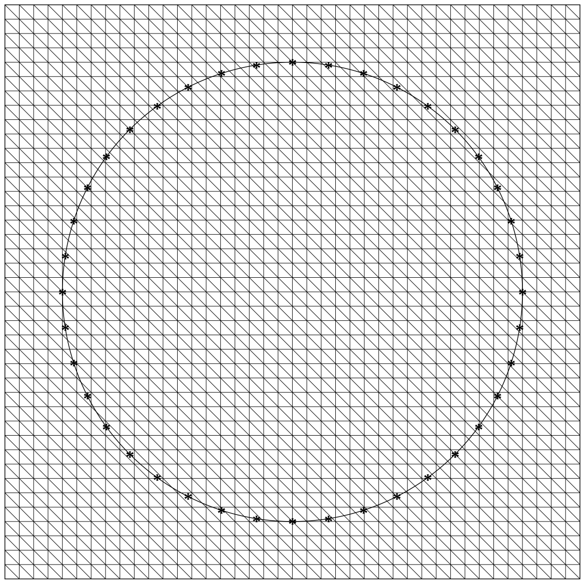

Figure 3. Part of the triangulation of Ω with $h = 1/128$ and mesh points marked by "*" on disk.

Concerning the space Λ^s_h approximating Λ^s, we define it by

$$\Lambda^s_h = \{\mu_h | \mu_h \in (L^\infty(\gamma^s))^2, \mu_h \, is \, constant \\ on \, the \, segment \, joining \, 2 \, consecutive \\ mesh \, points \, on \, \gamma^s\}.$$

A particular choice for mesh points on γ is visualized on Figure 3.

In the numerical simulation, mesh size for velocity field is $h_v = 1/128$, mesh size for pressure is $h_p = 1/64$, and time step is $\triangle t = 0.005$. Also the boundary of the disk is cut into 80 pieces. The initial condition is obtained with the mesh given above with time step $\triangle t = 0.01$ after 400 time steps. For the moving disk case, at each time step we need 12-13 iterations for algorithm (16)-(28) described in Section 4 with $\epsilon = 10^{-14}$ in (26). For elliptic problem (12)-(13) we need 38-63 iterations for the conjugate gradient algorithm discussed in [1] with $\epsilon = 10^{-12}$. Figures 4-7 show a sequence of frames for the vorticity density and stream lines during one period of the disk motion.

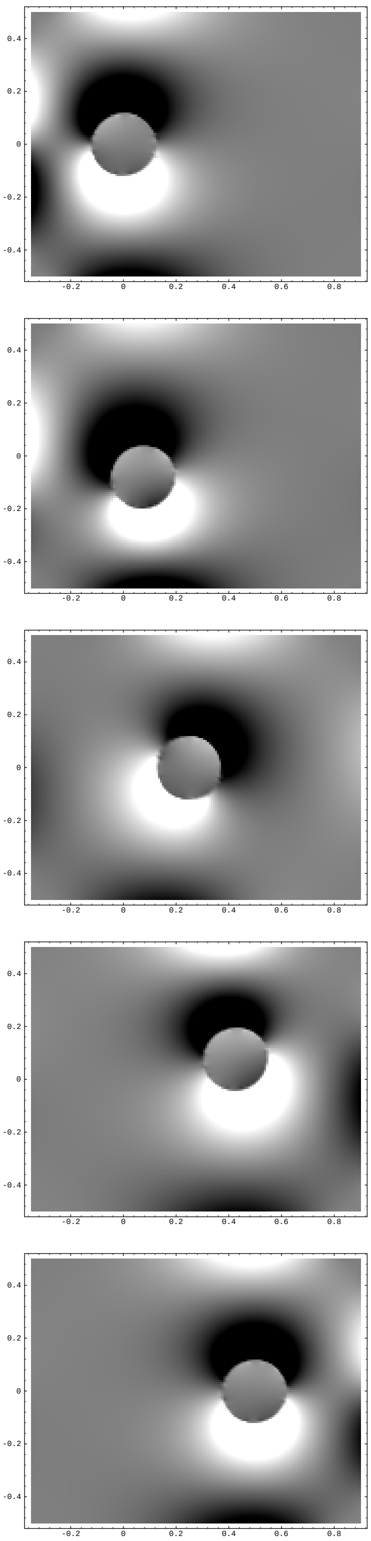

Figure 4. Vorticity density at time $t = 4$, 4.5, 5, 5.5, 6 during first half of one period of the disk motion.

