PROBLEM OF THE WEEK - FALL 2014 - WEEK 4

Question 1

[*Proposed by Dr. Paulsen*] Let A and B be two $n \times n$ matrices with real entries, so $A, B \in M_n(\mathbb{R})$. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \det(A + Bx)$$

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(i) Show that $f^{(3)}(x) = 3! \det B$.
(ii) Show that in general $f^{(n)}(x) = n! \det B$.

For example, when $n = 3$ and $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $\det B = 2$ and

$$f(x) = \det \begin{bmatrix} 1+x & 0 & 2 \\ 2+3x & 2+2x & 0 \\ 0 & 0 & 1+x \end{bmatrix} = (1+x)(1+x)(2+2x)$$

Then

$$f'(x) = 6(1+x)^2$$
$$f''(x) = 12(1+x)$$
$$f^{(3)}(x) = 12 = 3! \cdot \det B$$

Question 2

[*Proposed by Alex Bearden*] Find a sequence (x_n) of non-negative real numbers such that the series $\sum_n x_n$ converges, but the sequence (nx_n) does **not** converge to 0.