Problem 1: Find $\sum_{k=1}^{50} k^2$.

Solution 1: We know by induction that $\sum_{k=1}^{n} k = \frac{1}{2} n(n+1)$. (Or derive this as Gauss did by pairing off $(n, 1), (n-1, 2)$, etc. each with sum $n+1$.) Note that $(k+1)^3 = k^3 + 3k^2 + 3k + 1$.

\[
\sum_{k=1}^{n} (k+1)^3 - \sum_{k=1}^{n} k^3 = 3 \sum_{k=1}^{n} k^2 + 3 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1
\]

\[
\sum_{k=1}^{n} ((k+1)^3 - k^3) = 3 \sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n
\]

Note the left-hand side is a telescoping sum, which evaluates to $(n+1)^3 - 1$.

\[
3 \sum_{k=1}^{n} k^2 = (n+1)^3 - 1 - n - \frac{3}{2} n(n+1) = \frac{1}{2}(2n^3 + 3n^2 + n) = \frac{1}{2} n(n+1)(2n+1)
\]

\[
\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1).
\]

Back to the original problem:

\[
\sum_{k=1}^{50} k^2 = \frac{1}{6} (50)(51)(101) = 42925.
\]

Problem 2: Show $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x \coth(x)$ is strictly increasing.

Solution 2: By definition of coth(),

\[
f(x) = x \coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}.
\]

We will apply the quotient rule of differentiation, and have:

\[
f'(x) = \frac{(e^x - e^{-x})[(e^x + e^{-x}) + x(e^x - e^{-x})] - x(e^x + e^{-x})^2}{(e^x - e^{-x})^2}
\]

\[
= \frac{(e^x - e^{-x})(e^x + e^{-x}) + x[(e^x - e^{-x})^2 - (e^x + e^{-x})^2]}{(e^x - e^{-x})^2}
\]

\[
= \frac{e^{2x} - e^{-2x} - 4x}{(e^x - e^{-x})^2}.
\]

The denominator is positive for $x > 0$ due to the square, and that $e^x = e^{-x}$ only when $x = 0$.

Note that $g(x) := e^{2x} - e^{-2x} - 4x$, and $g'(x) = 2(e^x - e^{-x})^2$. $g'(x) > 0$ for all $x > 0$. Since $g(0) = 0$ and that $g$ is strictly increasing, $g(x) > 0$ for $x > 0$. $f'(x) > 0$ for all $x > 0$, hence it is strictly increasing.