Problem of the Week - Fall 2014 - Week of 11.10.14

Question 1

For which real numbers c does the inequality

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \leq e^{cx^2}$$

hold for all $x \in \mathbb{R}$?

Question 2

Show that $\mathbb{Z} + \sqrt{2}\mathbb{Z} = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is dense[1] in $\mathbb{R}$, but $\mathbb{Z} + \frac{1}{2}\mathbb{Z} = \{a + \frac{b}{2} : a, b \in \mathbb{Z}\}$ is not.

[1] By dense in $\mathbb{R}$ we mean that every real number can can be approximated by elements in the set. Specifically: $\forall y \in \mathbb{R}$ and for any $\epsilon > 0$, we can find $a, b \in \mathbb{Z}$ such that $|y - (a + b\sqrt{2})| < \epsilon$.