University of Houston | **Tin Lam**

Problem of the Week | **Fall 2014, Week 5**

Problem 5.1: To begin, we will prove a lemma. Consider an $m \times m$ board (with m possibly infinite), and a game where two players, I and II, take turn marking squares, and the first player marks N squares in a row, column or diagonal wins.

Lemma 1: *The second player, II, does* **not** *have a winning strategy.*

Proof: Suppose player II does have a winning strategy. After player I makes the first move by marking an arbitrary spot, player I becomes the second player, and can simply execute the winning strategy, and thus win before player II. So long, player I does not run into the case where he/she needs to mark a now-occupied spot. However, this does not affect player I's strategy, as he/she can mark another arbitrary square. The extra square cannot adversely impact player I in the execution of the winning strategy. ∎

Now we will prove the claim.

Claim 2: *A game of a 4×4 tic-tac-toe will always end in a draw if both players play optimally.*

Proof: We know from Lemma 1 that player II does not have a winning strategy. What we need to show is that player I does not have one either as player II does have a drawing strategy. Up to symmetry, player I has three possible first moves: (corner, center-square, outer-edge)

$$\begin{bmatrix} \times & \square & \square & \square \\ \square & \bigcirc & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \begin{bmatrix} \square & \square & \square & \square \\ \square & \times & \bigcirc & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \begin{bmatrix} \square & \times & \square & \square \\ \square & \bigcirc & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$$

Regardless of the position of the first $\times$, player II can always plaice his $\bigcirc$ in the inner 4 squares. Note that there are 10 possible winning possibilities (4 horizontal, 4 vertical, and 2 diagonals). However, once player II places the $\bigcirc$, three of the ten options can no longer allow player I to win, with 14 squares left.

At this point, we can just come up with a matching move strategy. For every move player I makes, maker II can make a blocking move that close down at least two (possible three) winning options for player I. Hence, player II can always force a draw by putting a $\bigcirc$ occupying every possible winning line.

To be more precise, consider the the scenarios above:

$$\begin{bmatrix} \times & A & B & A \\ G & \bigcirc & E & D \\ G & E & F & F \\ C & C & B & D \end{bmatrix} \begin{bmatrix} G & D & A & A \\ E & \times & \bigcirc & K \\ F & F & G & K \\ E & D & B & B \end{bmatrix} \begin{bmatrix} C & \times & A & A \\ E & \bigcirc & K & F \\ D & K & C & D \\ E & F & B & B \end{bmatrix}$$

If Player I puts an $\times$ into the square labeled by the letter Z, Player II can always put a matching $\bigcirc$ into the other square label with the same letter Z, causing a draw. ∎