***Problem* 1**: Find $\sum_{k=1}^{50} k^2$.

***Solution* 1**: We know by induction that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$. (Or derive this as Gauss did by pairing off $(n,1)$, $(n-1,2)$, etc. each with sum $n+1$.) Note that $(k+1)^3 = k^3 + 3k^2 + 3k + 1$.

$$\sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n} k^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$
$$\sum_{k=1}^{n}\Big((k+1)^3 - k^3\Big) = 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n$$

Note the left-hand side is a telescoping sum, which evaluates to $(n+1)^3 - 1$.

$$3\sum_{k=1}^{n} k^2 = (n+1)^3 - 1 - n - \frac{3}{2}n(n+1) = \frac{1}{2}(2n^3 + 3n^2 + n) = \frac{1}{2}n(n+1)(2n+1)$$
$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1).$$

Back to the original problem:

$$\sum_{k=1}^{50} k^2 = \frac{1}{6}(50)(51)(101) = 42925.$$

***Problem* 2**: Show $f : (0,\infty) \to \mathbb{R}$ defined by $f(x) = x\coth(x)$ is strictly increasing.

***Solution* 2**: By definition of coth(),

$$f(x) = x\coth(x) = x\frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

We will apply the quotient rule of differentiation, and have:

$$\begin{aligned} f'(x) &= \frac{(e^x - e^{-x})\big[(e^x + e^{-x}) + x(e^x - e^{-x})\big] - x(e^x + e^{-x})^2}{(e^x - e^{-x})^2} \\ &= \frac{(e^x - e^{-x})(e^x + e^{-x}) + x\big[(e^x - e^{-x})^2 - (e^x + e^{-x})^2\big]}{(e^x - e^{-x})^2} \\ &= \frac{e^{2x} - e^{-2x} - 4x}{(e^x - e^{-x})^2}. \end{aligned}$$

The denominator is positive for $x > 0$ due to the square, and that $e^x = e^{-x}$ only when $x = 0$. Note that $g(x) := e^{2x} - e^{-2x} - 4x$, and $g'(x) = 2(e^x - e^{-x})^2$. $g'(x) > 0$ for all $x > 0$. Since $g(0) = 0$ and that g is strictly increasing, $g(x) > 0$ for $x > 0$. $f'(x) > 0$ for all $x > 0$, hence it is strictly increasing.