PROBLEM OF THE WEEK - FALL 2014 - WEEK 3 - DUE SEPTEMBER 17

Question 1

(*Proposed by Dr. Climenhaga*) Consider the square lattice of $\mathbb{R}^2$ whose points are the points in $\mathbb{R}^2$ with coordinates that are both integers; e.g., $(0,0), (1,-3), (400,16)$.

Is there are regular hexagon (six edge polygon whose edges have equal length and with equiangular vertices) whose vertices all lie on the lattice?

Question 2

Consider the function which takes two points z_1, z_2 in the unit disc $\mathbb{B}^2 = \{z \in \mathbb{R} : |z| < 1\}$ and returns a real number:

$$d(z_1, z_2) = \cosh^{-1}\left(1 + \frac{2|z_1 - z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)}\right).$$

Show that $d : \mathbb{B}^2 \times \mathbb{B}^2 \to \mathbb{R}$ is a metric:

(i) $d(z,z) \geq 0$ with equality only when $z = 0$.
(ii) $d(z_1, z_2) = d(z_2, z_1)$
(iii) For $z_3 \in \mathbb{B}^2$, $d(z_1, z_2) \leq d(z_1, z_3) + d(z_3, z_2)$.

Bonus: What are the geodesics when we use this metric?