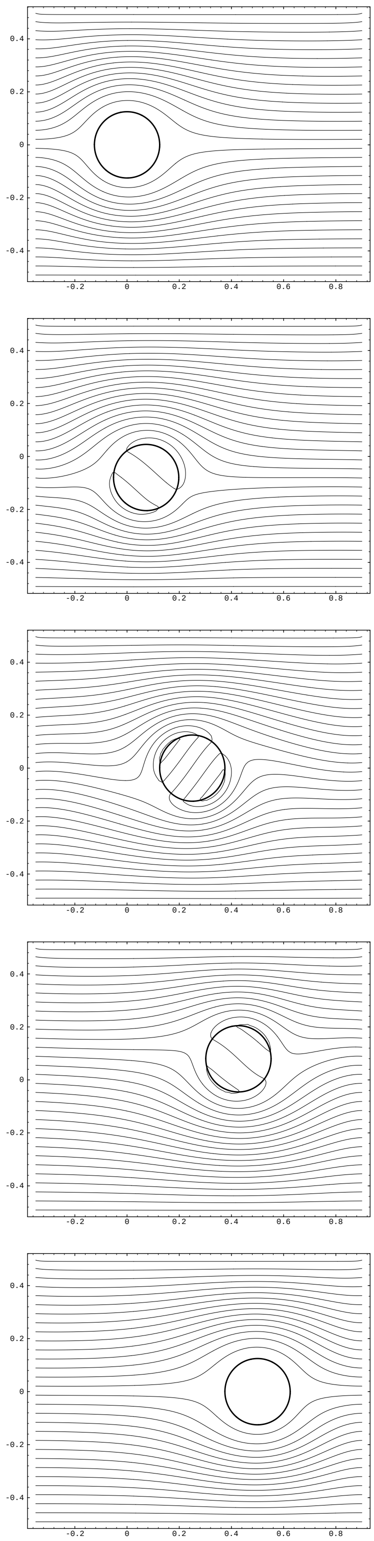

Figure 5. Stream lines at time $t = 4$, 4.5, 5, 5.5, 6 during first half of one period of the disk motion.

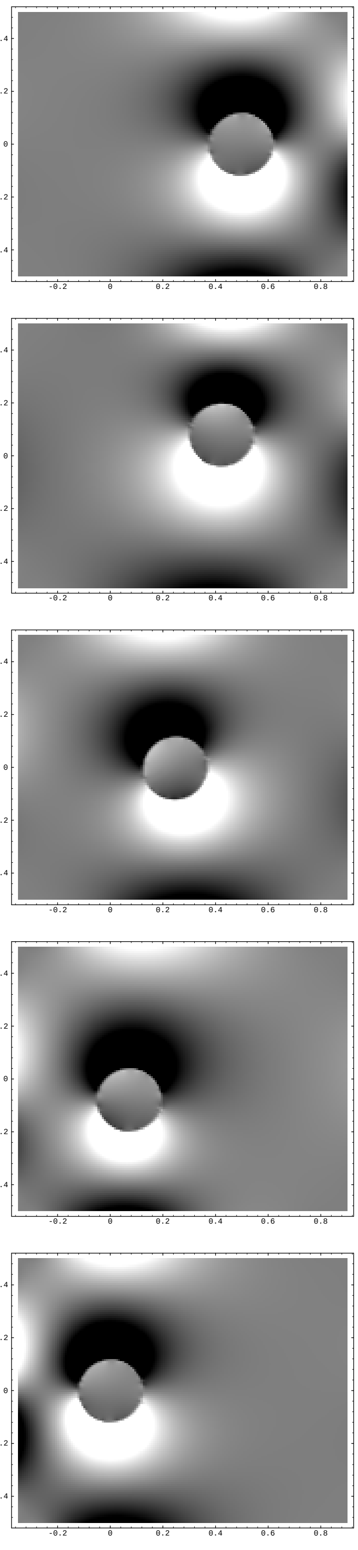

Figure 6. Vorticity density at time $t = 6$, 6.5, 7, 7.5, 8 during second half of one period of the disk motion.

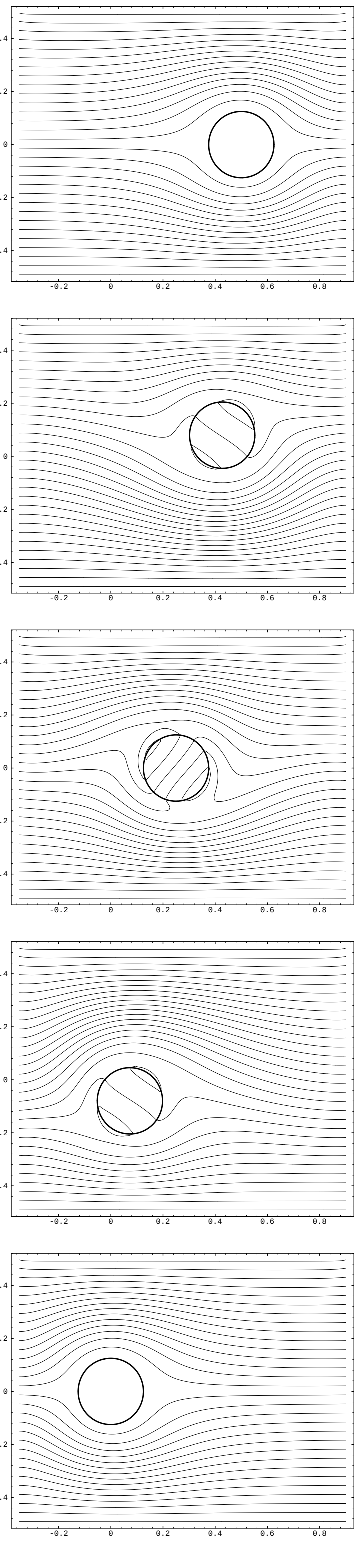

Figure 7. Stream lines at time $t = 6$, 6.5, 7, 7.5, 8 during second half of one period of the disk motion.

There is no basic difficulty to treat the case where the disk ω is also rotating.

Conclusion

This new methodology looks also promising for the simulation of time dependent solution of viscous flow problems around moving rigid bodies. However, further experiments are needed for the incompressible viscous flow past moving rigid bodies modelled by the Navier-Stokes equations for higher values of the Reynolds number.

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