Problem 10.1: Find values of $c \in \mathbb{R}$ where:

$$f(x) := \cosh x \leq \exp(cx^2) =: g(x).$$

Solution: $c \geq \frac{1}{2}$.

First note that for all $c \in \mathbb{R}$, f and g are even, and $f(0) = 1 = g(0)$. So, we just need to verify the inequality for $x > 0$. Moreover, we have:

$$\ln \cosh x \leq \ln \exp(cx^2) = cx^2.$$

Taking the derivative of both sides:

$$\frac{\sinh x}{\cosh x} = \tanh x \leq 2cx \tag{1.1}$$

Note that $\tanh' x = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x} > 0$, for all x, and that $\tanh'(0) = 1$. So for equation (1.1) to hold, $2k \geq 1$. We can also check that if $2c < 1$, then there exists $\varepsilon > 0$ where $\tanh x > 2cx$ for $x \in]0, \varepsilon[$.